PHYSICAL REVIEW A VOLUME 46, NUMBER 3 1 AUGUST 1992

Quantization of the nonlinear Schrödinger equation

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(Received 4 June 1992)

It is shown that the quantum form of the nonlinear Schrödinger equation does not call, in general, for the introduction of a finite relaxation time of the Kerr medium and of the concomitant quantum noise sources.

PACS number(s): 42.81.−i, 42.50.Dv, 42.50.Lc, 42.65.−k

The "squeezing" of solitons demonstrated experimentally [1] calls for a self-consistent quantum theory for the nonlinear Schrödinger equation (NLSE) [2–4]. When squeezing is performed at, or near, the zero dispersion wavelength of the fiber, the NLSE degenerates into a simpler form that involves only the action of the Kerr medium [5]. This nonlinear equation can be simply integrated. However, it was pointed out [6–9] that the quantized field calls for the assignment of a relaxation time to the Kerr medium. Such an assignment is equivalent to the introduction of a reservoir that calls for the introduction of quantum noise sources [7]. One may then raise the question of whether the assignment of such a relaxation time is necessary for a self-consistent quantum theory of the Kerr effect. It is the purpose of this paper to show that this is not necessary in general. The requirement for a finite relaxation time of the Kerr medium is the consequence of neglecting the fiber dispersion. The corollary of this conclusion is that the quantum noise associated with the Kerr medium is not required for the self-consistency of the quantized NLSE.

The NLSE involving creation and annihilation operators of the electromagnetic field [Eq. (2.10) of Ref. [3]]:

$$i\frac{\partial}{\partial t}\hat{\phi}(t,x) = -\frac{\partial^2}{\partial x^2}\hat{\phi}(t,x) + 2c\hat{\phi}^\dagger(t,x)\hat{\phi}(t,x)\hat{\phi}(t,x) \tag{1}$$

is the Heisenberg formulation of the nonlinear propagation problem in the slowly varying envelope approximation. Here $\hat{\phi}^\dagger(t,x)$ and $\hat{\phi}(t,x)$ are the creation and annihilation operators of photons at a "point" x and time t. The coefficient c was defined as

$$c = \frac{\kappa|k'|^2}{k''}I^2, \tag{2}$$

where the Kerr coefficient κ is expressed in terms of the nonlinear optical index n_2:

$$\kappa = \frac{2\pi}{\lambda}n_2\frac{1}{\mathcal{A}_{\text{eff}}} \tag{3}$$

with λ the free space wavelength, ω_0 the carrier frequency, $\mathcal{A}_{\text{eff}}$ the cross section of the mode, k' the first derivative (inverse group velocity $1/v_g$), and k'' the second derivative with respect to frequency of the propagation constant; I was defined as an intensity of normalization.

From these definitions it is not obvious whether a mode volume and/or normalization time has been used to define the quantum form of the NLSE. In order to ascertain whether this is the case, it is convenient to return to the classical NLSE and normalize it so as to arrive at its quantum counterpart. The correspondence principle then allows for a direct comparison.

The classical NLSE is of the form

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right)A(t,z) = i\tfrac{1}{2}k''\frac{\partial^2}{\partial t^2}A(t,z) - i\kappa|A(t,z)|^2A(t,z), \tag{4}$$

where $|A|^2$ is normalized so as to give the power and k is in units of m^{-1}. The coordinate transformation into a frame copropagating with the group velocity, and expressing $|A|^2$ in terms of photon density $|u|^2$ in units m^{-1},

$$|A|^2 = \hbar\omega_0 v_g|u|^2, \tag{5}$$

gives the normalized equation

$$i\frac{\partial}{\partial \tau}u(\tau,x) = -\frac{\partial^2}{\partial x^2}u(t,x) + 2c|u(\tau,x)|^2u(\tau,x) \tag{6}$$

where

$$x = v_g t - z \tag{7a}$$

and

$$\tau = \frac{k''}{2k'^2}z. \tag{7b}$$

Now the equation is of the form of the quantum-mechanical NLSE and the parameter c is

$$c = \frac{2\pi}{\lambda}n_2\frac{\hbar\omega_0}{\mathcal{A}_{\text{eff}}v_g k''}. \tag{8}$$

The c parameter does not involve a relaxation time of the medium. Thus, (1) is a self-consistent quantum formulation in which the finite relaxation time of the medium did not have to be taken into account. Why is this the case? Clearly some physical considerations must be envoked to settle the issue.

The finite relaxation time of the Kerr medium needs to be introduced when the dispersion term is absent. This follows from the fact that a pulse excitation of the medium can be viewed as an excitation by photons contained in time slots into which the pulse can be decomposed. When the time slots are made shorter and shorter, the fluctuations of the photon number in these time slots is larger and larger. In a linear system, such a decomposition does not present any problems. Indeed, decomposition into narrower and narrower time slots corresponds to the inclusion of fluctuations of broader and broader bandwidth. An eventual measurement uses a filter and passes only the fluctuations within the passband of the filter. The broadband noise does not affect the operation of the linear system. A nonlinear system reacts differently depending upon whether more and more noise is included by increasing the bandwidth of the electromagnetic field. The largest bandwidth processed by the nonlinear medium is the bandwidth proportional to the inverse relaxation time of the medium. Hence, this relaxation time acts as the ultimate filter that has to be included in the analysis.

The situation changes when the system is dispersive, since the dispersion itself imposes a bandwidth limitation. Pulses so short that they spread within a time that is too small for the nonlinear phase shift to take effect are not acted upon by the nonlinearity. Thus, the dispersion itself provides a bandwidth limitation. The finite relaxation time of the Kerr medium needs to be included only when analyzing pulses of duration comparable to the relaxation time. This implies, of course, a nonlinearity sufficiently large (and dispersion sufficiently small) so as to cause appreciable nonlinear phase shift within a distance small compared with the spreading time of the pulse due to dispersion.

Additional insight into the difficulties associated with the dispersionless case can be gained by looking at the Schrödinger picture corresponding to the NLSE. It is a many-particle problem with an attractive δ-function-like interaction, which models the instantaneous nonlinear action of the Kerr medium [3]. The limit of a dispersionless fiber corresponds to the limit of particles with infinite mass which attract each other. The quantum theory of such a system is completely meaningless, despite the fact that this limit makes perfect sense in the case of the classical optical system. In order to fix the quantum theory with respect to this singular limit, one has to reconsider the Hamiltonian generating the NLSE and taking into account higher-order effects, which are, e.g., the finite response time of the medium or higher-order dispersion and so on.

Of course, one can find states of the input field that are not properly treated in the frame work of the nonlinear Schrödinger equation, e.g., a higher-order soliton of very large order, so that the bandwidth occupied by the soliton is much larger than the bandwidth where the parabolic approximation of the dispersion profile and the instantaneous action of the Kerr medium is valid. But this is true in the classical as well as in the quantum treatment and is not an artifact of the corresponding quantum theory; it only shows that the modeling of a fiber by the NLSE has certain physical limits.

The above arguments show that one may build up a self-consistent quantum theory of nonlinear pulse evolution without inclusion of a finite relaxation time of the Kerr medium and of the accompanying quantum noise sources, provided the nonlinearity is small enough (or the dispersion is large enough) in the context discussed above.

The authors acknowledge stimulating discussions with Dr. M. Shirasaki, Professor J. H. Shapiro, and L. Joneckis. The work was supported, in part, by the ONR Grant No. N00014-92-J-1302 and by the Alexander von Humboldt Stiftung (F.X.K.).

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