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Relations of $1/f$ and $1/f^2$ power spectra to self-organized criticality

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We find that the discrete sandpile type of models in finite dimension under a uniform particle addition must result in $1/f^2$ or $1/(f+a)^2$ power spectrum depending on the existence of particle conservation law or not. However, their continuous counterparts may exhibit $1/f$ instead. This difference in power spectrum behavior can be explained in terms of rate of relaxation of the system in the continuous models.

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The idea of self-organized criticality (SOC) as introduced by Bak *et al.* [1] is very attractive because of its simplicity and its potential capability of explaining the $1/f$ noises of various systems without the use of any external fine tune parameters. In particular, they introduced a kind of discrete cellular automata models (which are now commonly known as the sandpile models) to demonstrate this idea. As a result, a vast number of authors have studied either numerically or analytically various different varieties of the discrete sandpile models [2,3]. Moreover, some authors began their study of SOC in their continuous counterpart, namely by means of stochastic partial differential equations [4–6]. In addition, some hybrid models, which are based on continuous local heights on a discrete lattice, are also constructed [4–6]. Experiments on real sandpiles are also carried out [7]. It is claimed by various authors that the theory of earthquakes, $1/f$ noises in electrical networks, growth models, and pattern formation may be explained in terms of SOC [1,3,4,6–8].

However, it has been pointed out that the original models of SOC formulated in terms of the discrete automata cannot produce the $1/f$ power spectrum. In fact, experimental stimulations on various systems and the analytic results on some exactly solvable ones suggest that the power spectra of these models should be $1/f^2$ [2]. Recently, we have proved using graph-theoretic method that all the sandpile type of cellular automata models which grid on finite-dimensional spaces and with discrete values of local heights are equivalent to the Abelian sandpile model [6].

Basically, we divide our system regularly into finite subsets of a space group of $\mathbb{R}^n$ and insist that all the toppling rules are translation invariant on this lattice (except possibly at boundary). To each grid, we assign an integer x_i called local height, a real-valued function f_i called toppling condition, and a vector Δ_i called toppling rule to it. We also assume that f_i is strictly increasing with x_i and increasing with x_j for all $j\neq i$. Whenever $f_i>0$, we say that site i becomes unstable and it will topple by the rule

$$x_j \longrightarrow x_j - \Delta_{ij}\,, \tag{1}$$

where Δ_{ij} is the jth element of the vector Δ_i which satisfies the following constraints:

$$\begin{aligned} &\Delta_{ii} > 0,\ \forall i\,,\\ &\Delta_{ij} \le 0,\ \forall i\neq j\,,\\ &\sum_j \Delta_{ij} \ge 0,\ \forall i\,. \end{aligned} \tag{2}$$

If we regard each system configuration as a point on the $\mathbb{Z}^N$ space (where N is the number of grid points) then Δ_i forms a linear independent set of $\mathbb{Z}^N$ [9]. In order that the scaling exponents be robust to perturbation, including the change of underlying particle addition probabilities at each site, we insist that a self-organized critical model must admit an absolute steady state: namely, we can find a subset of stable configurations of the system Ω such that $\mathscr{a}_i[\Omega]=\Omega$ for all i, where $\mathscr{a}_i$ denotes the addition of a particle to site i together the subsequent topplings [9]. Since the system can admit only a finite number of configurations upon the repeated particle addition onto it, we expect $\mathscr{a}_i^n(\alpha)=\alpha$ for some $n\in\mathbb{N}$ for each α. We can also impose a graphical structure onto the system in the following way: draw an arrow from site i to j whenever $i\neq j$ and $\Delta_{ij}\neq 0$, then by our assumption on translational invariance of Δ_i, a regular translational invariant digraph is obtained. In fact, the digraph must form a collection of directed paths [6]. Now we explicitly sort out all the possible method of toppling on this lattice.

Case (i): If f_i is simply a function of x_i only, by translational invariance, we know that this is also true for all other local sites. So we get back the well-known Abelian sandpile model.

Case (ii): If f_i is also a function of some x_j where all the j are located downstream to the directed path (this is to say, particles in site i may, after a number of toppling, arrive at j but not vice versa, then we consider the action of adding a single particle to i when the system is in the configuration where all the local sites are right at their critical level (at least one of the self-organized critical configurations of the system behaves like this). By our assumption, site i must topple first and its local height is reduced by $1-\Delta_{ii}$. Suppose that $\Delta_{ii}>1$, then the local height of i will decrease. Due to this decrease, together with the translational invariance of the lattice, we can always find a site k upstream such that f_k is decreasing with x_i. So we have induced a toppling in site k. Therefore the

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repeated addition of particles at site i can never generate this original system configuration again. As a result, Δ_{ii} can only be 1.

Case (iii): If f_i also depends on some upstream sites, then by means of a similar argument as in case (ii), it is not difficult to see that absolute steady state exists if and only if $\Delta_{ii}=1$ for all i.

We may regard both cases (ii) and (iii), as Abelian sandpile models with $\Delta_{ii}=1$ and with a single j ($\neq i$) such that $\Delta_{ij}=-1$ except possibly at system boundary. So in conclusion, we have proved that all the finite discrete sandpile type of cellular automata models are basically Abelian sandpiles. Further discussions on this issue can be found elsewhere [6].

Moreover, for both cases (ii) and (iii), the self-organized critical configuration shall only consist of a singleton (a set that contains only one element) so that every new particle added will just slide through the grid points and dissipate at the system boundary. In this respect, it behaves just like the one-dimensional model introduced by Bak, Tang, and Wiesenfeld [1]. Therefore, with a uniform (but random) particle addition onto the system, $1/f^2$ power spectrum is a natural consequence [1,6].

Finally, for the Abelian sandpile case [that is case (i)], we like to find all the eigenvalues of the toppling matrix Δ. If in the thermodynamic limit, none of them approaches 0, then the number of self-organized critical states $\det(\Delta)\to\infty$. In addition, it is straightforward to see that the two point correlation function [3] $G_{ij}=\Delta_{ij}^{-1}$ decreases exponentially as the separation between i and j increases. Since we are working in a finite-dimensional lattice, the number of connected cluster size cannot grow exponentially fast. Therefore we shall expect an $1/(f+a)^2$ type of scaling in the power spectrum. This is the case whenever there is no particle conservation laws in the system: the reason is that the value of $\sum_j \Delta_{ij}>0$ for all i and hence the Gerschgorin theorem [10], all the eigenvalues of Δ can approach 0 in the thermodynamic limit.

On the other hand, if some of the eigenvalues tends to 0 in the thermodynamic limit, then the two point correlation function need not die out exponentially. The reason is that some of the eigensubspaces of $\mathbb{R}^N$ becomes more and more ill defined. In this case, the G_{ij} may correlate either algebraically or power-law-like throughout the system. In the first case, an $1/f^2$ power spectrum is observed. Suppose that G_{ij} really correlate in the form of a power law, then the system must have evolved to the state where there is no natural length scale [1]. Thus the power spectrum of avalanche size of the system should be robust (and actually a fixed point) under the renormalization-group procedure in regrouping the lattice grid points. Under such a renormalization, the values of the elements of Δ may change in general. Suppose that Δ_{ij} is bounded in the sense that for sufficiently large separation between i and j, Δ_{ij} must vanish. Then we can always regroup the lattice points such that the renormalized toppling matrix Δ' contains only the nearest-neighbor interactions only. But we have recently proved by direct calculation of the two point correlation function that this type of Δ' can only produce $1/f^2$ power spectra [6]. On the other hand, if the toppling is so nonlocal that Δ_{ij} is not bounded in the above sense, then the value of Δ_{ii} must tend to $+\infty$ in the thermodynamic limit for almost all i. Hence almost all the eigenvalues of $\Delta\to+\infty$ so that $G_{ij}=\Delta_{ij}^{-1}\to 0$ exponentially fast as the separation between i and j increases. This contradicts with the assumption that G_{ij} scales like a power law.

In short, by examining all the possible cases, we find that the two point correlation function and hence the power spectrum of the system can either scale like $1/f^2$ or $1/(f+a)^2$ depending on the existence of a particle conservation law in the system.

On the other hand, various different continuous models, where the evolution of the system is governed by a single or a system of stochastic particle differential equations *do* exhibit $1/f$ scaling in the power spectra [4,5]. Similar behavior is also observed in some hybrid models [4,5]. In fact, the hybrid models is much more closely related to the continuous models than to the discrete cellular automata (CA). By taking the limit where the grid size goes to infinity, the continuous models is recovered. Also, all the numerical stimulations of the continuous models must grid the space into finite number of cells and therefore they are basically the hybrid case. Therefore, it is worthwhile to see why the discrete models never show the $1/f$ behavior even in the thermodynamic limit (where the total number of grids tend to infinity) while their continuous counterparts can. The answer of this problem is the following: when we take the thermodynamic limit, we see that the distance between adjacent sites becomes so small that we can replace the difference between the local heights by spatial partial derivatives ($\partial/\partial x$). However, the concept of toppling in discrete SOC models, which is a sudden catastrophic process for all discrete sandpile models, is fundamentally different from that in models using smooth stochastic partial differential equations which bear temporal partial derivatives ($\partial/\partial t$). That is to say, all the discrete models allow an instantaneous mechanism for the system to relax via toppling. This is surely not the case in all the continuous counterparts. Owing to this gradual relaxation process in the continuous models, in which the transition between stable and unstable configurations is a smooth but time consuming process. Therefore extra particles may (and is in fact likely) add onto the system before it is completely relaxed. Nevertheless the propagation of the avalanche in these models, even in the limit where the particle addition rate is just above zero, do not have a natural length scale. Therefore temporal scaling in power spectra are expected even in the continuous models. Thus the presence of $1/f$ power spectrum is not unusual especially when the dimension of the space is large [1,4] (say $d\geq 4$ in the model of Hwa and Kardar [5]). It is due to this fundamental difference in the rate of toppling (or relaxation) that the power spectra of these two system can differ so greatly.

Furthermore, the discreteness of the local heights for the discrete CA models also play an important role for the exhibition of $1/f^2$ power spectra in these models. The two point correlation function of these systems (when completely relaxed) will either decrease exponentially or simply do not decay as the separation between two sites increases. So the system can either get no scaling or $1/f^2$ scaling in the power spectrum under a uniform but random particle addition if it is finite dimensional. Detail discussions on this point can be found elsewhere [6]. On the other hand, the situation is completely different for the continuous and hybrid models. Owing to the continuous nature of the local heights, the two point correlation functions of these systems may have a much wider variety of behavior including the $1/d$ decrease where d is the distance between two successive sites. So $1/f$ scaling is possible in these kind of models.

Other possibilities in restoring the $1/f$ power spectra includes alternation of the underlying particle addition probability of the system [6], using a totally different type of toppling rules so that the number of particles topples each time in a given site is not fixed [11,12]. So far, we have discussed the difference in the power spectra between the continuous and discrete models. We have also seen that this fundamental difference is due to the rate of particle relaxation between these two types of models and the different nature of the local heights. Therefore, the $1/f$ power spectrum of a system may still be explained using the concept of self-organized criticality provided that one use the formalism of stochastic partial differential equations (or at most the hybrid models) instead of the discrete cellular automata or to change the rules so that the toppling matrix Δ is not a fixed quantity. In particular, recently a lattice gas type of discrete automata model introduced by Jensen does show $1/f$ scaling in power spectrum [12]. His model, which is driven by the dynamics of the system boundary, uses a set of toppling rules which are function of the local distribution of particles in the system.

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