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Performance measures of quantum-phase measurement

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Dispersion and peak likelihood—two performance measures recently used for the description of quantum phase—are compared in this Rapid Communication. The Susskind-Glogower phase operator represents the extremal quantum estimator with respect to both measures. The states with infinite peak likelihood and zero phase information explicitly demonstrate that peak likelihood is not relevant to the phase measurement, whereas dispersion represents a well-behaved performance measure of the phase variable. Peak likelihood restricts the lower limit of the variance of photon number, and the above-mentioned states represent the extremal states of the Shapiro-Shepard [Phys. Rev. A **43**, 3795 (1991)] uncertainty relation.

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The existence of some meaningful performance measure of quantum-phase measurement has been a long-standing problem for every phase concept used in the past. A performance measure is intuitively understood as a functional associated with the given probability density function (PDF) of phase measurement, which in principle makes it possible to compare the accuracy of different phase-shift measurements. The specification of an appropriate performance measure is crucial to the investigation of the ultimate limit of phase-shift measurements.

This Rapid Communication addresses the phase-measurement problem from the viewpoint of quantum estimation theory [1] along the same lines established in papers [2–4]. Specifically, we compare the recently introduced performance measures of phase variance, reciprocal peak likelihood, and dispersion. We will demonstrate that an optimum performance measure of the phase variable exists, according to the mathematical statistics, as dispersion. The relations between the above-mentioned quantities are specified and the reasons for the failure of the remaining ones are mentioned. Particularly, the weakness of the reciprocal peak likelihood explains the behavior of the PDF specified in Ref. [4], which achieves the infinite peak likelihood but contains no information about phase shift. Moreover, it is emphasized, that the Susskind-Glogower (SG) phase concept is not only the maximum likelihood estimator [2], but also the minimum dispersion estimator [1].

Consider the general quantum estimation problem of measurement of phase variable $\theta = (-\pi, \pi]$, which enters the input quantum state $|\psi\rangle_{\rm in}$, on which the measurement is performed, as

$$|\psi\rangle = \exp(i\hat{n}\theta)|\psi\rangle_{\rm in}, \qquad (1)$$

$\hat{n} = \hat{a}^\dagger\hat{a}$ being the photon-number operator of the single-mode boson field. According to the quantum estimation theory, every measurement can be described using a probability operator measure (POM) $d\hat{\Pi}(\phi)$, which provides the positively defined resolution of the identity operator

$$\int_{-\pi}^{\pi} d\hat{\Pi}(\phi) = \hat{1}, \qquad d\hat{\Pi}(\phi) \geq 0.$$

The conditional probability density $p(\theta|\phi)$ of finding the value ϕ , while the given shift is θ, is simply given as

$$p(\theta|\phi)d\phi = \langle\psi|d\hat{\Pi}(\phi)|\psi\rangle. \qquad (2)$$

The general question then is to optimize the whole measurement, i.e., to choose the best POM representing it. Of course, the criterion of the best choice is still rather vague, unless we specify how the deviations from the true value are important. For this purpose, the cost function $C(\theta, \phi)$, assessing the cost of errors in the estimates, is introduced. The optimum strategy needs to pick the POM, which minimizes the average cost

$$C = (1/2\pi)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\theta\ C(\theta,\phi)\ \langle\psi|d\hat{\Pi}(\phi)|\psi\rangle, \qquad (3)$$

and may be found by the method of Lagrange multipliers, as is done in Ref. [1], Chap. 8. This quantity represents the asked performance measure for the particular choice of cost function $C(\theta, \phi)$.

All considerations of this Rapid Communication are performed using the Susskind-Glogower phase concept due to its extremal properties. Particularly, we will assume that the phase measurement is matched to the measured input quantum state as

$$d\hat{\Pi}(\phi) = |e^{i\phi},\psi\rangle\langle\psi, e^{i\phi}|d\phi/2\pi, \qquad (4)$$

where

$$|e^{i\phi},\psi\rangle = \sum_{n=0}^{\infty} e^{i\phi n + i\chi_n}|n\rangle,$$

$$|\psi\rangle_{in} = \sum_{n=0}^{\infty} x_n e^{i\chi_n}|n\rangle, \ \ x_n \geq 0.$$

The state

$$|e^{i\phi}\rangle = \sum_{n=0}^{\infty} e^{i\phi n}|n\rangle$$

is the eigenstate of the SG operator $\hat{A} = (\hat{n}+1)^{-1/2}\hat{a}$. Another remarkable property of this phase concept is that the phase representation of $|\psi\rangle$ is intimately related to the number-ket expansion, since they are a Fourier transform pair [3]

$$\Phi(e^{i\phi}) = \sum_{n=0}^{\infty} \psi_n e^{-i\phi n}, \qquad \psi_n = (1/2\pi)\int_{-\pi}^{\pi} d\phi \Phi(e^{i\phi})e^{i\phi n}, \tag{5}$$

where $\Phi(e^{i\phi}) = \langle e^{i\phi}|\psi\rangle$ and $\psi_n = \langle n|\psi\rangle$. The conditional PDF (2) then has the form

$$p_{\rm SG}(\theta|\phi) = p_{\rm SG}(\phi-\theta) = \frac{1}{2\pi}\sum_{n,m=0}^{\infty} e^{i(\phi-\theta)(n-m)}x_n x_m. \tag{6}$$

Let us specify now the performance measures relevant to the phase measurement. The standard procedure tends to the criterion of the mean-square error, if the cost function in (3) is specified as the quadratic function $C(\theta,\phi) = (\theta-\phi)^2$. Then, as is detailed in [1], the minimum average cost is given as

$$C_{\rm min} = (\Delta\phi)^2 = \langle\phi^2\rangle - \langle\phi\rangle^2,$$

where the parentheses abbreviate the mean value over phase $\langle\cdots\rangle = \int_{-\pi}^{\pi} d\phi\ p_{\rm SG}(\phi)\cdots$. However, this functional is not appropriate as a performance measure of the phase variable, since it is evidently not invariant with respect to the phase shift transformation. Consequently, the two shifted PDF's $p(\phi)$ and $p(\phi+c)$, which differ in mean values of phase about an arbitrary value c, will not result in the same mean-square error, even if they have the same shape. This property will be used in the following treatment.

The investigations of Refs. [2,3] are relevant to the singular cost function $C(\theta,\phi) = -\delta(\theta-\phi)$. The SG POM (4) is shown to be an extremal estimator, which tends to the criterion of minimum of the reciprocal peak likelihood

$$\delta\phi = 1/p(\theta|\phi=\theta) = 2\pi/f^2 \tag{7}$$

or, equivalently, the maximum peak likelihood $p(\theta|\phi = \theta) = f^2/2\pi$, where f is an abbreviation for $\sum_{n=0}^{\infty} x_n$. This measure is evidently invariant with respect to the phase-shift transformation, since the peak likelihood, i.e., the height of the PDF, is independent of its position. Nevertheless, the peak likelihood does not represent a meaningful performance measure of the phase variable. In reality, the large value of peak likelihood does not necessarily imply that all the measured data are distributed sufficiently close to the mean value. An example of such behavior is given in Ref. [4] as the state, which provides the given average number of photons N, the infinite peak likelihood, but converges in norm to the vacuum state

$$|\psi\rangle = \lim_{M\to\infty}\left[|0\rangle + \left(N\Big/\sum_{n=0}^{M} 1\Big/n\right)^{1/2}\sum_{n=0}^{M} 1\Big/n|n\rangle\right]. \tag{8}$$

An explanation of this behavior may be given, if the cost function

$$C(\theta,\phi) = |e^{i\theta} - e^{i\phi}|^2 = 4\sin^2[(\theta-\phi)/2] \tag{9}$$

is taken into account. This tends to the criterion of dispersion

$$D^2 = 1 - |\langle e^{i\phi}\rangle|^2, \tag{10}$$

and the POM minimizing the average cost function is the same as in the previous case, see Ref. [1], Chap. 8, 2b, and therefore, the SG POM provides the quantum measurement for the minimum dispersion estimation. In this case, the dispersion may be specified as

$$D^2 = 1 - \sum_{n=0}^{\infty} x_n x_{n-1} = (\tfrac{1}{2})\sum_{n=0}^{\infty}(x_n - x_{n-1})^2, \tag{11}$$

where formally $x_{-1} = 0$. We will now show that the problems of previously mentioned performance measures disappear in the case of dispersion.

Some properties of dispersion are evident. When the errors of directional data are on the average very small, then the cost function (9) is approximately $(\theta-\phi)^2$. Consequently, the dispersion is equal to the mean-square error whenever the phase measurement is sufficiently accurate. Moreover, the dispersion is also invariant respective to the phase-shift transformation, as expected. Let us specify the relation between dispersion and peak likelihood. The existence of the state (8) shows that the upper limit of peak likelihood need not be relevant to the phase resolution. The physical justification is the following: the height of the PDF of the phase variable does not imply any restrictions to its width, since such a peak could be very narrow and therefore unimportant. But, on the contrary, if a PDF with a certain width is assumed, then this condition implies the existence of a minimal value of the height of the PDF. Therefore, it is reasonable to investigate the lower limit of peak likelihood or functional f in expression (7), under the constraints of the given width-dispersion (11) and condition of normalization $\sum_{n=0}^{\infty} x_n^2 = 1$. The method of Lagrange multipliers can be applied to this problem, yielding a difference equation with constant coefficients. For our purpose it is not necessary to carry out all of these lengthy calculations, since the value of $f_{\rm min}$ may be simply estimated for D small, if we notice that the dispersion is the width of the distribution and $f^2/2\pi$ is its height. The extremal PDF, for which the height is minimal with respect to its width could be easily concluded as the rectangular distribution

$$p_{\rm rect} = \begin{cases} f^2/2\pi, & |\phi| \le q, \\ 0, & q < |\phi| \le \pi. \end{cases}$$

The relations for small q follow as $\langle e^{i\phi}\rangle = \sin q/q$ and $D \approx q/\sqrt{3}$. The condition of normalization for $p_{\rm rect}$ then reads $f^2_{\rm min} q = \pi$, which implies, apart from an unimportant constant multiplier, the relation

$$f^2_{\rm min} D \approx \pi/\sqrt{3} \approx 1.$$

The rectangular PDF represents an ultimate limit, which can be approximated but not achieved due to the number-ket causality [3], a consequence of the Paley-Wiener theorem. The Gaussian PDF's well approximate this limiting shape, since

$$f^2_{\rm Gauss} \approx 2\sqrt{\pi K},\;\; D \approx \Delta\phi \approx 1/(2\sqrt{K}),$$

and therefore

$$f^2_{\rm Gauss}\, D \approx \sqrt{\pi},$$

K being N (N^2) for homodyne (TCS heterodyne) detections [3,5]. Thus we heuristically conclude the estimation

$$f^2 \geq f^2_{\rm min} \approx 1/D, \tag{12}$$

which resembles an uncertainty relation between phase and its canonically conjugated quantity. Therefore, the peak likelihood seems to be relevant to the photon-number variance.

This viewpoint may be supported by exact calculations. For this purpose, the number-phase uncertainty relation derived by Shapiro and Shepard [3] may be exploited. They found, using the phase representation, the relation

$$(\Delta N)^2(\Delta\phi)^2 \geq \tfrac{1}{4}[2\pi p(\pi,|\psi\rangle) - 1]^2 \tag{13}$$

where $p(\phi,|\psi\rangle) = |\Phi(e^{i\phi})|^2/2\pi$ and photon-number variance may be simply expressed in the phase representation as

$$(\Delta N)^2 = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \left|\frac{d\,\Phi'(e^{i\phi})}{d\phi}\right|^2,$$

where $\Phi'(e^{i\phi}) = \Phi(e^{i\phi})e^{iN\phi}$. Let us suppose without loss of generality that the maximum peak likelihood of the PDF $p(\phi,|\psi\rangle)$ is at the origin $\phi = 0$. The desired result can be easily obtained, if we specify the new phase wave function as

$$\Phi_n(e^{i\phi}) = \Phi(e^{i\phi+i\pi}), \tag{14}$$

i.e., if we simply shift the maximum peak likelihood of the new PDF $p_n(\phi,|\psi_n\rangle) = |\Phi_n(e^{i\phi})|^2/2\pi$ to the points $-\pi,\pi$. This manipulation evidently does not change the moments of the photon-number operator. The inequality (13) for the quantum state $|\psi_n\rangle$ with the new (shifted) phase wave function $\Phi_n(e^{i\phi})$ then has the form

$$(\Delta N)^2(\Delta\phi_n)^2 \geq 1/4[f^2 - 1]^2, \tag{15}$$

since the photon-number variance is invariant with respect to the phase-shift transformation. On the contrary, the mean-square error is not invariant and can be estimated as [6]

$$\Delta\phi_n \leq \pi.$$

This implies the desired relation between photon-number variance and peak likelihood

$$(\Delta N)^2 \geq [(f^2-1)/2\pi]^2. \tag{16}$$

The difference between peak likelihood and the background noise $1/2\pi$ estimates the lower bound of the photon-number noise. This result may be further simplified for large peak likelihoods as the uncertainty relation

$$\Delta N\;\delta\phi \geq 1. \tag{17}$$

The variational formulation makes it possible to appreciate the lower bounds of the inequality (16) with better accuracy. The extremalization of the peak likelihood under the constraints of a given photon number and photon-number variance tends to the specification of the extremal states with number-ket decomposition

$$x_n = \frac{A}{n^2 + pn + r}, \tag{18}$$

A, p, and r being Lagrange multipliers. This treatment clarifies the meaning of the extremal states discussed in Refs. [2–4]—they yield the extremal photon-number noise, but not necessarily phase noise.

Let us also notice that the relations (12) and (17) imply the inequality

$$(\Delta N)^2 D^2 \geq 1/(2\pi)^2.$$

This estimation may be compared with the Heisenberg uncertainty relation following from the commutation rule $[\hat{n},\hat{A}_{\rm SG}] = -\hat{A}_{\rm SG}$,

$$(\Delta N)^2 D^2 \geq 1/4[1 - D^2].$$

The former inequality provides a less effective estimation than the latter one, since the equality signs in (12) and (17) do not occur simultaneously.

We have dealt with the problem of ideal phase measurement using the SG phase concept. Nevertheless, the same conclusions are valid also for the Pegg-Barnett phase concept representing an alternative but physically equivalent treatment [7]. The dispersion was shown to be the only well-behaved performance measure of the phase variable. The small value of dispersion implies a reciprocal peak likelihood at least of the order $1/D$ but not vice versa; a high reciprocal peak likelihood does not necessarily restrict the value of dispersion. On the other hand, the peak likelihood of the PDF associated with the ideal phase measurement estimates the lower bound of photon-number noise. The quantum state with infinite peak likelihood and zero information about the phase shift explicitly demonstrates the nonequivalence of reciprocal peak likelihood and dispersion as performance measures of phase. Moreover, this state belongs to the broader class of states, which minimize the photon-number noise under the constraint of a given peak likelihood. The knowledge of the PDF of phase measurement using the ideal phase concept allows us not only to specify the statistics of the phase variable, but also to conclude the minimum possible noise of the photon number. Quantum estimation theory establishes the primacy of the ideal phase operator, since it represents the minimum dispersion and maximum peak likelihood estimator. The physical interpretation of the former property is clear—it means that for a given quantum state the ideal phase measurement provides the most accurate information about the phase shift compared to other phase concepts. Nevertheless, the latter question—how to interpret the maximum likelihood—has remained open. The main result of this contribution, inequality (16), indicates that the explanation has to be found within the framework of the photon-number measurement. Let us conclude therefore that even if the ideal phase measurement seems to be the highly formal and yet not the feasible phase concept, it represents the most accurate phase measurement, including also some piece of information about the canonically conjugated variable.

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