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Soliton chaos in the nonlinear Schrödinger equation with spatially periodic perturbations

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We perturb the one-dimensional nonlinear Schrödinger equation with a time-independent spatially periodic potential with a period large compared to the spatial width of solitons present in the system. A collective-coordinate approximation maps the system to a nonintegrable many-particle dynamics with an effective Hamiltonian, which we derive under the assumption that we can neglect three-soliton collisions as well as two-soliton collisions with vanishing relative velocity. We give an estimate for the power radiated by a single soliton in the presence of the perturbation and show that radiative effects can be neglected for perturbations with sufficiently small amplitude and large spatial period. We show that the nonintegrability of this perturbed nonlinear Schrödinger equation manifests itself already in the two-soliton sector. We use the effective many-particle Hamiltonian to investigate soliton depinning. The effective Hamiltonian results are compared with numerical simulations of the full perturbed equation.

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To understand the physical properties of nonlinear, spatially inhomogeneous materials is of considerable practical importance [1–4]. Applications can be found in many fields: nonlinear optics [5], long Josephson junctions [6], charge-density waves [7], dislocations in crystalline materials [8], and so on. In the absence of spatial modulations the dynamics of low-dimensional materials can often be approximated by completely integrable nonlinear partial differential equations such as the nonlinear Schrödinger (NLS) or the sine-Gordon equation.

These completely integrable dynamics take into account the interaction of the nonlinear excitations (waves, solitons, breathers) solely through phase and space shifts: All scattering events leave the momenta of the excitations unchanged. In real nonlinear materials the interactions will certainly have an anharmonic component. This can give rise to interconversion of excitations: solitons can emit radiation, radiation can self-focus to coherent, localized excitations, and solitons can scatter inelastically.

In this paper we examine the inelastic scattering of solitons in the presence of a spatially periodic perturbation. As the underlying unperturbed dynamics we choose the nonlinear Schrödinger equation because of its direct physical relevance as well as it describing certain limiting cases of other nonlinear dynamics. We restrict our study to the case when the spatial width of solitons present in the system is much smaller than the spatial period of the perturbation (for other cases see [9]). In this limit the radiation emitted by the solitons is negligible on the timescale that the anharmonic interaction between the solitons manifests itself.

The perturbed NLS equation we study is of the form:

$$i\psi_t+\psi_{xx}+2\psi|\psi|^2=\epsilon\psi\cos(kx)\,. \tag{1}$$

For a *single* soliton moving in the presence of the perturbation we make a collective variable ansatz [10–12]:

$$\psi(x,t)=2\eta\frac{e^{i\dot{q}x/2-i\Phi}}{\cosh[2\eta(x-q)]}\,. \tag{2}$$

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Equation (1) possesses two integrals of motion, the norm $\mathcal{N}$ and the energy E. For the $\psi(x,t)$ given in Eq. (2) they have the following values:

$$\mathcal{N}=\int_{-\infty}^{+\infty}|\psi|^2dx=4\eta\,, \tag{3}$$

$$E=\int_{-\infty}^{+\infty}[|\psi_x|^2-|\psi|^4+\epsilon|\psi|^2\cos(kx)]$$
$$=\eta\left[\dot{q}^2-\frac{16}{3}\eta^2\right]+\frac{\epsilon k\pi}{\sinh(k\pi/4\eta)}\cos(kq)\,. \tag{4}$$

The conservation of the norm of ψ leads to $\eta=\text{const}+O(\epsilon^2)$. The time dependence of the phase Φ decouples from the time dependence of the soliton position q, which is the relevant dynamical variable on which we will focus below.

The conservation of the energy E leads to an equation of motion for q which can be derived from the following effective single-particle Hamiltonian (for a single soliton):

$$H_1=\frac{p^2}{2\eta}+\frac{\epsilon k\pi}{2\sinh(k\pi/4\eta)}\cos(kq)\,. \tag{5}$$

The amplitude of the potential is nontrivially rescaled. Only in the limit $k/\eta\rightarrow 0$ does one recover the prefactor $2\epsilon\eta$ which one would expect naively by expanding the perturbing potential locally up to linear order in position and then solving the perturbed NLS exactly.

As the perturbed NLS equation (1) with $\cos(kx)$ replaced by $ax+b$ is still completely integrable [13] one expects radiation mainly to originate from the nonvanishing curvature of the perturbing potential. The radiative power emitted can be made arbitrarily small by reducing the amplitude ϵ of the potential as well as the ratio k/η of the width of the soliton and the period of the potential. Numerical evidence shows that one can easily achieve the situation with negligible radiation on the time scale of the soliton motion, for example the period of oscillation around a potential minimum.

In the next step we neglect the perturbation ($\epsilon=0$) and focus on a *two-soliton* collision. When two NLS solitons collide their centers of inertia q_i, as well as their phases Φ_i, suffer a shift. As the perturbed NLS equations (1) is U(1) invariant we neglect the dynamics of the phases. The shifts of the soliton positions q_i ("space shifts") amount to an attractive interaction between the solitons. For example, the soliton that was the right one for $t\rightarrow-\infty$ and the left one for $t\rightarrow+\infty$ has the form

$$\lim_{t\rightarrow\pm\infty}|\psi|=\text{sech}[2\eta(x-q)\pm a]\,. \tag{6}$$

For the other soliton the shift a should be replaced by $-a$. The shift a is given by (see, for example, [14])

$$a=\tfrac{1}{2}\ln\left[\frac{16(\eta_1+\eta_2)^2+(v_1-v_2)^2}{(v_1-v_2)^2}\right], \tag{7}$$

where v_1 and v_2 are the asymptotic velocities for the two separated solitons and η_1 and η_2 their amplitude parameters [Eq. (2)].

A simple calculation shows that the following two-particle Hamiltonian gives rise to the same space shifts Eq. (7):

$$H_2=\frac{p_1^2}{2\eta_2}+\frac{p_2^2}{2\eta_2}-8\eta_1\eta_2(\eta_1+\eta_2)\text{sech}^2\left[\frac{2\eta_1\eta_2(q_1-q_2)}{\eta_1+\eta_2}\right]. \tag{8}$$

Our third step is to combine the single-particle Hamiltonian H_1 with the attractive two-soliton interaction from Eq. (8): This leads to the following effective N-particle Hamiltonian:

$$H_{\text{eff}}=\sum_{i=1}^{N}\left[\frac{p_i^2}{2\eta_i}+\frac{\epsilon k\pi\cos(kq_i)}{2\sinh(k\pi/4\eta_i)}\right]-8\sum_{1\le i<j\le N}\eta_i\eta_j(\eta_i+\eta_j)\text{sech}^2\left[\frac{2\eta_i\eta_j(q_i-q_j)}{\eta_i+\eta_j}\right]. \tag{9}$$

This Hamiltonian describes the motion of N unharmonically coupled nonlinear pendula which is nonintegrable already for $N=2$. We stress that the nonintegrability of the perturbed NLS equation (1) can already be seen in the two-soliton sector before radiative effects become important.

We illustrate our analytical results by numerical simulations. We use the integrable discretization of the NLS equation given by Ablowitz and Ladik [15] (see also [12]) near the continuum limit. We integrate the resulting differential-difference equations with the help of a fifth- and sixth-order Runge-Kutta-Verner method, monitoring the accuracy of the numerical integration with the help of the conserved quantities $\mathcal{N}$ and E [see Eqs. (3) and (4)]. For well-separated solitons their positions are tracked numerically by a three-point fit around local maxima having an amplitude exceeding a given minimum value.

Figure 1 shows an example for two solitons with $\eta_1\neq\eta_2$ interacting in a cosine potential. We see that the motion of the soliton positions (solid lines) and the effective two-particle dynamics (crosses) compare well over many collisions.

In Fig. 2 we show a Poincaré section for two effective particles with $\eta_1=\eta_2$ with fixed total energy ($H_{\text{eff}}=E_0$) and $q_2=0$. The chaotic region around the single-soliton separatrix determined by the energy $E_1=\epsilon k\pi/2\sinh(k\pi/4\eta_1)$ is clearly visible.

A plausible explanation for the chaotic region is as follows. If the first soliton has an energy slightly below the separatrix energy E_1 it will be trapped in a potential minimum. Collision with the second soliton leads to a space shift which can move the first soliton over a potential maximum. Whether this happens depends sensitively on the initial conditions of both solitons, leading to a positive Lyapunov exponent.

The approximation of the full NLS soliton dynamics by the N-particle dynamics fails in certain special cases, including the following.

(i) When the curvature of the perturbing potential excites additional modes which are not taken into account

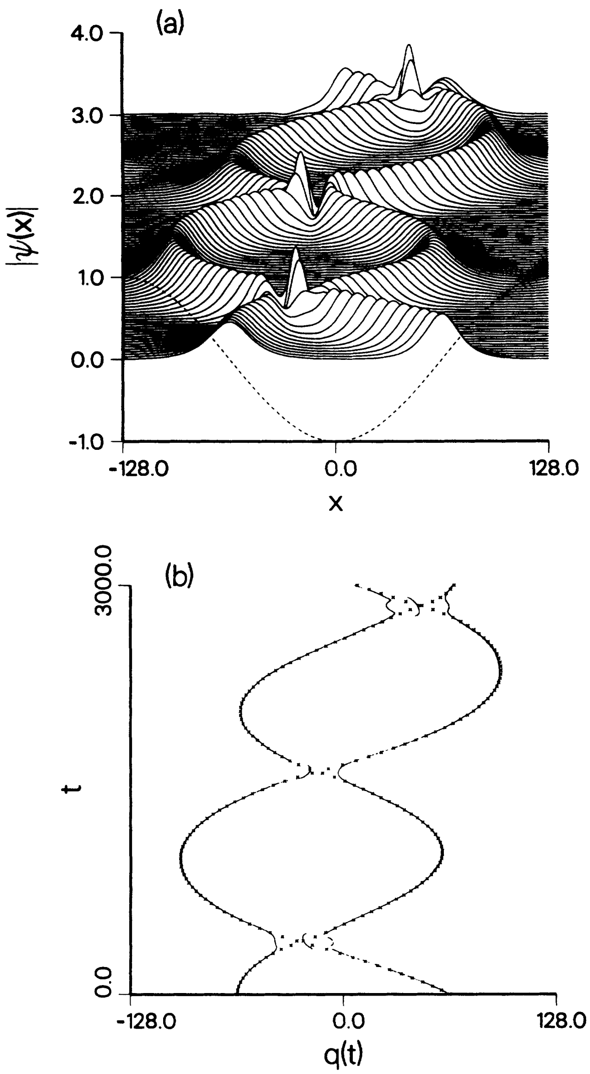

FIG. 1. Two interacting solitons on a cosine potential. Parameters: $V(x)=\epsilon\cos(2\pi x/256)$ with $\epsilon=-0.01$, $\eta_1=0.045$, $\eta_2=0.055$. Initial values: $q_1(0)=-64$, $q_2(0)=64$, $p_1(0)=0$, $p_2(0)=-0.0088$. (a) The full solution (multiplied by a factor of 5) of the perturbed NLS equations (1). (b) Comparison between the soliton positions of the full solution (solid lines) and the effective two-particle dynamics (crosses).

by the effective particle dynamics. For sufficiently small coupling parameter ϵ, we found numerically that the approximation works for times $T=O(\epsilon^{-\nu})$ with $\nu>1$.

(ii) When two solitons collide with vanishing relative velocity. This happens only for a small fraction of initial conditions. Our approximation could be improved in this case by using the exact two-soliton solution [16] of the unperturbed dynamics ($\epsilon=0$) with appropriate time-dependent parameters instead of the single-soliton ansatz (2). Notice that the conservation of the norm $\mathcal{N}=\mathcal{N}_1+\mathcal{N}_2$ allows for a change of the soliton parameters η_1 and η_2 which determine the shape of the solitons. Numerical results show that η_1 and η_2 are indeed changing during a collision with vanishing relative velocity.

(iii) When three (or more) solitons collide at the same time. In a sufficiently dilute gas of solitons these are rare events, which we neglect.

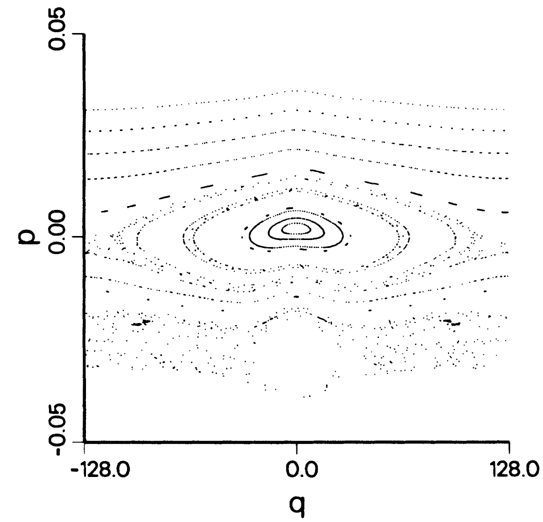

FIG. 2. Poincaré section for the effective two-particle dynamics. Parameters: $V(x)=\epsilon\cos(2\pi x/256)$ with $\epsilon=-0.01$ and periodic boundary conditions, $\eta_1=\eta_2=0.05$. Fixed total energy: $E_0=0.0152152$. Section at $q_2=0$ for $p_2<0$. Shown are the phase space coordinates of the "slow" particle ($q=q_1$, $p=p_1$). Total integration time $T=40\,000$.

Excluding these special cases, the effective N-particle dynamics can be used to analyze the behavior of the N-soliton dynamics in the full perturbed NLS equation. As an application we discuss the depinning of trapped solitons by a beam of fast solitons.

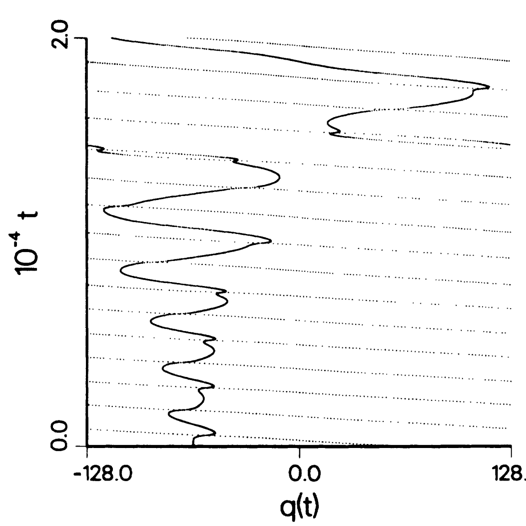

FIG. 3. Depinning of a soliton initially at rest in a potential well by a "fast" soliton. Parameters: $V(x)=\epsilon\cos(2\pi x/128)$ with $\epsilon=0.002$ and periodic boundary conditions, $\eta_1=\eta_2=0.05$. The two minima of the potential are at $q=\pm 64$. Initial values: $q_1(0)=-64$, $q_2(0)=64$, $p_1(0)=0$, $p_2(0)=-0.0115$. Shown are the positions of both effective particles (determined numerically). The straight dotted lines give the position of the "fast" soliton, the irregular lines the position of the soliton undergoing depinning.

A constant flux of fast moving solitons entering a system containing N solitons from one side leads to space shifts of the solitons in the system periodically in time directed opposite to the flux. For an appropriate choice of the density of the solitons in the flux as well as their parameters η_i and v_i, this drag will be strong enough for the solitons in the potential wells to overcome the potential barrier. This depinning leads to a flux of solitons directed oppositely to the incoming flux.

We calculate the depinning threshold in a simple case when the solitons in the well and the flux solitons have equal masses η but the flux solitons move much faster. Then the collisions can be assumed to happen instantaneously and the solitons in the well will be shifted in space by $\Delta q = a/\eta$ [see Eq. (8)] with negligible change of velocity. This leads to a change in the energy of the well solitons which is compensated by the beam solitons. A soliton located in a well with energy $E = H_1(q,p)$ will leave the well after one collision if

$$H_1(q+a/\eta,p) > \frac{\epsilon k\pi}{2\sinh(k\pi/4\eta)} . \tag{10}$$

The solitons in the well can lose as well as gain energy during the collision depending on their position in the well at the time of the collision. Under suitable conditions the collisions with the beam solitons can lead to depinning through a resonance. Figure 3 illustrated this process with the help of the effective two-particle approximation through a period-two resonance.

In summary, we have shown how a nonintegrably perturbed NLS equation with soliton excitations can be mapped, using a particle ansatz, to a many-particle Hamiltonian which takes into account the single-soliton potentials and two-soliton interactions. The dynamics is nonintegrable and shows chaos already for two solitons. Away from the chaotic region it can be used for long-time predictions. A large number of interesting many-soliton effects can be investigated and understood with the help of this effective many-particle dynamics. We have given soliton depinning as an example.

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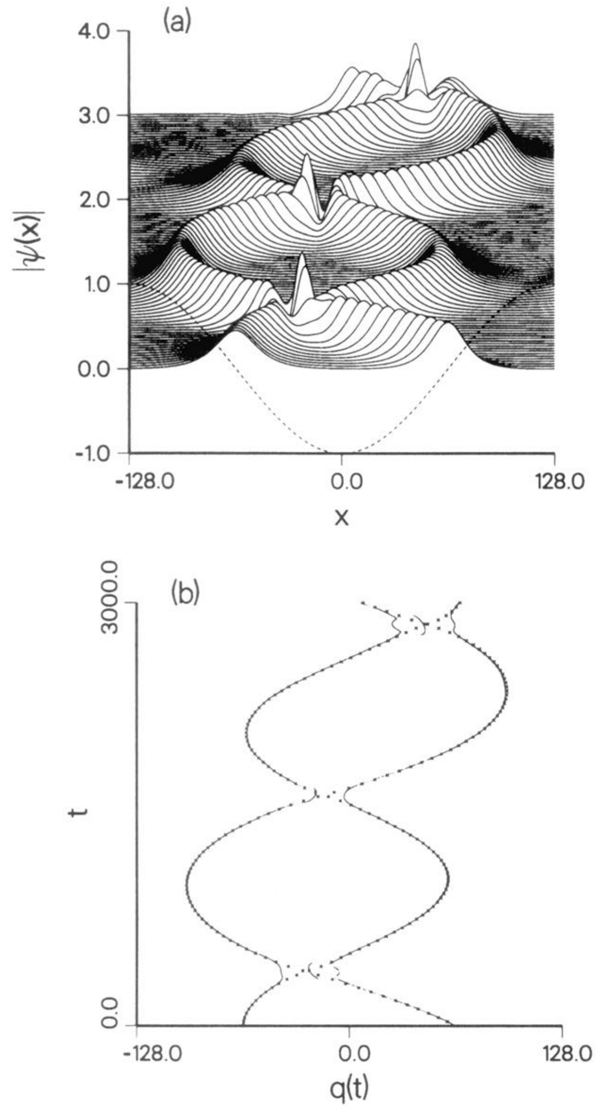

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