Bethe solution for the dynamical-scaling exponent of the noisy Burgers equation

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We approximate the noisy Burgers equation by the single-step model, alias the asymmetric simple exclusion process. The generator of the corresponding master equation is identical to the ferromagnetic Heisenberg spin chain with a purely imaginary XY coupling. We Bethe diagonalize this nonsymmetric Hamiltonian. We show that the gap between the ground state and first excited state scales as $N^{-3/2}$ for large system size N. The gap between the largest and next-largest eigenvalue scales as N^{-1}. This property hints at conformal invariance. We also explain the connection to the six-vertex model.

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I. THE NOISY BURGERS EQUATION AND AN ASYMMETRIC SPIN HAMILTONIAN

The noisy Burgers equation appears in a variety of problems in nonequilibrium statistical mechanics. Originally [1], Burgers was looking for a simple model in turbulence and proposed a one-dimensional version of the incompressible Navier-Stokes equations. Denoting the velocity field by $u(x,t)$ his equation reads

$$\frac{\partial}{\partial t}u=\frac{\partial}{\partial x}u^2+\nu\frac{\partial^2}{\partial x^2}u(+\text{ noise})\ , \tag{1.1}$$

with ν the viscosity. Burgers studied the noiseless equation with random initial data (for more recent progress, see [2–6]). In turbulence one often considers a randomly stirred fluid. In analogy one would have to add space-time white noise to (1.1) [7]. Here we are interested in the case where u is locally conserved. Therefore the noise is of the form $\partial/\partial x\,\eta(x,t)$ with η space-time white noise. Equation (1.1) is then a prototype for a driven diffusive system [8]. If we reinterpret $u=h'$ as the slope of a (one-dimensional) surface, then (1.1) becomes the Kardar-Parisi-Zhang equation, which governs the shape fluctuations of various growth models, e.g., of the Eden and the Williams-Bjerknes model [9,10]. Of course, the properties to be studied will depend on the particular application. However, from a theoretical point of view the central issue is the behavior of the *stationary* correlation $\langle u(x,t)u(0,0)\rangle-\langle u(0,0)\rangle^2$ for large x and t.

For many purposes it is useful to have a discretized version of (1.1), keeping only the essential elements, namely, the local conservation law together with a nonlinear systematic current, dissipation, and noise. There are many ways to discretize (1.1), all with their own merits. Perhaps the simplest version is to put x on a lattice and to have $u(x,t)$ take only values ±1. We can think then of a spin configuration $\sigma(j,t),\sigma(j,t)=\pm1$, which changes according to a stochastic time evolution. In order to respect the conservation law we update a pair of neighboring spins during short time interval dt as follows:

$$\sigma(j,t+dt),\sigma(j+1,t+dt)=\begin{cases}\sigma(j,t),\sigma(j+1,t) & \text{with probability } 1-\frac{1}{2}[1+\sigma(j,t)\epsilon]dt\\ \sigma(j+1,t),\sigma(j,t) & \text{with probability } \frac{1}{2}[1+\sigma(j,t)\epsilon]dt\ ,\end{cases} \tag{1.2}$$

$|\epsilon|\le1$. Pairs of neighboring spins are updated independently. Let ρ_t be the probability distribution of spin configurations at time t. With the rule (1.2) it is governed by the master equation

$$\begin{aligned}\frac{d}{dt}\rho_t(\sigma)=\tfrac{1}{4}\sum_j\{&[1-\sigma(j)\sigma(j+1)-\epsilon\sigma(j)\\&+\epsilon\sigma(j+1)]\rho_t(\sigma^{j,j+1})\\&-[1-\sigma(j)\sigma(j+1)+\epsilon\sigma(j)\\&-\epsilon\sigma(j+1)]\rho_t(\sigma)\}\ ,\end{aligned} \tag{1.3}$$

where $\sigma^{j,j+1}$ is the spin configuration σ with spins at site j and $j+1$ exchanged.

In the probabilistic community (1.2) is known as the asymmetric simple exclusion process [11]. If + corresponds to a particle and − to a vacant site, then particles hop with rate $\frac{1}{2}(1+\epsilon)$ to the right and rate $\frac{1}{2}(1-\epsilon)$ to the left, provided the final site is empty. In surface growth (1.2) is known as the single-step model [12,13]. If we define height variables through $h(j+1,t)-h(j,t)=[\sigma(j+1,t)+\sigma(j,t)]/2$, then the rule (1.2) describes how material attaches to and evaporates from the surface.

To give our model a quantum-mechanical flavor, we denote a spin configuration by $|\sigma\rangle$ and a state by $\psi=\sum_\sigma \rho(\sigma)|\sigma\rangle$. Introducing the standard Pauli matrices $\boldsymbol{\sigma}_j=(\sigma_j^x,\sigma_j^y,\sigma_j^z)$ at site j satisfying $[\sigma_j^x,\sigma_l^y]=\delta_{jl}2i\sigma_j^z$ plus cyclic permutations, we can write the generator in the master equation (1.3) as

$$H=-\tfrac{1}{4}\sum_{j=1}^{N}[\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_{j+1}-1+i\epsilon(\sigma_j^x\sigma_{j+1}^y-\sigma_j^y\sigma_{j+1}^x)]\ . \tag{1.4}$$

Here and in the following we assume periodic boundary conditions, i.e,. $\boldsymbol{\sigma}_{N+1}=\boldsymbol{\sigma}_1$. $\langle\sigma|e^{-tH}|\sigma'\rangle$ is the (properly normalized) transition probability from the spin configuration σ to σ' in time t. We note that H is nonsymmetric and reduces to the ferromagnetic Heisenberg chain for $\epsilon=0$.

Since H is a real matrix, eigenvalues come in complex-conjugate pairs. They all lie in the positive half plane. The "ground state" has eigenvalue zero and is N-fold degenerate. If we fix the number of up-spins, say M, then in the ground state every spin configuration has equal weight. Let us denote this state by $|0\rangle_M$. The correlation function of interest can then be written as

$${}_M\langle 0|\sigma_0^z e^{-|t|H}\sigma_j^z|0\rangle_M\ . \tag{1.5}$$

To avoid boundary effects we first have to take the limit $N\to\infty$ with the spin density $m=(2M-N)/N$ kept fixed. We would like to understand the behavior for large $|j|$ and t.

It is rather obvious that our discrete version of the noisy Burgers equation has dissipation and an essentially uncorrelated random current. Let us check then on the systematic current. For fixed magnetization m the steady-state current is $j(m)=\epsilon(1-m^2)/2$; in fact, it is even quadratic as in (1.1). As additional evidence, the noisy Burgers equation has spatial white noise as a formal invariant measure, which corresponds to the fact that in the stationary measure for (1.3) spins are independent. Therefore we expect the noisy Burgers equation and the spin exchange dynamics (1.2) to be in the same universality class.

II. STATIONARY CORRELATIONS AND SCALING

The large-scale asymptotics in (1.5) is more easily discussed through the dynamic structure function defined by

$$S(k,t)=\sum_j e^{ikj}(\langle 0|\sigma_0^z e^{-|t|H}\sigma_j^z|0\rangle-\langle 0|\sigma_0^z|0\rangle^2)\ , \tag{2.1}$$

where fixed m is understood. For the symmetric case $\epsilon=0$, $\langle 0|\sigma_0^z e^{-tH}\sigma_j^z|0\rangle$ satisfies a discrete diffusion equation. Therefore

$$S(k,t)=(1-m^2)\exp[-(1-\cos k)|t|]\ . \tag{2.2}$$

Now, for a system of finite size N, (2.2) indicate that the slowest mode should decay as $\exp(-t2\pi^2/N^2)$. Of course, there could be modes with an even slower decay whose matrix elements with σ_j^z vanish, however, and therefore do not contribute to $S(k,t)$. In fact, this happens already in the two magnon sector, where there are modes with a decay rate proportional to $1/N^2$ but a prefactor smaller than implied by (2.2). Still it can be proved that the energy gap E_N (i.e., the energy of the first excited state, since $E_0=0$) satisfies the bounds

$$\frac{2}{N^2}\le E_N\le\frac{2\pi^2}{N^2}\ . \tag{2.3}$$

The upper bound follows from (2.2). The lower bound is proved in [14]. Since this result seems to be essentially unknown, we repeat its proof in Appendix A.

There is another way to understand why only spin-wave excitations contribute to (2.2). Using a projection operator (to M up-spins) P_M the state $|0\rangle_M$ in (1.5) can be written as

$$|0\rangle_M=P_M|\tilde{1}\rangle\ ,\quad |\tilde{1}\rangle=\bigotimes_{j=1}^{N}\begin{bmatrix}1\\1\end{bmatrix}_j\ . \tag{2.4}$$

Since P_M commutes with H and σ_j^z we write (1.5) as

$$\langle\tilde{1}|\sigma_0^z e^{-|t|H}\sigma_j^z P_M|\tilde{1}\rangle-\langle\tilde{1}|\sigma_0^z P_m|\tilde{1}\rangle\ . \tag{2.5}$$

Rotating each $\binom{1}{1}$ to $\binom{1}{0}$ can transform $|\tilde{1}\rangle$ to

$$|1\rangle=\bigotimes_{j=1}^{N}\begin{bmatrix}1\\0\end{bmatrix}_j\ . \tag{2.6}$$

Let U be the corresponding unitary operator; U transforms between the z representation and the x representation $U\sigma_j^z U=\sigma_j^x$. By rotational invariance of H, we have $[U,H]=0$. Therefore (1.5) becomes

$$\langle 1|\sigma_0^x e^{-|t|H}\sigma_j^x UP_M U|1\rangle-\langle 1|\sigma_0^x UP_M U|1\rangle^2\ . \tag{2.7}$$

But σ_0^x creates a single spin excitation in the all up state $\langle 1|$ and, by conservation of magnetization, $e^{-|t|H}$ leaves the subspace of single spin excitations invariant. Thereby we obtain (2.2).

For $\epsilon\neq 0$ a variety of theoretical arguments and extensive numerical simulations [10,13] suggest that the structure function has the scaling form

$$S(k,t)=(1-m^2)e^{-ik\epsilon mt}g([\epsilon^2(1-m^2)]^{1/3}k|t|^{2/3})\ . \tag{2.8}$$

Here g is a universal scaling function with $g(0)=1$ and $g(x)=g(-x)$, and decaying at infinity. Unless there are strange cancellations we expect then that for $\epsilon\neq 0$ and $|m|<1$, H has an energy gap

$$E_N\approx N^{-3/2} \tag{2.9}$$

for large N. (Since H is nonsymmetric, we mean here the real part of the energy of the first excited state.)

The main goal of our paper is to show how (2.9) follows from the Bethe diagonalization of H in the particular case $\epsilon=1$ and $m=0$. We will also have a look at the top of the spectrum. Such high-energy states are of no

relevance for the large-scale behavior, but they could be of interest in understanding whether the conformal invariance at $\epsilon=0$ extends to the nonsymmetric case. It is known that H commutes with the transfer matrix of the six-vertex model [15,16]. We will exploit this connection in more detail in order to arrive at novel predictions for the polarization correlations.

III. THE BETHE ANSATZ

We label the configurations by the location of the + spins $\mathbf{m}=(m_1,m_2,\ldots,m_n)$, $1\le m_1<m_2<\cdots<m_n\le N$. In this basis a state is expanded as $|\phi\rangle=\sum_{\mathbf{m}} f(\mathbf{m})|\mathbf{m}\rangle$ and the right eigenvalue equation $H|\phi\rangle=E|\phi\rangle$ reads

$$-\sum_{j=1}^{n}\{\epsilon_+[1-\delta(m_{j+1}-m_j,1)][f(\ldots,m_j+1,\ldots)-f(m_1,\ldots,m_n)] +\epsilon_-[1-\delta(m_j-m_{j-1},1)][f(\ldots,m_j-1,\ldots)-f(m_1,\ldots,m_n)]\}=Ef(m_1,\ldots,m_n)\ . \tag{3.1}$$

Here $\delta(m,n)$ is the Kronecker symbol and $\epsilon_\pm=\frac{1}{2}(1\pm\epsilon)$. The Bethe ansatz tries to solve (3.1) through a generalization of the plane-wave ansatz for a single particle. One sets

$$f(\mathbf{m})=\sum_{\mathcal{P}}\mathcal{A}(\mathcal{P})\prod_{j=1}^{n}(z_{p(j)})^{m_j}\ , \tag{3.2}$$

where the summation is over all permutations $\mathcal{P}=(p(1),\ldots,p(n))$ of $(1,\ldots,n)$. z_l are complex numbers. There is no reason for them to be of the special form e^{ik}, k real. Still, for convenience, we refer to them as wave numbers.

If particles are at least one spacing apart, the eigenvalue is given by

$$E=\sum_{j=1}^{n}(1-\epsilon_+z_j-\epsilon_-z_j^{-1})\ . \tag{3.3}$$

The expansion coefficients $\mathcal{A}(\mathcal{P})$ are now fixed by taking two particles next to each other in (3.1). This yields the relation

$$\mathcal{A}(\ldots,p,q,\ldots)=W(z_p,z_q)\mathcal{A}(\ldots,q,p,\ldots) \tag{3.4}$$

with

$$W(z_j,z_l)=-\frac{z_j-\epsilon_+z_jz_l-\epsilon_-}{z_l-\epsilon_+z_jz_l-\epsilon_-}\ . \tag{3.5}$$

Equations (3.4) and (3.5) determine recursively $\mathcal{A}(\mathcal{P})$ up to an overall phase factor. Finally considering a jump between sites N and 1 in (3.1), we must have

$$\mathcal{A}(p(1),\ldots,p(n))=(z_{p(1)})^N\mathcal{A}(p(2),\ldots,p(n),p(1))\ . \tag{3.6}$$

By combining (3.6) with (3.4) we arrive at the *Bethe equations*

$$(z_j)^N=\prod_{j\neq l=1}^{n}\left[-\frac{z_j-\epsilon_+z_jz_l-\epsilon_-}{z_l-\epsilon_+z_jz_l-\epsilon_-}\right], \tag{3.7}$$

$j=1,\ldots,n$. Any solution to (3.7) with distinct wave numbers $\{z_j,j=1,\ldots,n\}$ gives a right eigenvector of H with energy (3.3).

Let e^{iP} be the operator of left translation. It satisfies $e^{iPN}=1$ because the system has N sites. Since $[H,P]=0$, the eigenvectors constructed above are also eigenvectors of the momentum P. We have

$$e^{iP}|\psi\rangle=\left[\prod_{j=1}^{n}z_j\right]|\psi\rangle\ , \tag{3.8}$$

in particular,

$$\prod_{j=1}^{n}(z_j)^N=1\ . \tag{3.9}$$

If $\{z_j\}$ is a solution to the Bethe equations (3.7) with energy E and momentum P, so is the complex conjugate $\{\bar{z}_j\}$ with energy $\bar{E}$ and momentum $-P$. Also, if $\{z_j\}$ is a solution for the Hamiltonian with bias ϵ, so is $\{1/z_j\}$ for the Hamiltonian with bias $-\epsilon$, which is just the adjoint of H. Thus the right and left eigenvectors of H are related in a simple fashion. As we know already, the normalized ground state is just

$$|0\rangle=(n!)^{-1/2}\sum_{\mathbf{m}}|\mathbf{m}\rangle\ . \tag{3.10}$$

This corresponds to $z_j=1,j=1,\ldots,n$ and $\mathcal{A}(\mathcal{P})=1$. The ground state has zero energy and momentum.

IV. THE COMPLETELY ASYMMETRIC, HALF-FILLED SECTOR

The Bethe equations (3.7) are a set of n nonlinear coupled equations. As discussed at the end of Sec. II, for the symmetric case only single spin excitations, i.e., $n=1$, have to be studied. The Bethe ansatz reduces then to plane waves and yields the structure function (2.2). For $\epsilon\neq0$ we must solve the coupled Bethe equations, however.

Fortunately for $\epsilon=1$ the Bethe equations simplify to a nonlinear equation for a single complex variable, which is further simplified by considering the half-filled system $2n=N$. It is convenient to introduce the shifted wave numbers $Z_j\equiv2z_j-1$. Then for $\epsilon=1$ the Bethe equations (3.7) become

$$(1+Z_l)^{N-n}(1-Z_l)^n=-2^N\prod_{j=1}^{n}\frac{Z_j-1}{Z_j+1}\ , \tag{4.1}$$

$l=1,\ldots,n$, with energy and momentum

$$E=\sum_{j=1}^{n}\frac{1-Z_j}{2}\ ,\quad e^{iP}=\prod_{j=1}^{n}\frac{1+Z_j}{2}\ . \tag{4.2}$$

The corresponding eigenfunction can be written as a determinant. To verify this we note that for $\epsilon=1$

$$W(z_j,z_l)=-w_l/w_j\ ,\quad w_j\equiv 1-z_j^{-1}\ . \tag{4.3}$$

Choosing the standard normalization for $\mathcal{A}(\mathcal{P})$ [15], we have

$$\begin{aligned}\mathcal{A}(\mathcal{P})&=\delta_{\mathcal{P}}\prod_{j<l}[-W(z_{p(j)},z_{p(l)})]^{1/2}\\&=\delta_{\mathcal{P}}\prod_{j=1}^{n}(w_{p(j)})^{j-(n+1)/2}\ ,\end{aligned} \tag{4.4}$$

with $\delta_{\mathcal{P}}$ the signature of the permutation $\mathcal{P}$. Thus the eigenfunction is given by

$$f(\mathbf{m})=\sum_{\mathcal{P}}\delta_{\mathcal{P}}\prod_{j=1}^{n}(w_{p(j)})^{j-[(n+1)/2]}(z_{p(j)})^{m_j}\ ,$$

equivalently

$$f_{\mathbf{m}}=\det\mathcal{M}\ ,\quad \mathcal{M}_{jl}=(z_l)^{m_j}(1-z_l^{-1})^{j-[(n+1)/2]}\ . \tag{4.5}$$

The right-hand side of (4.1) is independent of the index l. If we now require that $N=2n$, then we obtain the single polynomial equation

$$(1-Z^2)^n=Y\ . \tag{4.6a}$$

It has $2n$ solutions. Out of them we pick $Z_1,\ldots,Z_n$. Y has then to be determine through

$$Y=-4^n\prod_{j=1}^{n}\frac{Z_j-1}{Z_j+1}\ . \tag{4.6b}$$

[We remark that for a fractional filling $n=(q/p)N$, q and p are the integer and the relative prime, and the quadratic Z polynomial in (4.6a) would be replaced by a polynomial of degree p.]

Let us label explicitly the solutions to (4.6a). We set $Y=-a^n e^{in\theta}$, $a\geq 0$, $\theta\in[-\pi/n,\pi/n)$ and denote the roots by

$$y_m=ae^{i\theta}e^{2\pi i[m-(1/2)]/n}\ , \tag{4.7}$$

y_j are evenly spaced on a circle with radius a. In fact, it will be a more natural to regard y as lying on a two-valued Riemann surface. Therefore we set $m=1,\ldots,2n$ with $y_1,\ldots,y_n$ located on the first sheet and $y_{n+1},\ldots,y_{2n}$ located on the second sheet. The $2n$ solutions to (4.6a) are then given by

$$Z_m=(1-y_m)^{1/2}\ , \tag{4.8}$$

$m=1,\ldots,2n$. If y_m is on the first (second) sheet, we take the phase of Z_m to be between $-\pi/2$ and $\pi/2$ (between $\pi/2$ and $3\pi/2$.) Note that $Z_m=-Z_{n+m}$. For a particular solution of the Bethe equations we have to select n roots. Therefore we introduce a strictly increasing function l on $(1,\ldots,n)$, taking values in $(1,\ldots,2n)$. There are $\binom{2n}{n}$ such functions. A solution is then the set $\{Z_{l(m)},m=1,\ldots,n\}$. Once $l(m)$ is fixed, we insert in (4.6b) and obtain a nonlinear equation for Y, i.e., for the amplitude a and the phase θ.

To summarize, a solution to the Bethe equations for the half-filled $\epsilon=1$ case is determined by the parameters $a\geq 0$ and $\theta\in[-\pi/n,\pi/n)$, which satisfy

$$\left[\frac{a}{4}e^{i\theta}\right]^n=\prod_{m=1}^{n}\frac{Z_{l(m)}-1}{Z_{l(m)}+1}\ . \tag{4.9}$$

Here $l(m)$ is strictly increasing and $Z_l=(1-y_l)^{1/2}$ is defined on the two-valued Riemann surface as explained after Eq. (4.8).

In the following, pairs such as Z_m and Z_{n+m} will play a special role. We note that $Z_m+Z_{n+m}=y_m-1$ and $Z_m+Z_{n+m}=0$. Therefore each pair Z_m, Z_{n+m} contributes a factor 1 to $-Y/4^n$, a factor $y_m/4$ to $\exp iP$, and a summand 1 to E.

The momentum of a solution set $\{Z_l\}$ follows from

$$\begin{aligned}\prod_{m=1}^{n}(-y_m)&=a^n e^{in\theta}=4^n\prod_{m=1}^{n}\frac{Z_{l(m)}-1}{Z_{l(m)}+1}\\&=e^{-2iP}\prod_{m=1}^{n}[(Z_{l(m)})^2-1]\\&=e^{-2iP}\prod_{m=1}^{n}(-y_{l(m)})\ ,\end{aligned} \tag{4.10}$$

where we used (4.2) and (4.9). The products on the left- and right-hand sides of (4.10) are defined on the two-value Riemann surface. Thus equating phases yields

$$P=\frac{\pi}{n}\sum_{m=1}^{n}[l(m)-m]\ \ (\mathrm{mod}2\pi)\ . \tag{4.11}$$

There is a simple pictorial rule that reproduces (4.11), cf. Fig. 1: We indicate the solution set by putting "markers" at the labels of the corresponding roots. For example, in the ground state the n markers are at the labels 1 to n with the labels $n+1$ to $2n$ empty. Another eigenstate is characterized by some other marker configuration. The momentum of this state, in units of $2\pi/N$, is then just the total signed number of shifts (-1 for each clockwise and $+1$ for each counterclockwise shift) necessary to move the markers from the configuration corresponding to the ground state of the configuration corresponding to the state under consideration.

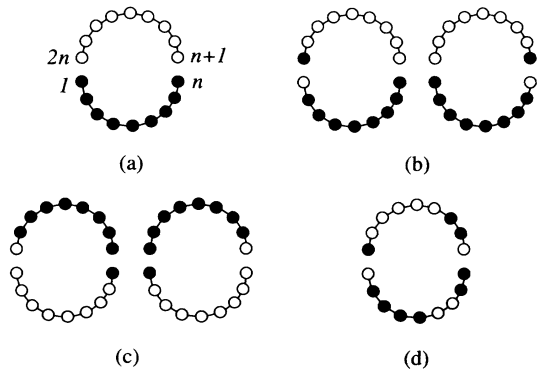

FIG. 1. The roots $y_1,\ldots,y_{2n}$ on the two-dimensional Riemann surface and marker configurations. (a) Ground state $P=0$; (b) first excited states $P=\pm 2\pi/N$; (c) next-highest states $P=0$ (n odd), π (n even); (d) a state with $P=9(2\pi/N)$.

The ground-state eigenfunction is constant. This corresponds to taking the first sheet, i.e., $l(m)=m$, $m=1,\ldots,n$ in (4.9). Solving for Y yields the amplitude $a=0$. A natural guess is that the first excited state is obtained through a minimal modification of the ground-state configuration of markers. On our two-valued Riemann surface the minimal shifts are n to $n+1$ and 1 to $2n$. This yields the solution sets $\{Z_m, m=2,\ldots,n-1\}$ and either Z_1, Z_{n+1}, or Z_n, Z_{2n}. Since a pair contributes a factor 1 in (4.9), both sets have the same amplitude a and phase θ. A pair contributes a summand 1 to the energy. Since the remaining roots are equal in both sets, this implies that they must have the same energy. Eigenvalues come in complex-conjugate pairs. Therefore, unless there is some further degeneracy, the eigenvalue of the first excited states must be real, hence $\theta=0$. By (4.11) the total momentum of the first excited states is $\pm 2\pi/N$. We verified our rule for the first excited state by solving numerically the Bethe equations for small size systems (up to $N=20$). One may further guess that the wave numbers of the first excited state are close to those of the ground state. This is not the case. In fact the amplitude will be shown to be $a=1+O(1/n)$.

The state with largest energy corresponds to taking the second sheet, i.e, to $l(m)=n+m, m=1,\ldots,n$. Our numerical results indicate that the next-largest states (in $|E|$) correspond to $\{Z_{n+m}, m=2,\ldots,n-1\}$ and either Z_1, Z_{2n}, or Z_{n+1} and Z_n. In contrast to the first excited state, this is not a minimal shift. To understand the difference we note that the contribution to the energy from a pair is always 1. However, for the high-lying states the average energy per particle is greater than $\frac{1}{2}$. Therefore, to obtain minimal changes in energy in the upper portion of the spectrum, it is more important to avoid pairs than to make small shifts on the Riemann surface. The two next-highest eigenstates are nondegenerate and have complex-conjugate eigenvalues.

The degeneracy of eigenvalues is closely related to the number of pairs. For example, in the ground state and in the highest state there are no pairs and both states are nondegenerate. On the other hand, if all Z_l's are paired (n even), then $E=n/2$ always, which is the state with the highest degree of degeneracy.

In the following two sections we will compute the energies for the first excited state and the two highest states as $n\to\infty$. For this purpose it is convenient to convert the logarithm of (4.10) and the sum in (4.2) to complex contour integrals by using the following identity [17]:

$$\sum_{j=m}^{n} f(j)=\int_m^n dt\, f(t)+\tfrac{1}{2}[f(m)+f(n)] +2\int_0^\infty dt\frac{F(n,t)-F(m,t)}{e^{2\pi t}-1}\ , \tag{4.12a}$$

where

$$F(s,t)=\frac{1}{2i}[f(s+it)-f(s-it)]\ . \tag{4.12b}$$

Also, we shall do an analysis on the complex y plane with a square-root branch cut and write $Z_m=+\sqrt{1-y_m}$ for $m=1,\ldots,n$ and $Z_m=-\sqrt{1-y_m}$ for $m=n+1,\ldots,2n$, where the real part of the square root is positive. The results of the asymptotic analysis are given in (5.26) for the first excited state, in (6.9) for the highest state, and in (6.29) for the next-highest state.

V. FIRST EXCITED STATE—THE $n^{-3/2}$ GAP

According to our discussion above we set $Z_m=+\sqrt{1-y_m}$, $m=2,\ldots,n-1$, and $\theta=0$. Let us define

$$q(m)=ae^{i2\pi[m-(1/2)]/n}\ ,\quad h(m)=\sqrt{1-q(m)}\ . \tag{5.1}$$

Then, according to (4.9),

$$\left[\frac{a}{4}\right]^n=\prod_{m=2}^{n-1}\frac{h(m)-1}{h(m)+1}\ , \tag{5.2}$$

equivalently,

$$n\ln 4-n\ln a=\sum_{m=2}^{n-1}\{2\ln[1+h(m)]-\ln[-q(m)]\}\ . \tag{5.3}$$

Using $-q=qe^{-i\pi}$, the second part of the sum on the right-hand side is simply $(2-n)\ln a$ and therefore (5.3) reduces to

$$n\ln 2=\ln a+\sum_{m=1}^{n}\ln[1+h(m)]-\ln(1+h_1)-\ln(1+h_n)\ , \tag{5.4}$$

where $h_1=h(1)$ and $h_n=h(n)$. The energy is given by

$$2E=n-\sum_{m=1}^{n}h(m)+h_1+h_n\ . \tag{5.5}$$

Substituting (5.4) and (5.5) in (4.11) we obtain

$$n\ln 2=\ln a+\frac{1}{2i\varepsilon}\int_C\frac{dq}{q}\ln(1+\sqrt{1-q}\,) -\tfrac{1}{2}[\ln(1+h_1)+\ln(1+h_n)]+J_1\ , \tag{5.6}$$

$$2E=n-\frac{1}{2i\varepsilon}\int_C\frac{dq}{q}\sqrt{1-q}+\tfrac{1}{2}(h_1+h_n)-J_2\ . \tag{5.7}$$

The contour of integration C is from $q(1)$ to $q(n)$ counterclockwise along the circle with a branch cut $[1,\infty)$ along the real axis, cf. Fig. 2. J_1 and J_2 are the right-hand integrals in (4.12a) with $f(x)$ equal to $\ln[1+h(x)]$ and $h(x)$, respectively. The parameter ε is defined

$$\varepsilon=\frac{\pi}{n}\ . \tag{5.8}$$

To evaluate the integrals (5.6) and (5.7), we close the contour C by adding a contour C_2 from $q(1)$ to $q(n)$ around the branch cut if $a>1$, cf. Fig. 2. Denoting the closed contour as C_1, we have $C=C_1+C_2$. Since there is only a simple pole inside C_1 for either integral, their values follow from the residue theorem. Thereby $n\ln 2$ and n in (5.6) and (5.7), respectively, are canceled.

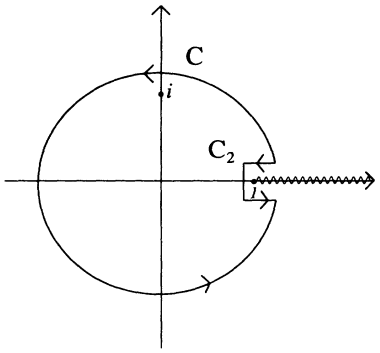

FIG. 2. Complex contour integration.

Next we want to show that $a \to 1$ as $\varepsilon \to 0$. Note that the integral J_1 is $o(1)$ because of (4.12b) and because

$$q\left[1+\frac{it}{2}\right]=ae^{\varepsilon(i-t)}, \qquad q\left[n+\frac{it}{2}\right]=ae^{2\pi i-\varepsilon(i+t)}. \tag{5.9}$$

Now suppose that $a-1=O(1)$. Then

$$\frac{1}{2i\varepsilon}\int_{C_2}\frac{dq}{q}\ln(1+\sqrt{1-q}) = \begin{cases} O(\varepsilon^{-1}), & a>1 \\ -\ln(1+\sqrt{1-a})+o(1), & a<1 . \end{cases} \tag{5.10}$$

An $O(\varepsilon^{-1})$ term cannot be balanced in (5.6), hence the first case cannot hold. For the second case, the $O(1)$ solution for a is 1, which contradicts the assumption $a-1=O(1)$. We will set then

$$a=1+\varepsilon\beta , \tag{5.11}$$

with $\beta=O(1)$.

As will be shown, (5.6) and (5.7) can be expanded, respectively, in the forms

$$0=\varepsilon^{1/2}A(\beta)+\varepsilon^{3/2}B(\beta)+\cdots , \tag{5.12}$$

$$2E=\varepsilon^{1/2}A(\beta)+\varepsilon^{3/2}D(\beta)+\cdots . \tag{5.13}$$

Let $\beta=\beta_0+\varepsilon\beta_1$. Then (5.12) becomes

$$0=\varepsilon^{1/2}A(\beta_0)+\varepsilon^{3/2}[B(\beta_0)+\beta_1 A'(\beta_0)]+\cdots , \tag{5.14}$$

which implies

$$A(\beta_0)=0 , \quad B(\beta_0)+\beta_1 A'(\beta_0)=0 . \tag{5.15}$$

Expanding also the energy in (5.13),

$$2E=\varepsilon^{1/2}A(\beta_0)+\varepsilon^{3/2}[D(\beta_0)+\beta_1 A'(\beta_0)]+\cdots , \tag{5.16}$$

and therefore, by (5.15),

$$2E\approx\varepsilon^{3/2}[D(\beta_0)-B(\beta_0)] . \tag{5.17}$$

Thus β_1 is not needed explicitly.

In order to evaluate (5.6) and (5.7) to the required order in ε, we need that

$$\begin{aligned} &q(j)\approx 1-\varepsilon(\pm i-\beta)+O(\varepsilon^2) , \\ &h_j\approx\pm i\varepsilon^{1/2}\sqrt{\beta\mp i}+O(\varepsilon^{3/2}) , \\ &h\left[j+i\frac{t}{2}\right]\approx\pm i\varepsilon^{1/2}\sqrt{\beta-t\mp i}+O(\varepsilon^{3/2}) , \end{aligned} \tag{5.18}$$

$j=1,n$, where the upper sign is taken for $j=n$ and the lower sign for $j=1$. Note that opposite signs are due to the appearance of the branch cut. Also, to order ε^2,

$$\begin{aligned} &\ln a=\varepsilon\beta+O(\varepsilon^2) , \\ &\ln(1+h_1)+\ln(1+h_n)=\varepsilon\beta+(h_1+h_n) \\ &\qquad\qquad +\tfrac{1}{3}(h_1^3+h_n^3)+O(\varepsilon^2) . \end{aligned} \tag{5.19}$$

Since $a\approx 1$, we expand q around 1 for the integral along C_2. For arbitrary s, $s>0$, we have

$$-\tfrac{1}{2}\int_{C_2}\frac{dq}{q}(1-q)^{s/2}=\int_{h_1}^{h_n}dh\frac{h^{s+1}}{1-h^2} =\left[\frac{h^{s+2}}{s+2}+\frac{h^{s+4}}{s+4}+\cdots\right]_{h_1}^{h_n} . \tag{5.20}$$

Finally expanding the logarithm in the integrand in (5.6) and using the above yields

$$\frac{1}{i\varepsilon}\int_{C_2}\frac{dq}{q}\ln(1+\sqrt{1-q})=-\varepsilon\beta-\frac{2}{3i\varepsilon}(h_n^3-h_1^3) -\frac{8}{15i\varepsilon}(h_n^5-h_1^5)+O(\varepsilon^2) . \tag{5.21}$$

Substituting (5.19) to (5.21) in (5.6) gives

$$\tfrac{1}{2}(h_1+h_n)+\frac{1}{3i\varepsilon}(h_n^3-h_1^3)+\tfrac{1}{6}(h_1^3+h_n^3) +\frac{4}{15i\varepsilon}(h_n^5-h_1^5)-J_1=O(\varepsilon^2) . \tag{5.22}$$

Similarly, substituting (5.20) in (5.7) results in

$$2E=\tfrac{1}{2}(h_1+h_n)+\frac{1}{3i\varepsilon}(h_n^3-h_1^3) +\frac{1}{5i\varepsilon}(h_n^5-h_1^5)-J_2+O(\varepsilon^2) . \tag{5.23}$$

Note that both h_1 and h_n are $O(\varepsilon^{1/2})$. Thus we are halfway in showing (5.12) and (5.13).

It remains to analyze J_1 and J_2. Since J_1 and J_2 are given by the right integral of (4.12a) with $f(x)$ equal to $\ln[1+h(x)]$ and $h(x)$, respectively, and since $h(x)$ is small, we can expand the logarithm in J_1 as $J_1=J_2-\frac{1}{2}J_3+\frac{1}{3}J_4+O(\varepsilon^2)$, where J_3 and J_4 are given by the same integral with integrand $f(x)$ equal to $h(x)^2$ and $h(x)^3$, respectively. The leading order of J_3 vanishes and only $J_2=O(\varepsilon^{1/2})$ and $J_4=O(\varepsilon^{3/2})$ are of relevance. Thus the $O(\varepsilon^{1/2})$ terms in (5.22) and (5.23) are the same, as claimed in (5.12) and (5.13). Using (5.18) in (5.22) the

$O(\varepsilon^{1/2})$ equation $A(\beta_0)=0$ becomes

$$\mathrm{Re}[(2\beta_0+5i)\sqrt{\beta_0+i}\,] = 3\,\mathrm{Re}\int_0^\infty dt \frac{\sqrt{\beta_0+t-i}-\sqrt{\beta_0-t-i}}{e^{\pi t}-1}\ , \tag{5.24}$$

which determines β_0. As explained in (5.12) to (5.17), only the leading order of the difference between (5.23) and (5.22) is relevant for the leading order of E. Therefore,

$$6E \approx J_4 - \tfrac{1}{2}(h_1^3+h_n^3) - \frac{1}{5i\varepsilon}(h_n^5-h_1^5) \approx \frac{\varepsilon^{3/2}}{5}\mathrm{Re}\left[5\int_0^\infty dt \frac{(\beta_0+t-i)^{3/2}-(\beta_0-t-i)^{3/2}}{e^{\pi t}-1} - (2\beta_0+7i)(\beta_0+i)^{3/2}\right]\ . \tag{5.25}$$

Evaluating (5.24) and (5.25) numerically, we obtain

$$\beta_0 = 1.119\,068\,8\ldots,\quad E \approx (2.301\,345\,96\ldots)n^{-3/2}\ . \tag{5.26}$$

VI. HIGHEST AND NEXT-HIGHEST STATES—THE n^{-1} GAP

Reversing the sign of the Hamiltonian H turns it into an asymmetric antiferromagnetic spin chain. Although the connection with the fluctuating Burgers equation is lost, it could still be of interest as a non-Hermitian generalization of the *XXX* model. In particular, since the *XXX* spin chain and the anisotropic *XXZ* chain have been used as a paradigm for conformal invariance [18,19], one could raise the question whether this property extends to non-Hermitian operators such as the asymmetric spin chain. Since a mass gap of order $1/N$ is characteristic of conformal invariance, we shall examine the top of the spectrum in the asymmetric spin chain.

The ground state and the first excited state for $-H$ are just the highest and the next-highest eigenstates for H, respectively. The highest state is determined by the Bethe equation

$$\left[\frac{a_0}{4}\right]^n = \prod_{m=1}^{n}\frac{h(m)+1}{h(m)-1}\ ,\quad q(m)=a_0 e^{i2\pi[m-(1/2)]/n}\ , \tag{6.1}$$

$$2E_0 = n + \sum_{m=1}^{n} h(m)\ . \tag{6.2}$$

The two next-highest states form a pair related through complex conjugation. We consider only one of them, which is determined by

$$\left[\frac{a_1}{4}e^{i\theta}\right]^n = \frac{h(n)-1}{h(n)+1}\prod_{m=1}^{n-1}\frac{h(m)+1}{h(m)-1}\ ,\quad q(m)=a_1 e^{i\theta}e^{i2\pi[m-(1/2)]/n}\ , \tag{6.3}$$

$$2E_1 = n - h(n) + \sum_{m=1}^{n-1} h(m)\ , \tag{6.4}$$

where $h(m)=\sqrt{1-q(m)}$, as before. The asymptotic analysis of the first excited state used an expansion around the square-root branch point, resulting in a half-integer exponent. Here, the analysis involves an expansion around a regular point and hence an integer exponent should be expected.

Let us first consider the highest state. We repeat the steps leading from (5.2) to (5.4) and obtain

$$n\ln\frac{a_0}{2} = \sum_{m=1}^{n}\ln[1+h(m)]\ . \tag{6.5}$$

Using the identity (4.12) we transform (6.5) and (6.2) into contour integrals. Again, the term corresponding to the last integral of (4.12a) is of order $o(1)$. Due to a change of sign the $O(N)$ terms do not cancel in either equation. Therefore to leading order only the contour integrals are relevant. As before, we close the contour along the branch cut and obtain $n\ln 2$ and n. Therefore, Eqs. (6.5) and (6.2) become, to leading order,

$$\ln\frac{a_0}{4} = \frac{1}{2\pi i}\int_1^{a_0}\frac{dq}{rq}\ln\frac{1+i\sqrt{q-1}}{1-i\sqrt{q-1}} + o(1)\ , \tag{6.6}$$

$$2E_0 = 2n + \frac{n}{\pi}\int_1^{a_0}\frac{dq}{q}\sqrt{q-1} + O(1)\ , \tag{6.7}$$

where for the contours along the branch cut we have used $\sqrt{1-q}=+i\sqrt{q-1}$ for the one below the cut and $\sqrt{1-q}=-i\sqrt{q-1}$ for the one above, cf. (5.18). After simplification and evaluation of the integral in (6.7), we obtain

$$\ln\frac{a_0}{4} \approx \frac{2}{\pi}\int_{a_0^{-1/2}}^{1}\frac{dx}{x}\arccos x\ ,\qquad \frac{E_0}{n} \approx \frac{1}{2}+\frac{1}{\pi}\left[\sqrt{a_0-1}+\arcsin\frac{1}{\sqrt{a_0}}\right]\ , \tag{6.8}$$

and therefore

$$a_0 \approx 6.595\,724\,1\ldots,\quad E_0 \approx (1.380\,280\,23\ldots)n\ . \tag{6.9}$$

Our next task is to compute the difference between E_1 and E_0. We denote the variables associated with the highest state by a subscript 0. We have

$$q_0(m)=a_0 e^{i2\pi[m-(1/2)]/n}\ ,\quad h_0(m)=\sqrt{1-q_0(m)}\ . \tag{6.10}$$

We divide (6.3) by (6.1) and subtract (6.4) from (6.2) to arrive at

$$\left[\frac{a_1}{a_0}\right]^n e^{in\theta}=\left[\frac{h(n)-1}{h(n)+1}\right]^2 \times\prod_{m=1}^{n}\left[\frac{h(m)+1}{h_0(m)+1}\right]\left[\frac{h_0(m)-1}{h(m)-1}\right], \tag{6.11}$$

$$2(E_0-E_1)=2h(n)+\sum_{m=1}^{n}[h_0(m)-h(m)] . \tag{6.12}$$

Equation (6.11) determines the asymptotic behavior of a_1 and θ, which will be computed first. Note that $\theta\in[-\pi/n,\pi/n)$ is a small parameter. In addition, a_1 and a_0, as well as $q(m)$ and $q_0(m)$, are asymptotically identical. Therefore we define the small parameters α and ξ by

$$\frac{a_1}{a_0}=1+\alpha , \quad \frac{q(m)}{q_0(m)}=1+\xi . \tag{6.13}$$

Then $\xi\approx i\theta+\alpha-\frac{1}{2}\theta^2$. The following gives the asymptotic analysis in detail; the main results are (6.23) and (6.24) for θ and α, and (6.27)–(6.29) for E_0-E_1.

Expanding $h(m)$ around $h_0(m)$ we have

$$h(m)=h_0(m)-\tfrac{1}{2}\xi X[q_0(m)]+O(\xi^3) , \qquad X(q)=\frac{q}{h}+\frac{\xi q^2}{4h^3} , \tag{6.14}$$

and hence, dropping the arguments,

$$\frac{h+1}{h_0+1}\frac{h_0-1}{h-1}=1-\frac{\xi}{h_0}-\frac{\xi^2 q_0}{4h_0^3}+\frac{\xi^2(1+h_0)}{2h_0^2}+O(\xi^3) . \tag{6.15}$$

Substituting (6.13) and (6.15) in (6.11) and taking logarithms, we get

$$n\alpha+in\theta=2\ln\frac{h_0(n)-1}{h_0(n)+1}+\frac{2\xi}{h_0(n)} -\xi\sum_{m=1}^{n}\Delta(q_0(m))+O(\xi^2) , \tag{6.16}$$

where

$$\Delta(q)=\frac{1}{h}-\frac{\xi}{2h}+\frac{\xi q}{4h^3} . \tag{6.17}$$

We apply (4.12) to the sum in (6.16) and obtain

$$\sum_{m=1}^{n}\Delta(q_0(m))=\frac{n}{2\pi i}\int_{q_0(1)}^{q_0(n)}\frac{dq}{q}\Delta(q) +\tfrac{1}{2}[\Delta(q_0(1))+\Delta(q_0(n))]+\cdots . \tag{6.18}$$

The integral can be evaluated in closed form, i.e.,

$$\int\frac{dq}{q}\Delta(q)=\left[1-\frac{\xi}{2}\right]\ln\frac{h-1}{h+1}+\frac{\xi}{2h} . \tag{6.19}$$

Substituting back into (6.16) and dropping terms of order ξ^2, we have

$$n\alpha+in\theta=-\frac{n\xi}{2\pi i}\left[\left[1-\frac{\xi}{2}\right]\ln\frac{h_0(j)-1}{h_0(j)+1}+\frac{\xi}{2h_0(j)}\right]_{j=1}^{n} +2\ln\frac{h_0(n)-1}{h_0(n)+1}+\frac{\xi}{2}\left[\frac{3}{h_0(n)}-\frac{1}{h_0(1)}\right] +O(\xi^2) . \tag{6.20}$$

The leading orders of $h_0(n)$ and $h_0(1)$ are, respectively,

$$h_0(n)\approx is+\frac{\pi a_0}{2sn} , \quad h_0(1)\approx -is+\frac{\pi a_0}{2sn} , \quad s=\sqrt{a_0-1} . \tag{6.21}$$

Hence

$$\ln\frac{h_0(j)-1}{h_0(j)+1}\approx\pm 2i\phi-\frac{\pi}{ns} , \quad \phi=\arcsin\frac{1}{\sqrt{a_0}} , \tag{6.22}$$

$j=1,n$, where the + sign is taken for $j=n$ and the − sign for $j=1$. Substituting the above in the right-hand side of (6.2) we arrive at the asymptotics of θ and α,

$$\left[1+\frac{2\phi}{\pi}\right]n\theta\approx 4\phi , \quad \frac{s}{2}\left[1+\frac{2\phi}{\pi}\right]n^2\alpha\approx n\theta-\pi-\frac{n^2\theta^2}{4\pi} , \tag{6.23}$$

i.e., approximately

$$n\theta\approx 1.275\,140\,3\ldots , \quad n^2\alpha\approx -1.344\,982\,958\,6\ldots . \tag{6.24}$$

The mass gap will be computed as a function of $n\theta$ and $n^2\alpha$. Applying (4.12) to the sum in (6.12) and using (6.14) for the remainder, we obtain

$$2(E_0-E_1)=\frac{n\xi}{4\pi i}\int_{q_0(1)}^{q_0(n)}\frac{dq}{q}X(q) +\frac{\xi}{4}[X(q_0(1))-3X(q_0(n))] +2h_0(n)+O(\xi^2) . \tag{6.25}$$

The integral can be evaluated in closed form. We have, using (6.21),

$$\int\frac{dq}{q}X(q)=\left[\frac{\xi}{2}-2\right][h_0(n)-h_0(1)] +\frac{\xi}{2}\left[\frac{1}{h_0(n)}-\frac{1}{h_0(1)}\right] \approx -4is+i\xi(s-s^{-1}) . \tag{6.26}$$

Similarly substituting (6.21) in the remaining terms of (6.25) we arrive at

$$\mathrm{Im}(E_0-E_1)\approx s\left[1-\frac{n\theta}{2\pi}\right], \tag{6.27}$$

$$\mathrm{Re}(E_0-E_1)n\approx\frac{n^2\theta^2}{8\pi}(s+s^{-1})+\frac{a_0}{2s}(\pi-n\theta)-\frac{sn^2\alpha}{2\pi}. \tag{6.28}$$

Combining (6.9), (6.21), (6.22), and (6.24), we finally obtain

$$E_0-E_1\approx(3.288\,839\,35\ldots)n^{-1}+i(1.885\,456\,427\ldots). \tag{6.29}$$

VII. RELATION TO THE SIX-VERTEX MODEL

Let us consider the standard six-vertex model on a square lattice [15,20,21]. We denote the spin (vertical arrow) configuration in the row n by $\sigma^{(n)}$ (in the ferroelectric interpretation this would be the vertical polarization configuration in row n). The transfer matrix T goes from row n to $n+1$ in the positive y direction and has matrix elements $\langle\sigma^{(n)}|T|\sigma^{(n+1)}\rangle$. As remarked already, T commutes with H of (1.4) provided the vertex weights satisfy

$$\omega_5\omega_6=(\omega_1-\omega_4)(\omega_2-\omega_3), \tag{7.1}$$

cf. Fig. 3. The asymmetry parameter ϵ is given by [15]

$$\epsilon=(\omega_2\omega_4-\omega_1\omega_3)/(\omega_2\omega_4+\omega_1\omega_3). \tag{7.2}$$

Since $\langle\sigma|T|\sigma'\rangle>0$ and $\langle\sigma|e^{-tH}|\sigma'\rangle>0$ in a sector a fixed magnetization and since $[T,H]=0$, T and e^{-tH} have the same maximal eigenvector by the Perron-Frobenius theorem. Thus

$$\sum_{\sigma'}\langle\sigma|T|\sigma'\rangle=h(m). \tag{7.3}$$

In approximation, $h(m)=e^{-\lambda m}$ for large N with suitable λ depending on the vertex weights. Therefore, in the parameter subspace defined by (7.1), the free energy of the six-vertex model depends linearly on m (i.e., on the vertical polarization).

Since in (7.3) h depends only on the conserved total spin,

$$p(\sigma\to\sigma')=h(m)^{-1}\langle\sigma|T|\sigma'\rangle \tag{7.4}$$

is a properly normalized transition probability. We identify then the y axis as time direction and interpret the six-vertex model as a Markov chain with transition probability (7.4). In the usual parlance, the condition (7.1) defines a codimension one manifold of *disorder points* for the six-vertex model [22,23]. The Markov chain (7.4) has a direct particle interpretation: Arrows pointing to the right and upwards form the world lines of particles, cf. Fig. 4. Each history has a probability given through the normalized Boltzmann weight of the six-vertex model.

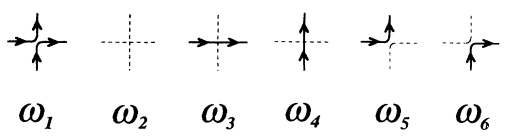

FIG. 3. Six-vertex weights.

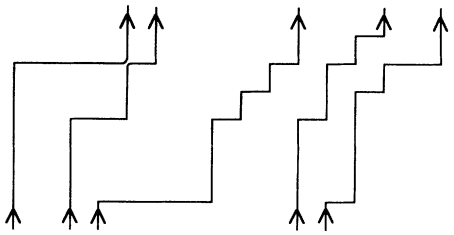

FIG. 4. A space-time history for the six-vertex model.

Our way of viewing the six-vertex model is strongly reminiscent of the noisy Burgers equation. In particular we may anticipate that the stationary correlations have the same large-scale asymptotics. To offer some credibility we must however identify the parameters appearing in the structure function (2.8). The prefactor is just the static susceptibility. Since spins are uncorrelated, it is again $(1-m^2)$. The remaining parameters are given through the steady-state current $j(m)$. The speed of propagation is $j'(m)$ and the dissipation time scale is set by $|j''(m)|^{2/3}$ [for the master equation (1.3) we have $j(m)=\epsilon(1-m^2)/2$ and hence $j'(m)=-\epsilon m$]. $j(m)$ is the average distance traveled by the particles in one time step. In order not to interrupt the main line of reasoning we defer this one-step calculation to Appendix B. The final result is

$$j(m)=2(1+m)/[a(1-m)+(1+m)], \tag{7.5}$$

with $a=\omega_1(\omega_2-\omega_3)/\omega_2(\omega_1-\omega_4)$.

If $\epsilon=0$, then $j(m)=1+m$. The Burgers equation with this systematic current is linear and up to the factor $\exp[i(1+m)kt]$ we recover the structure function (2.2). For $\epsilon>0$, $j(m)$ lies above and for $\epsilon<0$, $j(m)$ lies below the $\epsilon=0$ current. In particular $j''(m)\neq0$. Thus the structure function is of the form

$$S(k,t)=(1-m^2)e^{ikj'(m)t}g([j''(m)^2(1-m^2)]^{1/3}k|t|^{2/3}). \tag{7.6}$$

for small k and large t.

It is a small step to translate (7.6) to the six-vertex model. We consider the correlations in the vertical polarization between two points with relative distance r and slope $1/j'(m)$. (We again emphasize that in the equilibrium ensemble m is fixed.) Then, if $\epsilon=0$, these correlations decay as $r^{-1/2}$ whereas for $\epsilon\neq0$ they decay as $r^{-2/3}$. In the nearby cone, one sees the scaling function g of (7.6) and away from the special direction, the correlations decay exponentially.

The horizontal polarization in the six-vertex model corresponds to the current for the Burgers equation. Because of the conservation law $\partial/\partial tu+\partial/\partial xj=0$ the large-scale behavior of the stationary current-current correlations follows from (7.6). From this we conclude that to leading order the horizontal correlations decay just as the vertical ones.

As noted in [24–26] another way to set up the correspondence between the six-vertex model and the Burgers equation is by considering the diagonal transfer matrix. We choose the 45° diagonal as the time axis and space axis orthogonal to it. The particle (or spin) interpretation is as before. Note that now the dynamics is just the familiar parallel updating. We block the spatial lattice in cells of two sites each and update each cell independently. In the next time step the blocking is shifted by one lattice spacing. Clearly we have to assign the probability 1 to the vertex configurations ω_1 and ω_2. If $p(q)$ is the probability to jump to the right (left), then $\omega_3=q$ and $\omega_4=p$. In order to conserve probability we must set $\omega_5=1-q$ and $\omega_6=1-p$. Thus we have the two constraints

$$\omega_5=\omega_2-\omega_3\ ,\quad \omega_6-\omega_1-\omega_4\ , \tag{7.7}$$

which we recognize as a particular case of (7.1). If $p=q$, then $\epsilon=0$, as to be expected. In the stationary measure for the diagonal transfer matrix, spins are independent with period 2 [26]. Correlations propagate at most with speed 1 [27]. Thus a given spin is uncorrelated with its "spacelike" spins. This is consistent with our previous findings, which show that on the "light cone" there are no correlations.

VIII. CONCLUSIONS

As our main result we have shown that the dynamical scaling exponent for the stationary correlations of the noisy Burgers equation is $z=\frac{3}{2}$. Certainly, this comes as no surprise. Still it is gratifying to understand how the dynamical scaling exponent follows from an exact Bethe diagonalization of the generator in the master equation. (In a one-page announcement, such a result was claimed before [28].) We used here a particular discretization. It would be of interest to understand whether the Fokker-Planck equation associated to the noisy Burgers equation can also be handled by the Bethe ansatz. In fact, if one converts the noisy Burgers equation via the Cole-Hopf transformation to the (imaginary time) Schrödinger equation with a space-time random potential, then in the replica solution the δ-Bose gas appears [29]. As is well known, it is solved again by the Bethe ansatz. Possibly this solution translates back to the noisy Burgers equation.

As we demonstrated, the Bethe ansatz can be used to compute some properties of a spin chain with a nonsymmetric Hamiltonian. Of course, the connection to quantum mechanics is lost. But it is of interest to speculate how much of the structure obtained for quantum spin chains persists as we let the coupling constants wander off into the complex plane.

ACKNOWLEDGMENT

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APPENDIX A: SPECTRAL GAP FOR THE FERROMAGNETIC HEISENBERG HAMILTONIAN

We consider only the case $\epsilon=0$ and want to prove that for any function f orthogonal to the zero subspace

$$\langle f|H|f\rangle \geq 2N^{-2}\langle f|f\rangle\ . \tag{A1}$$

For any two sites we introduce the exchange operator $T_{ij}=(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+1)/2$ and note that

$$T_{ij}=T_{ii+1}T_{i+1i+2}\cdots T_{j-1j}T_{j-2j-1}\cdots T_{ii+1}\ . \tag{A2}$$

By telescoping we have then

$$\begin{aligned}T_{ij}-1=&(T_{ii+1}-1)T_{i+1i+2}\cdots\\&+(T_{i+1i+2}-1)T_{i+2i+3}\cdots+\cdots\\&+(T_{ii+1}-1)\ .\end{aligned} \tag{A3}$$

We apply both sides to $|f\rangle$, square, and use the Schwarz inequality,

$$\langle f|(T_{ij}-1)^2|f\rangle \leq 2|i-j|\sum_{u=i}^{j-1}\langle f|(T_{uu+1}-1)^2|f\rangle\ . \tag{A4}$$

Finally, we sum over the ring employing the shortest distance between i and j,

$$\begin{aligned}\sum_{i\neq j=1}^{N}&\langle f|(T_{ij}-1)^2|f\rangle\\&\leq 2\left[\frac{N}{4}+1\right]^3\sum_{u=1}^{N}\langle f|(T_{uu+1}-1)^2|f\rangle\ .\end{aligned} \tag{A5}$$

Using spin operators (A5) becomes

$$\left\langle f\left|\left[\tfrac{1}{2}\sum_{j=1}^{N}\boldsymbol{\sigma}_j\right]^2-\frac{N}{2}\left[\frac{N}{2}+1\right]\right|f\right\rangle\leq\frac{(N+1)^3}{2}\langle f|H|f\rangle\ . \tag{A6}$$

We recognize on the left-hand side the square of the total spin operator. Since f is orthogonal to the subspace with total spin $j=N/2$, the left-hand side is bounded below by $N\langle f|f\rangle$. This proves (A1).

APPENDIX B: STEADY-STATE CURRENT FOR THE SIX-VERTEX MODEL

We follow the notation in [15,16] for the transfer matrix of the six-vertex model. The transfer matrix T can be written as an N-fold product of 2×2 matrices, whose basis vectors are the horizontal-arrow state, labeled by $|R\rangle$ and $|L\rangle$, in order to be distinguished from the vertical-arrow state $|\pm1\rangle$. Each matrix has entries consisting of spin operators referring to a single site only. For simplicity, we use left-left boundary conditions; since T is applied only once, this results in an error of order $1/N$ for the average current per site. The total current operator for the transition from configuration σ to configuration σ' is given by

$$J(\sigma,\sigma')=\sum_{j=1}^{N} j(\sigma'_j-\sigma_j)\ . \tag{B1}$$

It is simpler to use a grand canonical average. Therefore we fix the magnetization through an external field h. The average current per site is then given by

$$j_N=\frac{1}{NZ}\frac{\partial}{\partial\alpha}\sum_{\sigma}\sum_{\sigma'}\langle\sigma|T^{LL}|\sigma'\rangle \times\exp\left[h\sum_{j=1}^{N}\sigma_j+\alpha J(\sigma,\sigma')\right]\Bigg|_{\alpha=0}$$
$$=\frac{1}{NZ}\frac{\partial}{\partial\alpha}\langle\phi_1\otimes\cdots\otimes\phi_N|T^{LL}|\psi_1\otimes\cdots\otimes\psi_N\rangle\Bigg|_{\alpha=0}\ . \tag{B2}$$

Here Z is the normalization, i.e., the double sum for $\alpha=0$. The two-vectors ϕ_j and ψ_j are given by

$$\phi_j(\pm)=\exp[\pm(h-\alpha_j)]\ ,\quad \psi_j(\pm)=\exp[\pm\alpha j]\ . \tag{B3}$$

Since the states in (B2) as well as the transfer matrix are of product form, one can carry out the summation over σ and σ'. At site j this yields the matrix

$$A_j=\begin{pmatrix}\omega_1 e^{h}+\omega_3 e^{-h} & \omega_5 e^{-h}e^{2\alpha j}\\ \omega_6 e^{h}e^{-2\alpha j} & \omega_4 e^{h}+\omega_2 e^{-h}\end{pmatrix}\ . \tag{B4}$$

The current is then

$$j_N=\frac{2}{NZ}\frac{\partial}{\partial\alpha}\langle L|A_1\ldots A_n|L\rangle\Big|_{\alpha=0} =\frac{2}{N\langle L|A^N|L\rangle}\sum_{j=1}^{N} j\langle L|A^{j-1}BA^{N-j}|L\rangle \tag{B5}$$

with

$$A=\begin{pmatrix}\omega_1 e^{h}+\omega_3 e^{-h} & \omega_5 e^{-h}\\ \omega_6 e^{h} & \omega_4 e^{h}+\omega_2 e^{-h}\end{pmatrix},\quad B=\begin{pmatrix}0 & \omega_5 e^{-h}\\ -\omega_6 e^{h} & 0\end{pmatrix}\ . \tag{B6}$$

We diagonalize A and obtain the eigenvalues $\lambda_+=\omega_1 e^{h}+\omega_2 e^{-h}$ and $\lambda_-=\omega_3 e^{-h}+\omega_4 e^{h}$. Without loss of generality we assume that $\lambda_+>\lambda_-$. Since $\langle\lambda_+|B|\lambda_+\rangle=0$, the leading term for large N is

$$j_N\approx\frac{2\langle\lambda_+|B|\lambda_-\rangle\langle\lambda_-|L\rangle}{N\lambda_+^N\langle\lambda_+|L\rangle}\sum_{j=1}^{N} j\lambda_+^{j-1}\lambda_-^{N-j}\ . \tag{B7}$$

Equation (7.5) is obtained by carrying out the algebra and using

$$m=\frac{\partial\ln Z}{\partial h}=\frac{\omega_1 e^{h}-\omega_2 e^{-h}}{\omega_1 e^{h}+\omega_2 e^{-h}}\ . \tag{B8}$$

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