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Fractional revivals in the Jaynes-Cummings model

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It is shown that the phenomenon of fractional revivals, which was observed in recent laser experiments on atomic wave packets, determines the long-time behavior of the Jaynes-Cummings model for a two-level atom interacting with a quantized cavity field. This provides an alternative mechanism for the generation of coherent superpositions of macroscopically distinguishable states of the field (so-called "optical Schrödinger cats") via resonant processes. The emergence of these states is associated with the appearance of atomic inversion revivals following each other 2, 3, 4,... times faster than the usual ones.

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One of the simplest nontrivial systems in quantum optics is the Jaynes-Cummings model (JCM) for a two-level atom interacting with a single mode of the quantized electromagnetic field [1]. Investigations of dynamical properties of the model are of great importance due to its recent realization in experiments done with Rydberg atoms in a high-Q microwave cavity [2]. Although this model can be solved exactly within the rotating-wave approximation, it demonstrates a surprisingly rich dynamics. In particular, the atomic inversion undergoes a series of collapses and revivals [3], the latter being a direct manifestation of the quantum character of the field. The predicted collapses and revivals are in agreement with experimental observations [2].

Theoretical investigations of the revival effect for the cavity field initially in a coherent state have shown that the revivals become broader with time and their overlap and interference lead to a complicated beat pattern in the time dependence of atomic inversion [3].

Similar interference phenomena were independently examined lately in connection with the problem of generation and detection of quantum wave packets in atomic and molecular systems (see Refs. [4,5] for a recent review). Recent developments in the technique of producing ultrashort laser pulses made it possible to use wave packets in the study of molecular dissociation and predissociation reactions, and for "real-time" observation of molecular vibrations [so-called femtosecond transition state spectroscopy (FTS) [6]]. In another series of experiments [7,8] wave packets were produced from highly excited Rydberg states of atoms. Although these studies are stimulated by practical needs of spectroscopy they are of principal importance from the point of view of foundations of quantum mechanics by being the first direct experimental measurements of the transition from quantum to classical dynamics of atomic particles.

It was shown both theoretically [9–11] and experimentally [12] that small deviations from equidistant energy spacing cause the wave packet to collapse and revive after a time. Moreover, it was pointed out that for a wide class of atomic and molecular systems the long-term evolution of highly excited wave packets shows some universal features and is governed by the so-called "scenario of fractional revivals" [10,11]. Each revival of this type appears as a coherent superposition of several wave packets moving along the same classical trajectory and shifted in time from each other by a fractional part of the classical period. The phenomenon of fractional revivals was recently observed in an experiment on laser-pulse excitation of atomic Rydberg states [13].

I would like to show in this Rapid Communication that under certain conditions the same mechanism of fractional revivals determines the long-time dynamics of atomic inversion in the JCM and leads to the generation of coherent superpositions of macroscopically distinguishable states of the quantized electromagnetic field (so-called "optical Schrödinger cats" [14–16]).

The resonant JCM interaction Hamiltonian may be written in the form

$$H_I = \hbar\lambda(|e\rangle\langle g|a + a^\dagger|g\rangle\langle e|)\,, \tag{1}$$

where $|g\rangle$ and $|e\rangle$ are the ground and excited states of the two-level atom, respectively, a and $a^\dagger$ are the annihilation and creation operators of the field mode, and λ is a coupling constant. For an arbitrary initial state of the field and atom

$$\begin{aligned} |\psi(0)\rangle_{\text{field}} &= \sum_{n=0}^{\infty} C_n|n\rangle\,, \\ |\psi(0)\rangle_{\text{atom}} &= a|g\rangle + b|e\rangle \end{aligned} \tag{2}$$

($|n\rangle$ is the n-photon state), the time-dependent state vector of the system is given by

$$\begin{aligned} |\Psi(t)\rangle &= \sum_{n=0}^{\infty} \{[bC_n\cos(\lambda\sqrt{n+1}t) - iaC_{n+1}\sin(\lambda\sqrt{n+1}t)]|e\rangle + [-ibC_{n-1}\sin(\lambda\sqrt{n}t) + aC_n\cos(\lambda\sqrt{n}t)]|g\rangle\}|n\rangle \\ &= |\psi_g(t)\rangle|g\rangle + |\psi_e(t)\rangle|e\rangle\,. \end{aligned} \tag{3}$$

Here $|\psi_{g,e}(t)\rangle$ are the field state vectors corresponding to the ground and excited atomic states, respectively. Suppose for simplicity that the atom is initially in the ground state ($a=1$, $b=0$). Then, for instance, $|\psi_g(t)\rangle$ may be written as

$$|\psi_g(t)\rangle=\sum_{n=0}^{\infty}C_n\cos(\lambda\sqrt{n}t)|n\rangle=\tfrac{1}{2}[|\psi_g^+(t)\rangle+|\psi_g^-(t)\rangle]\,, \quad (4)$$

$$|\psi_g^{\pm}(t)\rangle=\sum_{n=0}^{\infty}C_n\exp(\mp i\lambda\sqrt{n}t)|n\rangle\,. \quad (5)$$

The time dependence of atomic inversion $w(t)=n_e(t)-n_g(t)$ (n_g and n_e are the populations of the ground and excited atomic states) is determined by the overlap of $|\psi_g^{\pm}(t)\rangle$:

$$w(t)=1-2n_g(t)=1-2\langle\psi_g(t)|\psi_g(t)\rangle=-\mathrm{Re}\langle\psi_g^+(t)|\psi_g^-(t)\rangle \quad (6)$$

(here we used the normalization condition $\sum_n|C_n|^2=1$).

Let us suppose now that the moduli of the coefficients C_n have a sharp maximum near some $\bar{n}\gg 1$ and the width Δn of their distribution is rather small ($\Delta n\ll\bar{n}$). The phases in (5) may be expanded as

$$\lambda\sqrt{n}t=\lambda\bar{n}^{1/2}t\left(1+\frac{k}{2\bar{n}}-\frac{k^2}{8\bar{n}^2}+\frac{k^3}{16\bar{n}^3}+\cdots\right),\quad k=n-\bar{n}\,. \quad (7)$$

For $(\Delta n)^2\ll\bar{n}$ only the linear term in the expansion (7) is important in the early stages of the JCM evolution and the field wave functions may be reduced to a harmonic form

$$|\psi_g^{\pm}(t)\rangle_R=\exp(\mp i\bar{\Omega}_R t/2)\times\sum_k C_{\bar{n}+k}\exp\left[\mp\pi ik\frac{t}{T_R}\right]|\bar{n}+k\rangle\,. \quad (8)$$

Here $\bar{\Omega}_R=2\lambda\bar{n}^{1/2}$ is the mean Rabi frequency, $T_R=2\pi\bar{n}^{1/2}/\lambda$ is the revival time [3]. The wave packets $|\psi_g^{\pm}(t)\rangle_R$ behave similarly to the coherent states $|\alpha\rangle$ and rotate in opposite directions along a circle with the radius $|\alpha|\sim\bar{n}^{1/2}$ in the complex α plane. The angular velocity of their rotation is equal to $\pm\pi/T_R$ for $|\psi^{\pm}(t)\rangle_R$. According to Eq. (6) the relative divergence of these packets leads to the collapse of the Rabi nutations and their overlaps cause periodic revivals:

$$w_R(t)=-\mathrm{Re}\exp(-i\bar{\Omega}_R t)\times\sum_k|C_{\bar{n}+k}|^2\exp\left[-2\pi ik\frac{t}{T_R}\right]. \quad (9)$$

We see from Eqs. (8) and (9) that the revival period of the field is twice that of the atom. Thus, when the inversion $w(t)$ is revived, the field completes only a half of the cycle. The connection between the splitting of the field state vector and the revival phenomenon at the initial stage of evolution of the JCM was pointed out in [5,17-20].

In this Rapid Communication we will concentrate on the *long-time behavior* of the JCM, where the dephasing due to the terms in Eq. (7), which are quadratic in $(n-\bar{n})$, begins to play a major role. If $\lambda t\ll 16\pi\bar{n}^{5/2}/(\Delta n)^3$ the effect of all subsequent terms in the expansion may be neglected, which results in

$$w(t)=-\mathrm{Re}\exp(-i\bar{\Omega}_R t)\sum_k|C_{\bar{n}+k}|^2\exp\left[-2\pi i\left(k\frac{t}{T_R}-k^2\frac{t}{T_{FR}}\right)\right], \quad (10)$$

$$|\psi_g^{\pm}(t)\rangle=\exp(\mp i\bar{\Omega}_R t/2)\sum_k C_{\bar{n}+k}\exp\left[\mp 2\pi i\left(k\frac{t}{2T_R}-k^2\frac{t}{2T_{FR}}\right)\right]|\bar{n}+k\rangle\,. \quad (11)$$

Here $T_{FR}=4\bar{n}T_R$ is a new time scale—the fractional revival time [9-11]. After the time of the order of $T_{FR}/(\Delta n)^2$ the dephasing produced by the second-order term in Eq. (10) destroys the periodical revivals. However, this destruction is not fully irreversible. Near $t=T_{FR}$ the additional "nonlinear" phases are exact multiples of 2π and the initial sequence of the inversion revivals recommences again (as in the case of the wave packets in laser-pulse excited Rydberg atoms [9]). Let us consider now the development of $|\psi^{\pm}(t)\rangle$ in the vicinity of the time point $t=(p/s)T_{FR}$ (p and s are mutually prime integers). The additional phase shifts due to the terms $\propto k^2$ in Eq. (11) are equal to $2\pi\theta_k$, where

$$\theta_k=\frac{p}{2s}k^2 \pmod 1\,. \quad (12)$$

As was shown in [10,11] these extra phases are periodic in k for any p and s, and the length l of the period is determined only by the denominator of the fraction p/s ($l=s$ for even s and $l=2s$ for odd ones). The periodicity enables us to expand $\exp(\pm 2\pi i\theta_k)$ in a finite Fourier sum and to present $|\psi_g^{\pm}(t)\rangle$ as

$$|\psi_g^{\pm}(t)\rangle=\sum_{q=0}^{l-1}a_q^{\pm}\exp\left(\mp 2\pi i\frac{q\bar{n}}{l}\right)\left|\psi_g^{\pm}\left(t-\frac{q}{l}T_R\right)\right\rangle_R \quad (13)$$

where

$$a_q^{\pm}=\frac{1}{l}\sum_{k=0}^{l-1}\exp\left(\pm 2\pi i\theta_k\mp 2\pi i\frac{kq}{l}\right). \quad (14)$$

Because of the symmetry of the coefficients a_q, only half of them are different from zero for odd s and all nonzero a_q are of the same modulus $1/\sqrt{s}$ [11].

Thus, close to $t=(p/s)T_{FR}$ each of the field state vectors as $|\psi_q^{\pm}(t)\rangle$ is divided into s components of the same shape, separated in time by T_R/s. Both of these symmetrical sets rotate in the α plane in opposite directions.

Rabi nutations are revived only when the overlap of the sets occurs. Again, due to the symmetry of the coefficients a_q, only half of the overlaps results in the revivals of atomic inversion for even s. Similar conclusions are true, of course, for $|\psi_e(t)\rangle$.

The above consideration is illustrated by Fig. 1 which shows the results of direct numerical evaluation of $w(t)$ and the Q function of the electromagnetic field

$$Q(\alpha,\alpha^*) = \mathrm{Tr}_{\mathrm{atom}}\langle\alpha|\hat{\rho}|\alpha\rangle = |\langle\alpha|\psi_g(t)\rangle|^2 + |\langle\alpha|\psi_e(t)\rangle|^2 \quad (15)$$

(where $\hat{\rho}$ is the system density matrix) for a Gaussian distribution of C_n:

$$C_n^2 = \frac{1}{\sqrt{2\pi}\Delta n}\exp\left[-\frac{(n-\bar{n})^2}{2(\Delta n)^2}\right]. \quad (16)$$

Initially the Q function looks like a well-localized hump in the α plane [Fig. 1(a)]. At the early stage of evolution the population inversion demonstrates pronounced almost-periodic revivals with a period equal to T_R [Fig. 1(b)]. Figures 1(c) and 1(d) correspond to the particular region of fractional revivals near $t \simeq (1/3)T_{FR}$. The burst of Rabi oscillations are now following each other three times faster, in full agreement with the predictions. Each of the field state vectors $|\psi^{\pm}_{g,e}(t)\rangle$ evolves into a three-component structure. The structures overlap during the revivals of Rabi nutations [see Fig. 1(d)].

The results of this paper are not directly applicable to the case of the field initially in a coherent state [3]. In contrast to atomic and molecular systems, the repetition frequency of the revivals, their "anharmonicity," and the width $\Delta n \sim \bar{n}^{1/2}$ of the coherent state energy distribution are determined by a single parameter $\bar{n}$. The relatively large width of the Poissonian distribution results in a rapid broadening of the revivals at the initial stage [3] and provides a significant role of the cubic term in the expansion (7) in the fractional revivals domain. Equations (10) and (11) are valid for $t \sim T_{FR}$ only if $(\Delta n)^3 \ll \bar{n}$, i.e., for sub-Poissonian statistics of the field. However, preliminary investigations show intensive revival peaks of the atomic inversion with the repetition period equal to T_R/s near the time points p/sT_{FR} even for the field being initially in a coherent state. This case is a subject for a separate analysis.

In conclusion, it has been shown that under certain conditions the mechanism of fractional revivals, which is the same one that determines the behavior of atomic and molecular wave packets following their classical stages of the motion, governs also the long-time dynamics of the Jaynes-Cummings model. In the course of time the Q function of the field is split into a number of symmetrical sets of macroscopically distinguishable components. Having in mind an experimental realization similar to that of a one-photon micromaser [2], it is possible, in principle, to arrive at a certain fractional revival by adjusting the flight time t of the atom through the resonator. Subsequent probing of the state in which the atom leaves the cavity, causes the field to collapse into one of the coherent superpositions $|\psi_g(t)\rangle$ or $|\psi_e(t)\rangle$ depending on the result of the measurement. This may be considered as an alternative method for the generation of "optical Schrödinger cats"

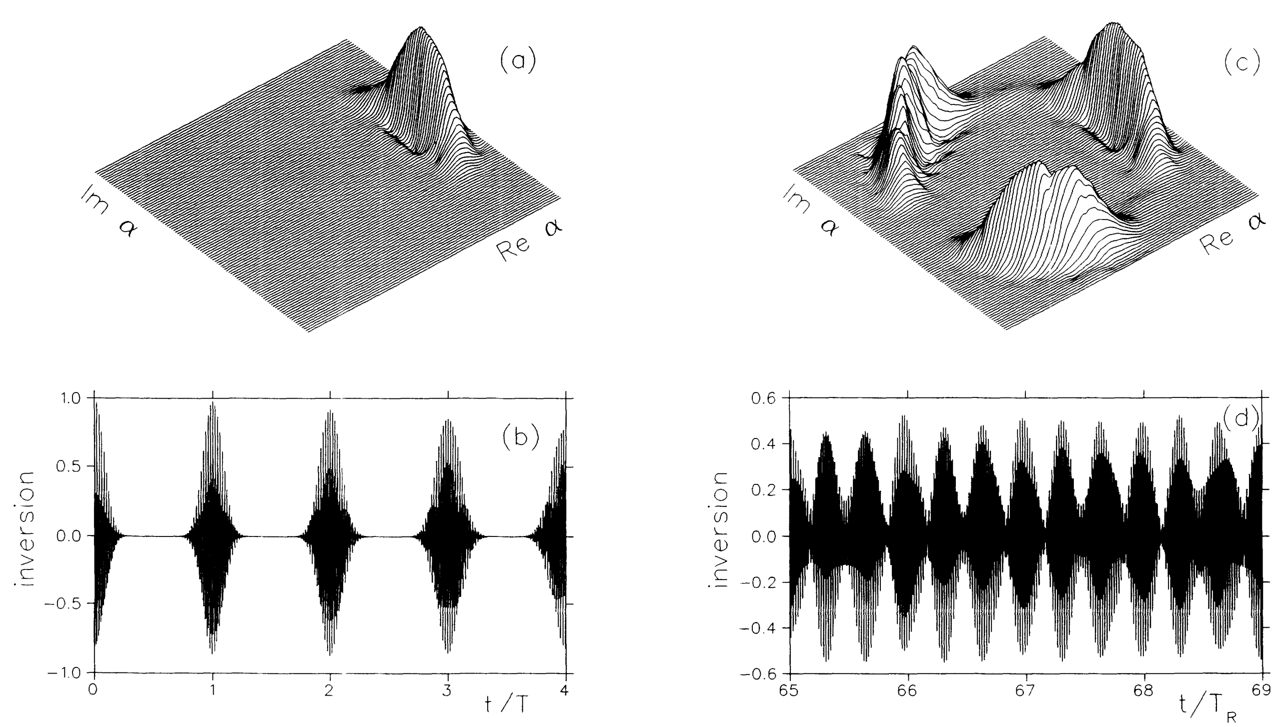

FIG. 1. Plots of the Q function with corresponding atomic population inversion for two time regions: (a),(b) the initial stage and (c),(d) the domain of fractional revivals near $t \simeq (1/3)T_{FR}$. The parameters used are $\bar{n}=50$, $\Delta n=2$; $t=0$ in (a) and $t=67T_R$ in (c).

via resonant atom-field interactions.

It should be noted that if the atom enters the resonator in a certain state (ground or excited) then the time evolution of the Rabi nutations depends only on the diagonal elements of the field density matrix. Hence, the fractional revivals of the inversion may be observed even for a field starting from a mixed initial state with sub-Poissonian statistics, similar to that achieved in recent micromaser experiment [21].

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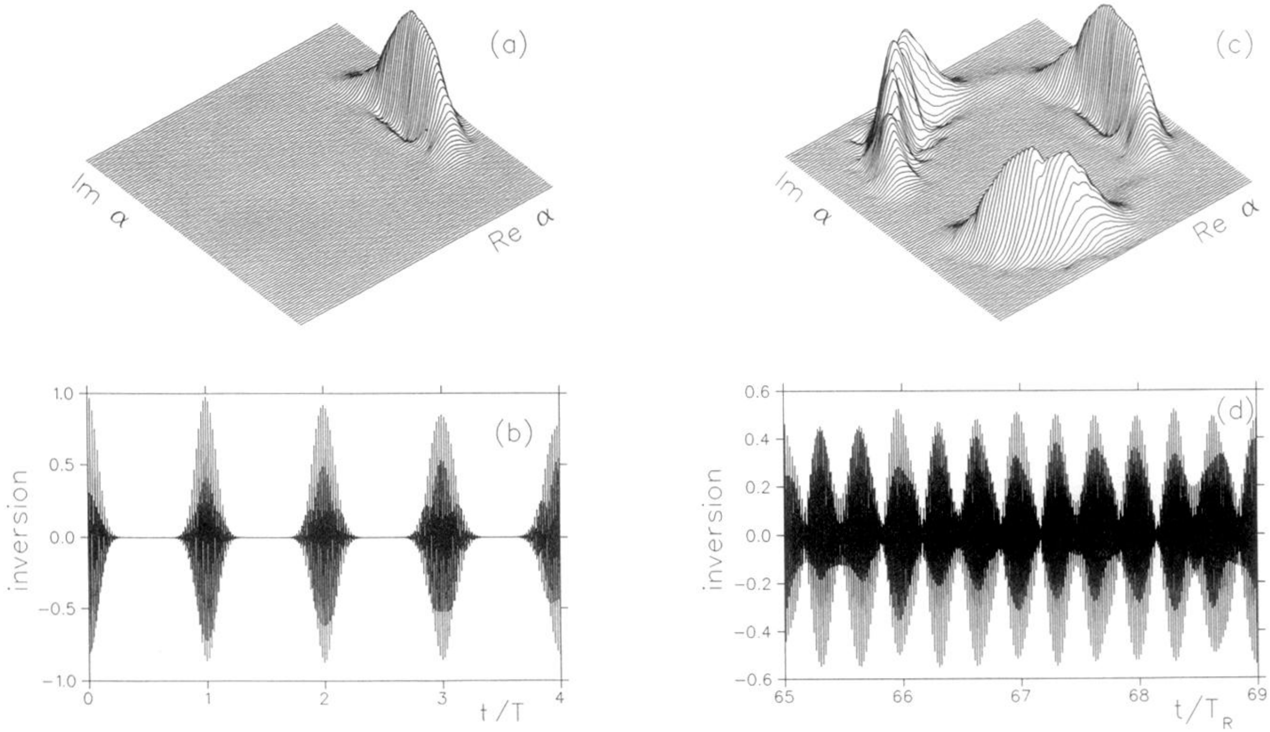

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