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Physical and numerical experiments on the wave mechanics of classically chaotic systems

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We study chaotic quantum billiards using both microwave cavities and numerical simulations. For the same geometry, viz., a Sinai billiard, agreement to remarkable precision is found for both the eigenvalue magnitudes and the spatial detail of the eigenfunctions. The association of the eigenfunctions with classical periodic orbits is demonstrated, and scarred states are identified. Desymmetrizing the Sinai billiard by slightly moving the central disk is shown to lead to strong localization of the eigenfunction. The calculated eigenstates of the symmetric billiard show an even- and odd-parity pair whose linear combination gives the localized state.

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The wave or quantum mechanics of a particle whose classical motion is chaotic is the subject of much recent interest [1–7]. Since the nonintegrable nature of the problem precludes exact analytical results for the eigenvalues and eigenfunctions, numerical approaches are required. Such numerical simulations have yielded significant information regarding eigenvalues and their statistics [8], and the eigenfunctions [1, 9]. From such simulations, the association of the eigenfunctions with classical periodic orbits was discovered [9].

Experiments and theory on atoms in magnetic fields have yielded some of the most definitive results to date, allowing direct comparison of experimental and theoretical results [10]. However, only the energy of the eigenstates, or at most an additional spectral intensity, are available from such experiments to date.

Recently, experiments on microwave cavity and scattering geometries [11, 12] have been reported that exploit the correspondence of the stationary solutions of the scalar Maxwell's equation and the Schrödinger's equation in two dimensions (2D), viz., $(\nabla^2 + k^2)\psi = 0$. Such experiments have also provided the first direct experimental observation and mapping [11] of scarred eigenstates. One of the unique aspects of the microwave cavity experiments is the ability to measure the entire eigenstate, a privilege usually accorded only to theorists. Although neither the experiments nor the theory on the Schrödinger equation in two-dimensional cavities is in much doubt, a detailed comparison of the two on the same cavity, mapping details of wave functions, has not been done before to our knowledge.

The microwave experiments were carried out on Cu cavities in the form of Sinai billiards. The cavity consists of a rectangle of dimensions 21.8 cm $\times$ 44 cm $\times$ 6 mm, with a circular disk 10 cm diameter $\times$ 6 mm height placed at the center. Because of the small 6 mm height, all modes occurring at frequencies below 25 GHz obey the 2D wave equation. The wave functions of nearly 200 modes were experimentally mapped out in two dimensions using a small bead perturbation [14] technique. As the metallic bead is magnetically moved from place to place inside the cavity, the resonant frequency of the cavity is tuned slightly, and the detuning is proportional to the square of the electric field strength [13] at the location of the bead. A contour plot or density map of the frequency detuning thus yields a map of $|\psi|^2$.

Three examples of Sinai billiard eigenstates are shown in Fig. 1. These are fairly low frequency modes which seem least susceptible to experimental error, such as slight imperfections of the cavity. The association of the wave functions with classical periodic orbits is just emerging at these low frequencies. The wave functions are compared below with numerical calculations, which are described next.

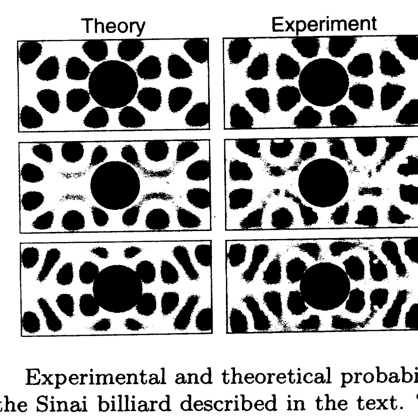

FIG. 1. Experimental and theoretical probability density plots for the Sinai billiard described in the text. The experimental frequencies are, from top to bottom, 3.112, 3.278, and 3.663 GHz; and the theoretical values are 3.114, 3.280, and 3.664 GHz.

The method we use to find the eigenstates of bounded enclosures like the stadium is simple and accurate. We pick a value of k, and suppose temporarily that an eigenstate with energy $E = \hbar^2 k^2/2$ exists. Plane waves are chosen as a basis; all have wave-vector magnitude k. For the Sinai billiard, the basis functions $\sin(k_x x)\sin(k_y y)$ are used for the states reported here; the x and y axes are boundaries and thus nodes of the eigenstates.

The Sinai billiard is solved separately for each of the four symmetries (even-even, even-odd, etc.) by using appropriate boundary conditions. For example, the wave function at the points shown as dots in Fig. 2 are set to zero for the odd-odd states, etc.

We try to force L' points on the boundary to be zero using $L < L'$ plane waves as a basis. If those points are closer together than a wavelength (say by a factor of 3 or so), the boundary will be specified sufficiently for the quantum mechanics. We set a single arbitrary point inside the boundary to 1, and the resulting inhomogeneous set of equations can be solved by singular value decomposition [15, 16].

At each trial energy, the wave function inside the boundary is normalized, after which the wave function is evaluated on the boundary at about four times as many points as were initially set to zero. The sum of the squares of the values on these points is a positive number which we call the "tension." The tension is found to have deep minima as a function of k, approaching very small values at what are obviously the eigenvalues.

The results of the experiments and the numerical calculations are shown for three eigenstates in Fig. 1. Note that the calculated and measured eigenfrequencies agree to better than 0.1%. Note also that experiment and theory for the wave functions agree very well.

The comparison was carried out over the entire spectral range up to about 10 GHz, corresponding to several hundred levels. With a few exceptions, all the wave functions observed in the experiment were also present in the theory. The experiments may miss closely spaced eigenvalues due to the finite resonance widths caused by absorption in the walls. As we note later, the absorption does not affect the spatial aspects of the measured wave functions.

One of the eigenfunctions (Fig. 1 top) is clearly influenced by periodic trajectories, an example of which is shown in Fig. 2(b); these are nonisolated periodic orbits which avoid hitting the circular disk in the center. The orbits are nonisolated in the sense that they are part of a continuous one parameter family of orbits which all have the same slope but differ in their intercepts; they are the analog of the "bouncing ball" states first seen in the stadium by McDonald and Kaufman [1].

A principal result which emerges from examination of eigenfunctions such as shown in Fig. 1 is the association of the eigenfunctions with classical periodic orbits [9]. Figure 3 shows the calculated and experimental state at 2.505 GHz (calculated 2.503 GHz), which has a peculiar localization between the central disk and the walls. There is an unstable periodic orbit which bounces vertically exactly in the center; however, this orbit is an isolated periodic orbit of zero measure. This phenomenon can be seen in many other states in the Sinai billiard. The influence of isolated periodic orbits on eigenstates and the first theoretical explanation for scarring (in terms of the short-time quantum dynamics near the periodic orbit) were given in 1984 [9]. A somewhat different theoretical approach with new insights was given by Bogomolny [17] and Berry [18]. In Ref. [9] it was shown that those orbits with small enough geometrical Lyapunov exponent λ ($\lambda/2\pi < 1$) would generate scars; the vertical bounce orbit between the top of the circular disk and the side has $\lambda/2\pi = 0.6$, so it is indeed a scarring orbit. These microwave experiments have provided the first experimental mapping of the *wave functions* of scarred states [11].

When classical trajectories flow between regions of coordinate space or more generally phase space but quantum eigenstates do not, the phenomenon is deemed "quantum localization," which in many ways is the opposite of quantum tunneling.

This phenomenon is observed experimentally by slightly desymmetrizing the Sinai billiard, thereby breaking parity. The consequence of this is shown in Fig. 4, bottom, which displays the effect of introducing a small asymmetry (about 1%) to the Sinai billiard of Fig. 1, by displacing the disk to the right. The $f = 3.654$ GHz eigenfunction of the asymmetric cavity shown in Fig. 4

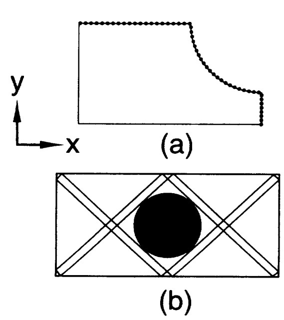

FIG. 2. (a) Typical set of points (dots) where boundary conditions are imposed; the x and y axes are automatically nodes because of the basis functions used. (b) A nonisolated periodic orbit (and its symmetry partner) in the Sinai billiard corresponding to the top state in Fig. 1.

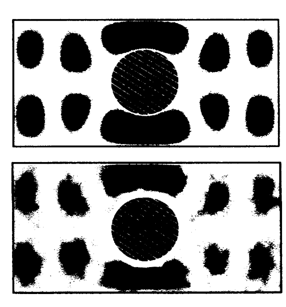

FIG. 3. Scarred state at 2.505 GHz (experimental, bottom), 2.503 GHz (calculated, top). The unstable vertical bounce trajectory is "responsible" for this scar.

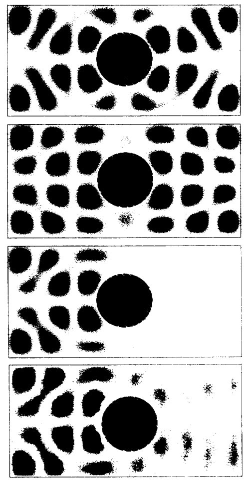

FIG. 4. Probability density plots of the 3.664 GHz state (top), its even partner at 3.695 GHz (down one); the addition of the two amplitudes squared (third state) and the experimental symmetry broken state for a displaced disk at 3.654 GHz (bottom).

not only has a lower frequency than the $f = 3.663$ GHz (Fig. 1) symmetric mode, but also has its wave function localized on one side. The lower frequency belongs of course to the state with the most area. This state displays a global localization, in addition to the scarring which is localization along (the diagonal) periodic orbits.

In an ironic turn of events, the quantum localized states may be made to "tunnel" into each other, thus regaining the delocalization of the classical trajectories, by bringing the disk to the center. (The use of the term "tunnel" is not really appropriate, since the classical trajectories cross from the left to the right with measure unity.) We shall call this phenomenon "retunneling." The phenomenon is illustrated in the numerical calculations, which show, in Fig. 4, the even and odd partners at 3.695 and 3.664 GHz, respectively. (Uncommonly, the even state has a *higher* energy.) When added, they give the nonstationary (for the symmetric billiard) state shown second from the bottom, which is very similar to the experimental slightly asymmetric stationary state at 3.654 GHz, Fig. 4, bottom. (All the plots are of $|\Psi|^2$; the bottom localized state was of course generated by adding amplitudes before squaring.)

The behavior of the energy levels as a function of a parameter has played a major role in the theory of eigenvalues of chaotic Hamiltonian systems [8,19-22]. As a parameter is varied, the levels of a given symmetry will interact and repel one another with an interaction that can sometimes be estimated [23]. Gaspard and Rice have introduced the important idea of the curvature of the energy levels plotted as a function of a parameter [21]. Takami [24] has given strong evidence that many of the avoided crossings in the stadium billiard (and doubtless other chaotic systems) are understandable as scarred states that have interacted in pairs, triplets, etc. The present situation involves states *of different symmetry*, states which in principle are allowed to be degenerate at some value of a parameter. Here they are necessarily split by the retunneling interaction.

The agreement between the numerical calculations and the experiments confirms the validity of both approaches. While no serious problems were expected from either, the mutual confirmation is significant, since it lends a physical reality to the numerical simulations, and verifies that the nonidealities (principally finite skin depth, about 1 μm, and absorption) present in the experiment do not affect the results (at least those discussed here).

There will be interesting geometries for scattering and cavities where calculations are not so easy. It is reassuring to see that when both experiments and theory can be directly compared, the agreement is more than satisfactory.

Beyond the good agreement of theory and experiment, this work has raised interesting questions about the relation between wave mechanics and the corresponding classical ray motion. The retunneling and its associated splitting is not fully understood, and is intimately tied up with the issue of quantum localization.

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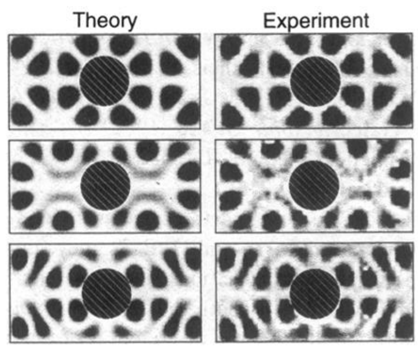

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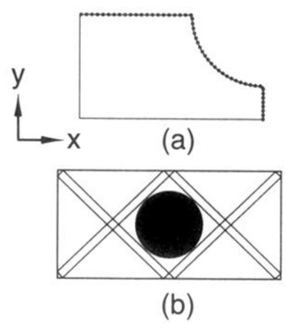

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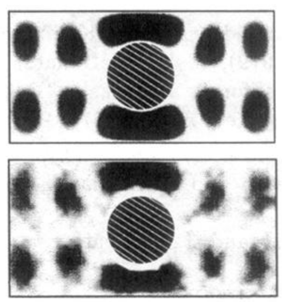

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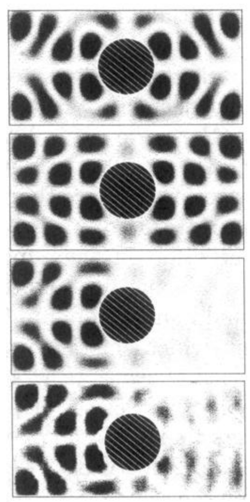

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