PHYSICAL REVIEW A

STATISTICAL PHYSICS, PLASMAS, FLUIDS,
AND RELATED INTERDISCIPLINARY TOPICS

THIRD SERIES, VOLUME 46, NUMBER 4 15 AUGUST 1992

RAPID COMMUNICATIONS

The Rapid Communications section is intended for the accelerated publication of important new results. Since manuscripts submitted to this section are given priority treatment both in the editorial office and in production, authors should explain in their submittal letter why the work justifies this special handling. A Rapid Communication should be no longer than 4 printed pages and must be accompanied by an abstract. Page proofs are sent to authors.

Collective response in globally coupled bistable systems

Peter Jung
Institut für Physik, Universität Augsburg, Memminger Strasse 6, W-8900 Augsburg, Germany

Ulrich Behn
Sektion Physik, Universität Leipzig, Augustus Platz 10, O-7010 Leipzig, Germany

Eleni Pantazelou and Frank Moss
Department of Physics, University of Missouri St. Louis, St. Louis, Missouri 63121
(Received 6 December 1991)

We consider a system of globally coupled bistable systems under the influence of noise and periodic modulations. The hopping process between the stable states is described by a nonlinear master equation. We observe an unusually large amplification of the periodic modulations for certain values of the noise strength due to collective dynamics of the coupled bistable elements.

PACS number(s): 05.40.+j

Rate processes in bistable and multistable systems have been studied for a long time in many fields of physics and other fields as well (for a review, see [1]). One widely used approach is stochastic modeling in terms of a Langevin equation [2–4]. These equations describe the motion of a particle in a low-dimensional phase space in the presence of fluctuations which simulate interactions with other degrees of freedom. The transition rates between basins of attraction can be obtained, in principle, from the corresponding Fokker-Planck equation. With these transition rates, the rate process can subsequently be described in terms of a master equation. These master equations are linear equations for the populations in the basins of attraction and possess unique asymptotic solutions for large times.

In this paper we consider a network of globally coupled bistable systems, wherein each individual system is described by a two-state master equation. We are especially interested in the collective aspects of the rate processes due to a possible cooperative interaction of the individual units. Our results demonstrate that there is a substantial collective effect near a spontaneous ordering transition which takes place at a critical noise intensity. This collective effect manifests itself as a substantial amplification of periodic modulation.

Spontaneous ordering transitions in noisy, globally coupled systems have been observed before [5], but the effects of periodic modulation, which have become increasingly important to a number of applications, were not previously studied. Amplification of periodic modulation near a deterministic bifurcation in a system of globally coupled oscillators has recently been reported in an interesting paper by Wiesenfeld [6], who observed simultaneously a suppression of the global response to the noise. Using perturbation theory and linearization of the dynamical equation near the bifurcation, he observed an amplification (of the modulation) and a suppression (of the noise) which occurred because of respectively coherent or incoherent superposition. In contrast, the system we study here is fully statistical, that is, the ordering transition is spontaneous and our treatment is based on a master equation. It is remarkable that we discover results which are qualitatively similar to those of Wiesenfeld. In our system, the individual bistable elements (numerated by $i=1,2,\ldots,N$) have stable states, + and −, and perform jumps, due to either environmental or internal fluctuations between the states with transition rates $r_i^{\pm}$. The dynamics of the ith single bistable element is described by a two-state master equation for the populations p_i^- and p_i^+ in the state − and +, respectively. Each element is coupled to all other elements with the same coupling constant g. The coupling between any two units is such that they prefer to be in the same states (ferromagnetic coupling). Originally, global coupling was studied using discrete maps [7], but recent interest in this area has been stimulated by an experiment with a multimode solid-state laser, wherein the interaction among the modes was successfully modeled by such a global coupling [8,9]. Moreover, the techniques are important for understanding arrays of lasers and/or Josephson junctions [9–11]. The transition rates of the ith system from the ∓ state to the ± state are then modeled as

$$r_i^{\pm}=r_0\exp\left[\pm\frac{g}{D}\frac{1}{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N}[p_j^+(t)-p_j^-(t)]\pm\frac{A}{D}\sin\Omega t\pm\frac{S}{D}\right],\quad r_0=\nu\exp\left[\frac{-\Delta U}{D}\right], \tag{1}$$

where g is a coupling constant, A is the strength of the external periodic modulation, S is a static bias, r_0 is the bare (uncoupled) transition rate, and D denotes the strength of the fluctuations. ΔU denotes the energetic barrier height that an uncoupled element has to overcome in order to jump from one stable state to the other. In contrast to conventional deterministic coupling (kinetic Ising model [12]), we have assumed probabilistic coupling. This means that a certain configuration of states of elements $j=1,\ldots,N$, $j\neq i$, does not deterministically generate a certain coupling to the ith element. This is reflected by the occurrence of the probabilities $\{p_j^{\pm}\}$ in (1) instead of the state variables $\sigma_j=\pm 1$ of the bistable elements. The dependence of the transition rates on the populations $\{p_j^{\pm}\}$ is chosen as the most simple one which guarantees ferromagnetic coupling. The parameters g, A, S, and D are already scaled quantities without dimension. Rate expressions of this form can be obtained from stochastic differential equations [13]. The *nonlinear* master equation then reads

$$\dot{p}_i^{\pm}=r_i^{\pm}(\{p_{j(\neq i)}^{\pm}\})p_i^{\mp}-r_i^{\mp}(\{p_{j(\neq i)}^{\pm}\})p_i^{\pm}\,. \tag{2}$$

Utilizing the mean-field approximation ($N\to\infty$)

$$\frac{1}{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N}(p_j^+-p_j^-)=p_i^+-p_i^-\equiv x\,, \tag{3}$$

we obtain the equation of motion for the order parameter x

$$\dot{x}=-2r_0x\cosh[\lambda x+a\sin(\Omega t)+\sigma]+2r_0\sinh[\lambda x+a\sin(\Omega t)+\sigma]\,, \tag{4}$$

where we have used the scaled quantities $\lambda=g/D$, $a=A/D$, and $\sigma=S/D$.

For $a=0$, the stationary solution of (4), $x_0=\tanh(\lambda x_0+\sigma)$, exhibits, in contrast to a single bistable element (described by a single-particle Fokker-Planck equation), a bifurcation at a critical value of the control parameter $\lambda_c(\sigma)$ with $\lambda_c(\sigma)=1/(1-x_0^2)$. Without bias ($\sigma=0$), the stationary solution x_0 shows a second-order (continuous) phase transition: The solution $x_0=0$ is stable for $\lambda<\lambda_c(0)=1$ and unstable for $\lambda>\lambda_c(0)$. In addition, two nonvanishing stable, symmetry-breaking, solutions exist for $\lambda>\lambda_c$ [Fig. 1(a)]. In the presence of bias ($\sigma\neq 0$) we find a (discontinuous) first-order phase transition: one solution remains stable [the upper branch in Fig. 1(b)], while a new stable solution and a new unstable solution occur below a critical value of the control parameter $\lambda_c(\sigma)$.

In a potential picture, Eq. (4) is the overdamped equation of motion in a symmetric potential for $\sigma=0$. For $\lambda>\lambda_c$ the potential is bistable with minima at the nonvanishing stationary solutions of (4), whereas for $\lambda<\lambda_c$ the potential is monostable with a minimum at the vanishing solution of (4). For $\sigma\neq 0$ the potential is asymmetric with two minima for $\lambda>\lambda_c(\sigma)$ and one minimum for λ

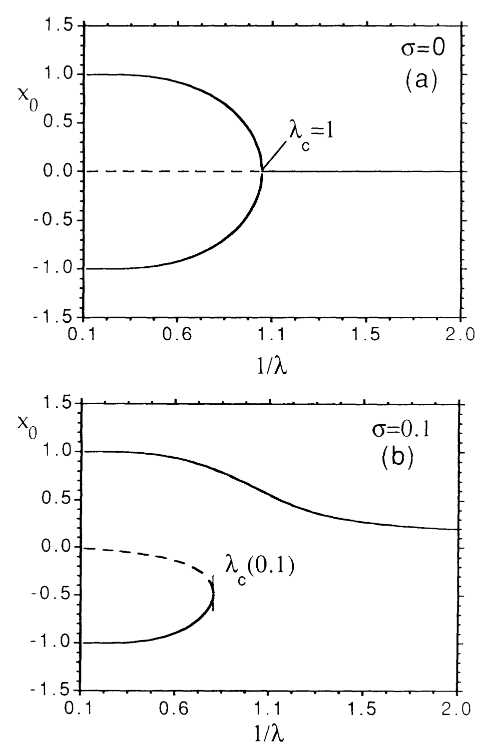

FIG. 1. Bifuraction diagram of the stationary solution of (4) (a) without bias and (b) with bias. The dashed lines represent unstable solutions, whereas the solid lines are stable solutions.

$<\lambda_c(\sigma)$. The minimum corresponding to the upper stable branch in Fig. 1(b) is the absolute minimum of the potential. This branch is thus termed the globally stable branch.

The critical behavior of the order parameter close to the stationary values is obtained for small periodic driving by expanding around the stationary points, i.e., $x(t)=x_0+\delta(t)$, where

$$\dot{\delta}(t)=-\Lambda\delta(t)+2ar_0(1-x_0^2)^{1/2}\sin(\Omega t) \tag{5}$$

and

$$\Lambda=2r_0(1-x_0^2)^{1/2}\left[\frac{1}{1-x_0^2}-\lambda\right]. \tag{6}$$

In Fig. 2, the relaxation coefficient Λ, which is the inverse time scale of the relaxation process of the control parameter, is shown. For $\sigma=0$, the inverse time scale Λ decreases with increasing control parameter (increasing coupling) until it vanishes at $\lambda=\lambda_c(0)$. On the stable branches ($x_0\neq 0$), the coefficient Λ increases again for $\lambda>1$ with increasing control parameter and diverges for $\lambda^{-1}\to 0$ (i.e., $g\to\infty$). The system thus exhibits critical slowing down.

In a physically realistic array of elements, not all of them are perfectly identical and much more important, none of them is perfectly symmetric. A subset of array elements may have a tiny preference to the + state, while the rest may have a certain preference to the − state with the result that the overall behavior is not fully symmetric. Thus, the case with bias $\sigma\neq 0$ is more realistic than the latter without bias. Since positive and negative bias leads to the same dynamical time scales we restrict ourselves to a constant positive bias. For $\sigma\neq 0$, the coefficient Λ has a minimum on the continuous globally stable branch. On the additional stable branch $[\lambda>\lambda_c(\sigma)]$, the coefficient Λ is positive, but becomes zero at $\lambda=\lambda_c(\sigma)$. The system thus shows critical slowing down on the additional stable branch.

The dynamical response of the system to periodic modulation is given by the long-time solution of Eq. (5), i.e., $\delta(t)\equiv\tilde{x}\sin(\Omega t+\phi)$, where $\tilde{x}=2r_0a(1-x_0^2)^{1/2}/(\Omega^2+\Lambda^2)^{1/2}$ and $\tan\phi=-\Omega/\Lambda$. The amplitude $\tilde{x}$ is, up to a phase factor, the dynamical response function of the system at the driving frequency, i.e, $\tilde{x}=\exp(-i\phi)\chi(\omega=\Omega)$, where $\chi(\omega)$ is the Fourier transform of the response function $R(t)=2r_0(1-x_0^2)^{1/2}\exp(-\Lambda t)$. Most interesting is the dependence of the response function on the strength of the fluctuations D. The spectral power amplification at the driving frequency, defined by $\eta=\tilde{x}^2/A^2$ [14], is then given by

$$\eta=\left[\frac{2r_0}{D}\right]^2\frac{1-x_0^2}{\Omega^2+\Lambda^2}. \tag{7}$$

In Fig. 3(a), η is shown as a function of the strength of the fluctuations D for $\sigma=0$. The quantity $\eta(D)$ shows a resonance whose height and position $D_0(g)$ depends strongly on the coupling g. For increasing coupling g the position of the maximum approaches very quickly the value of the coupling, i.e., $D_0(g)\to g$. The height of the resonance peak has a maximum for a certain value of the coupling $g=g_2$. This allows the system to be tuned such that the collective response to any particular periodic input can be maximized. The maximal collective response can exceed the maximal response of an individual bistable element by orders of magnitude (depending on the driving frequency Ω).

For $\sigma\neq 0$, the collective response also shows a resonance as a function of the noise strength [Fig. 3(b)]. The curve in Fig. 3(b) is obtained under the condition that the system always stays in the vicinity of the globally stable solution. The maximal response is reached here for a different

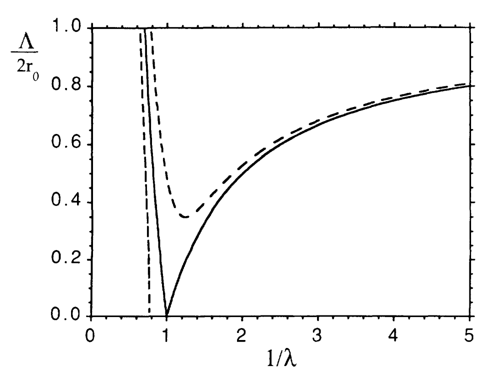

FIG. 2. The relaxation time scale of the order parameter Λ (5) on the stable branches with bias (dashed lines) and without bias (solid line).

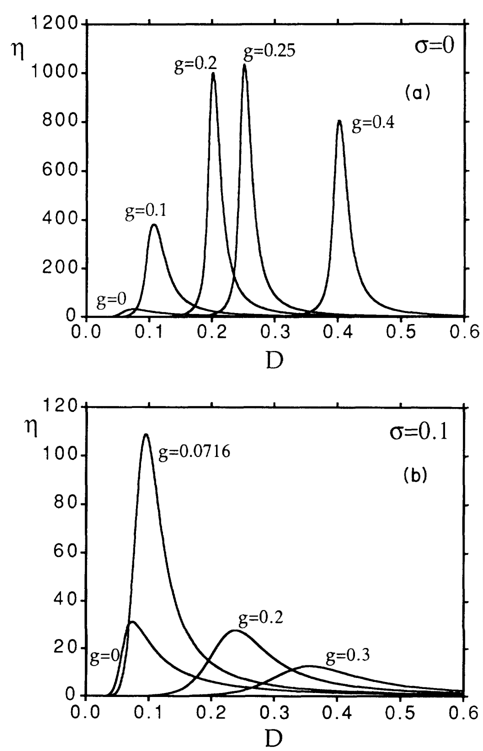

FIG. 3. The spectral amplification η is shown as a function of the noise D at $\Omega=0.1$ for various values of the coupling g (a) without bias and (b) with bias.

value of the coupling strength $g=g_1$ than in the case $\sigma=0$. The mechanism, leading to the large response for finite bias σ, is not the phase transition as in the case $\sigma=0$, but rather the minimum of $\Lambda/2r_0$ (see Fig. 2).

Also interesting is the shape of the response amplitude as a function of the noise strength shown in Fig. 3. For vanishing coupling, we do not observe a monotonous Curie-type shape of the response, but rather a maximum at a certain value of the noise strength $D_0(\Omega)$. This nonequilibrium effect has been termed stochastic resonance [14–20]. It occurs when the hopping time scale $r_0(D_0)$ between the stable states equals half the period of the periodic forcing and is connected with a significant improvement of the signal to noise ratio. For increasing coupling, critical slowing down (at the second-order phase transition for $\sigma=0$) generates a collective response which combines with the stochastic resonance effect to exhibit a *collective stochastic resonance* effect which is much more efficient than that of a single-element bistable system. The combined interaction of collective modes and nonequilibrium response becomes even more obvious for finite bias ($\sigma\neq0$) if we restrict ourselves to the vicinity of the globally stable stationary solution [the upper stable branch in Fig. 1(b)]. Here, the maximal response of the system is reached if the coupling constant g is chosen equal to the value of the noise strength $D=D_0$, for which the uncoupled system shows the largest response. In other words, the coupling must be tuned to the stochastic resonance of a single bistable element. At this point, the response of the individual system is maximal. Again, we obtain the collective stochastic resonance effect which is much larger than that of a single bistable oscillator. This effect suggests a variety of very interesting technical applications. For instance the detection of ultrasmall signals in biological systems (brain signals) might be significantly improved by using a network of globally coupled SQUID's or Josephson-junction arrays.

P.J. and U.B. wish to acknowledge the kind hospitality at the Department of Physics at the University of Missouri St. Louis. We thank P. Hänggi for valuable discussions and for carefully reading the manuscript. Valuable discussions with Lutz Schimansky-Geier on nondeterministic coupling are highly appreciated. This work has been supported by NATO Grant No. 1272/90 and by ONR Grant No. N00014-90-J-1327. P.J. wishes to acknowledge Stiftung Volkswagenwerk for financial support. U.B. gratefully acknowledges partial support by Deutsche Forschungsgesellschaft.

[1] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).

[2] H. Haken, *Synergetics, An Introduction* (Springer, Berlin, 1978).

[3] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).

[4] H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, 1984).

[5] M. Shiino, Phys. Rev. A **36**, 2393 (1987).

[6] K. Wiesenfeld, Phys. Rev. A **44**, 3543 (1991).

[7] K. Kaneko, Phys. Rev. Lett. **65**, 1391 (1990); **66**, 243 (1991).

[8] C. Bracikowski and R. Roy, Chaos **1**, 49 (1991).

[9] K. Wiesenfeld, C. Bracikowski, G. James, and R. Roy, Phys. Rev. Lett. **65**, 1749 (1990).

[10] K. Wiesenfeld and P. Hadley, Phys. Rev. Lett. **62**, 1335 (1989).

[11] K. Y. Tsang and K. Wiesenfeld, Appl. Phys. Lett. **56**, 495 (1990).

[12] R. J. Glauber, J. Math. Phys. **4**, 294 (1963).

[13] P. Jung, Z. Phys. B **76**, 521 (1989).

[14] P. Jung and P. Hänggi, Phys. Rev. A **44**, 8032 (1991).

[15] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).

[16] B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988).

[17] G. Vemuri and R. Roy, Phys. Rev. A **39**, 4668 (1989).

[18] P. Jung and P. Hänggi, Europhys. Lett. **8**, 505 (1989).

[19] F. Moss, in *Rate Processes in Dissipative Systems: 50 Years after Kramers,* edited by P. Hänggi and J. Troe [Ber. Bunsenges. Phys. Chem. **95**, 303 (1991)].

[20] L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. Lett. **62**, 349 (1989).