Gravity in one dimension: Stability of periodic orbits

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The failure of the one-dimensional gravitational system to relax to equilibrium on predicted time scales has raised questions concerning the ergodic properties of the dynamics. A failure to approach equilibrium could be caused by the segmentation of phase space into isolated regions from which the system cannot escape. In general, each region may have distinct ergodic properties. By numerically investigating the stability of two classes of periodic orbits for the N-body system, we have unequivocally demonstrated that stable regions in the phase space exist for $N \leq 10$. For populations $11 \leq N \leq 20$ we find numerical evidence for multiple, chaotic, invariant regions. Thus the failure of large systems (say, $N \geq 100$) to equilibrate may be a result of microscopic dynamical restrictions, rather than imposed macroscopic contraints.

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I. INTRODUCTION

During the past three decades, the one-dimensional (1D) self-gravitating system has been the subject of interesting theoretical predictions and extensive numerical simulations. The motivational factors for studying the one-dimensional system are numerous. Years ago Oort [1] and Camm [2] suggested that the system may be an appropriate model for the motion of stars in a direction normal to the disk of a highly flattened galaxy. Cuperman, Hartman, and Lecar [3] used the system to test conjectures concerning the mechanism for violent relaxation. Eldrige and Feix [4] have shown the relevance of this system to plasma physics. It has served as a source of insight into processes in gravitational systems. Perhaps its chief attraction is in the simplicity and accuracy with which it can be dynamically evolved.

Numerical simulations have been performed by a number of investigators in order to ascertain the relaxation time to equilibrium of the one-dimensional self-gravitating system. In one of the first numerical studies, Hohl [5] suggested that the system should relax on the order of $N^2 t_c$, where N is the system population and t_c is the characteristic time which is approximately the time for a member to make an oscillation of the system. Hohl's assertion was the accepted view referenced in the literature for nearly two decades. However, in the early 1980s Wright, Miller, and Stein (hereafter WMS) [6], using statistical tests based on the exact velocity and position equilibrium distribution functions derived by Rybicki [7], did not reach the same conclusion for the evolution of the initial states that they investigated. Their results indicated that one-dimensional systems do not even appear to approach relaxation after $2N^2 t_c$. A few years later, research by Luwel, Severne, and Rousseeuw (hereafter LSR) [8] suggested that, for a *specific class* of counterstreamed initial conditions in the $\mu(x,v)$ space and an initial virial ratio of 0.3, relaxation takes place within Nt_c. To test their conjecture, we studied the evolution of an assortment of initial states [9–12] including the conditions suggested by LSR. Relaxation was clearly not found for the majority of the cases [9]. Evidence against relaxation was found for the initial state suggested by LSR, but it was weaker. Using both a long-run ($4000t_c$) simulation of a single system [10,12] and an ensemble average of 500 systems [11] for $25t_c$ we concluded that the system in question enters a macrostate that mimics equilibrium and slowly drifts away from it.

Fundamental to relaxation is ergodicity. If a system is to experience relaxation then at a minimum it must explore its entire phase space; that is, as $t \to \infty$ the time averages of the dynamical quantities must converge to their equilibrium values. In earlier research [12], we found the existence of long-term weak correlations in position and velocity of the system members. The correlations appeared to last indefinitely with a lower bound $\approx 2000t_c$. If the system has strong ergodic properties and progresses towards equilibrium on a finite time scale, then the correlations must decay [13]. For small-N systems ($N \leq 10$) Froeschle and Scheidecker (hereafter FS) [14] studied the rate of divergence of nearby orbits and found no integrable orbits for $N > 5$. They conjectured that these systems are "ergodic." This was supported by Benettin, Froeschle, and Scheidecker (hereafter BFS) [15] who calculated Lyapunov characteristic numbers for $N \leq 10$ and demonstrated an increasing stochasticity with increasing N. The conclusions reached by FS and BFS were later supported by research performed by Wright and Miller (hereafter WM) [16]. They studied systems for $N < 10$ and found for $N > 4$ that relaxation seemed to occur in a time much greater than $N^2 t_c$ ($\approx 10^5 t_c$).

In the research performed by us and others to date, the point in phase space used to initiate a simulation was chosen "randomly," i.e., by sampling a specific distribution. FS used an algorithm devised by Henon to directly sample the microcanonical distribution. In contrast, our group, and later LSR, determined the initial particle coordinates and velocities by independently sampling specific single-particle distributions in the μ space. FS concluded that, for $N \geq 6$, the integrable region of the phase space was nonexistent or "too small" to be detected.

A central question then concerns the existence of a critical population, say N_c, above which the system is completely chaotic. In this context, chaotic means that the system dynamics is mixing and that the entire energy surface comprises a single ergodic component. A familiar example of such a system is Sinai's billiards [17]. If N_c does not exist, then the observed failure of the system to relax could be due to the existence of multiple invariant regions on the energy hypersurface in phase space. Although one of them may be dominant, it may be separated into weakly connected "lobes" by the boundaries of the other segments. The failure to relax would then result from the slow "Arnold" diffusion of a trajectory between the lobes [18].

In order to gain a better understanding of the geometry of the phase space of the one-dimensional system, we have constructed specific families of periodic orbits and examined their stability. These periodic orbits were perturbed, and we looked for the possibility of an empirical relationship between the population N and the size of the perturbation that leads to instability. Stability was determined by calculating the largest Lyapunov characteristic exponent of a trajectory. A strictly positive Lyapunov exponent indicates an unstable orbit. If the system is amenable to the methods of statistical physics, i.e., if it approaches equilibrium and correlation functions decay in time for almost every initial location on the energy hypersurface in phase space, then the space has a single connected ergodic component and every possible trajectory is unstable. If, on the other hand, the space is segmented into invariant components, stable or unstable, then it may not be a good candidate for this approach, and the meaning of equilibrium and finite memory will have to be defined in a different, more restricted, context.

In the following we examine two classes of periodic orbit which can be constructed analytically for every value of N. One class, which resembles a "breathing" mode for the system, is always found to be unstable. For populations $N \le 10$, the other class is stable and our results indicate that the size of the perturbation that produces instability decreases logarithmically with increasing population. However, for $N > 10$ this periodic orbit is no longer stable. In fact, a perturbation $\approx 10^{-8}\%$ leads to instability, whereas for $N = 10$ a perturbation of approximately 1% is required. Although it is tempting to conclude that the entire phase space becomes connected for $N > 10$, this is not the case. Our computations clearly indicate that, through $N = 20$ (the largest system considered) the Lyapunov exponent associated with each type of orbit is radically different, demonstrating that the phase space consists of at least two invariant components. Thus, at this juncture, we can state with confidence that $N_c > 20$. Complete details of our research follow.

II. DESCRIPTION OF THE MODEL

The one-dimensional self-gravitating system consists of N identical mass sheets, each of uniform mass density and infinite in extent in the (y,z) plane. The sheets move freely along the x axis and accelerate as a result of their mutual gravitational attraction. The ith sheet experiences a uniform gravitational field that provides a constant acceleration given by

$$A_i = (2\pi G/N)(N - 2i + 1) , \tag{1}$$

where N^{-1} is the mass of a sheet and G is the universal gravitational constant. When an encounter occurs between two sheets, they pass freely through each other. For simplicity the sheets are referred to as particles and treated as mass points. The energy of the system is expressed as

$$E = (1/2N) \sum_{i=1}^{N} v_i^2 + (2\pi G/N^2) \sum_{\substack{i,j \\ (i<j)}} |x_j - x_i| , \tag{2}$$

where v_i and x_i are the velocity and position of the ith particle, respectively.

The equilibrium velocity and position probability density functions have been developed by Rybicki [7] for this system. In the limit that $N \to \infty$ these functions are

$$\Theta(\eta) = \pi^{-1/2} \exp(-\eta^2) \quad \text{(velocity)} , \tag{3}$$

$$\rho(\xi) = (\tfrac{1}{2}) \operatorname{sech}^2 \xi \quad \text{(position)} , \tag{4}$$

where

$$\eta = (v/2)(3M/E)^{1/2} , \tag{5}$$

and

$$\xi = (3\pi G M^2 / 2E) x . \tag{6}$$

v, x, M, and E represent the velocity, position, total system mass, and total system energy, respectively.

A convenient dynamical characteristic time t_c has been employed in order to follow the chronological evolution of a one-dimensional self-gravitating system. Physically, this characteristic time represents the dynamical time required for a particle to traverse the system, and has been expressed by LSR [8] in terms of the maximum value of the equilibrium distribution function $\rho(\eta)$,

$$t_c = (GM\rho_{\max}/\pi)^{-1/2} , \tag{7}$$

III. LYAPUNOV CHARACTERISTIC NUMBERS

In a stochastic region of phase space, nearby orbits of dynamical systems diverge exponentially [19]. This exponential divergence may be determined by calculating the largest Lyapunov characteristic number [a general discussion is provided by Lichtenberg and Lieberman [20] (hereafter LL)]. This procedure has been described and used extensively by Contopoulos and Barbanis [21], Contopoulos, Galgani, and Giorgilli [22], and BFS [15]. The largest Lyapunov characteristic number is defined as

$$L = \lim_{\substack{d_0 \to 0 \\ t \to \infty}} [\ln(d/d_0)/t] , \tag{8}$$

where d and d_0 are the separations between two nearby orbits at times t and 0. For stable orbits the Lyapunov characteristic number is zero; it is strictly positive for unstable orbits. In stochastic regions of the phase space, the use of Eq. (8) to determine the Lyapunov characteristic number will eventually lead to an overflow in numerical simulations. This can be avoided if we periodically reset the location of the perturbed orbit so that its separation from the reference orbit is, once again, d_0. The Lyapunov characteristic number is then given by [20]

$$L=\lim_{n\to\infty}(1/n\Delta T)\sum_{j=1}^{n}[\ln(d_j/d_0)] \tag{9}$$

(where ΔT is the time interval between rescalings and $n=t/\Delta T$). In practice, the value of L obtained is insensitive to the choice of ΔT over a large range.

The separation in phase space of a particular reference orbit and some nearby orbit is given by

$$d_0=k\left\{\sum_{i=1}^{N}[(X_{0ir}-x_{0ip})^2+(V_{0ir}-v_{0ip})^2]\right\}^{1/2} . \tag{10}$$

Here, x_{0ir} and v_{0ir} refer to the initial position and velocity, respectively, of the ith particle of the reference orbit. Similarly x_{0ip} and v_{0ip} represent the initial perturbed orbit. The constant k is dependent on N. After some time ΔT the two orbits are separated by

$$d_{j=1}=k\left\{\sum_{i=1}^{N}[(X_{ir}-x_{ip})^2+(V_{ir}-v_{ip})^2]\right\}^{1/2} . \tag{11}$$

The perturbed orbit is then rescaled according to

$$\tilde{x}_{ip}=x_{ir}+(d_0/d_{j=1})(x_{ip}-x_{ir}) , \tag{12}$$

$$\tilde{v}_{ip}=v_{ir}+(d_0/d_{j=1})(v_{ip}-v_{ir}) , \tag{13}$$

where $\tilde{x}_{ip}$ and $\tilde{v}_{ip}$, respectively, represent the rescaled position and velocity of the ith particle. The evolution of the orbits is then resumed and rescaled after each interval of ΔT. The reference orbit is considered to be stable if the Lyapunov characteristic number in Eq. (9) converges to zero and unstable if it converges to some positive number.

IV. SELECTED PERIODIC ORBITS

As mentioned in the introduction, previous investigators have studied the evolution of one-dimensional (1D) systems where the initial conditions were randomly generated within some macroscopic constraints [3–6, 8–12]. For a particular population N, if only one orbit is stable then the system is not chaotic, regardless of how small the stable region is in the phase space. Since mechanical systems often have periodic modes of vibration that are stable, we wondered whether this is also true of 1D gravitational systems. We have selected two particular periodic orbits for study which are shown in Figs. 1(a) and 1(b). In one of the orbits (breathing mode), all of the system particles remain separated (do not cross) until they all simultaneously encounter each other at the origin. In the alternate mode (mode 1), encounters occur between adjacent pairs only. Three or more particles cannot occupy

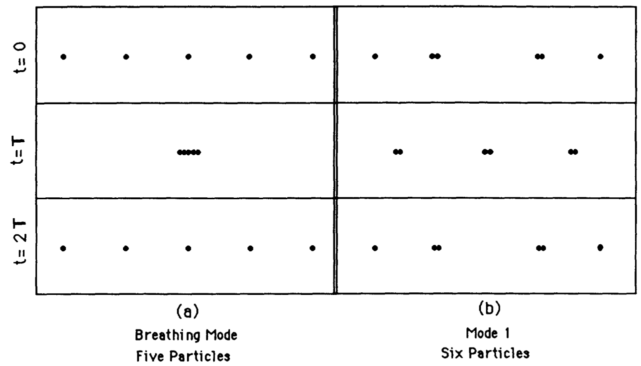

FIG. 1. Illustration of the collision sequence for the breathing mode and mode 1.

the same position. The motivation for this choice came from our earlier dynamical study of a system of three particles [16,23,24]. There it was found that orbits characterized by simultaneous triple encounters are divergent, whereas those which contained only pairwise crossings are stable.

Figure 1(a) illustrates what we refer to as the "breathing mode" for a five-particle system. Initially, all particles start from rest at appropriate locations on the x axis. After some time T the particles simultaneously converge at the origin. The particles pass through each other and after an additional time increment T the system, except for particle labels, returns to its initial configuration. Similarly, the periodic orbit, which we refer to as "mode 1," is shown in Fig. 1(b) for a six-particle system. Initially particles 1 and 6 have zero initial velocity, particles 2–5 have appropriate nonzero initial velocities, and the two "internal" pairs are crossing. After a specific time all three adjacent pairs cross simultaneously. The system then returns to the initial configuration. The initial conditions necessary for each mode as a function of N were determined analytically using basic kinematics.

It is difficult to conceptualize the orbits in a $(2N)$-dimensional phase space. However, if we project the phase-space orbit onto a two-dimensional space, $\mu(x,v)$, we obtain a comprehensible illustration of the periodic orbits. Figure 2(a) shows the $\mu(x,v)$-space trajectory of a five-particle system in the breathing mode. The trajectory is essentially two pairs of parabolas with a point at the

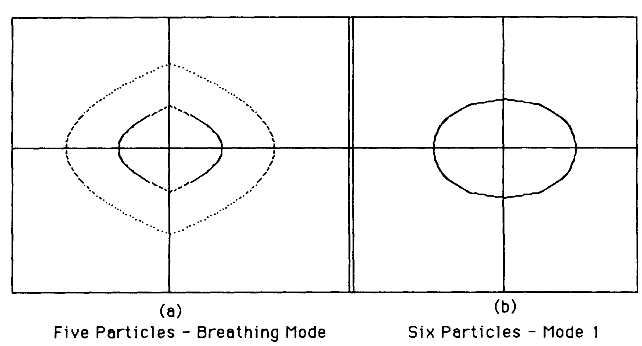

FIG. 2. $\mu(x,v,)$-space trajectories for the breathing mode and mode 1.

origin. The point at the origin results from the central particle which has a net force of zero acting on it and initial conditions of $\mu(0,0)$. Increasing the number of particles will increase the number of "pairs of parabolas." An odd number of particles will always have a stationary point at the origin. In real time, the positions of the particles trace out a rotating line in μ space. The trajectories of the particles only cross when the line is parallel to the v axis ("vertical"), when they all occupy the same position (the origin $x=0$).

Figure 2(b) depicts the trajectory of a six-particle system in mode 1. The orbit is a closed, approximately oval-shaped curve consisting of a set of parabolic segments. Increasing the number of particles will smooth the slope discontinuities at the crossing points. For an odd number of particles, the central particle, which experiences zero acceleration, will follow a straight-line path of constant velocity in $\mu(x,v)$ space. In real time, adjacent pairs of particles cross at the same time.

V. SCALING, SIMULATION, AND PERTURBATION

All initial positions and velocities were scaled according to Eqs. (5) and (6) with $M=1$ and $2\pi G=1$. This resulted in a characteristic time of 2π and forced a total dimensionless energy of three-fourths for all systems. This is important since all systems for a given N are then on the same energy surface in phase space. The evolution of each system was simulated using an exact code with updating occurring at each encounter. Most simulations were followed through $50\,000t_c$ and the nearby orbits used for calculating the Lyapunov characteristic numbers were rescaled according to Eqs. (12) and (13) every $0.1t_c$. As a check, some orbits were also rescaled every $0.01t_c$, but as expected there was no significant difference in the Lyapunov characteristic numbers. All calculations were performed in double precision (16 significant figures) on a VAX 6310 computer, and energy was conserved to better than one part in 10^{10}.

All perturbations to the periodic orbits were constructed to lie in a hypersphere of radius r around an initial point in phase space. For a particular perturbation $P>0$, the perturbed positions and velocities were selected as follows:

$$x_{i\,\text{perturbed}}=x_{i\,\text{periodic}}(1\pm P) \tag{14}$$

and

$$v_{i\,\text{perturbed}}=v_{i\,\text{periodic}}(1\pm P)\ . \tag{15}$$

These perturbed velocities and positions are then uniformly scaled so that the total energy of the perturbed system is $\frac{3}{4}$. The "radius" of the hypersphere in phase space is then given by

$$r=k\left\{\sum_{i=1}^{N}[(X_{i\,\text{periodic}}-x_{ip\,\text{scaled}})^2+(V_{i\,\text{periodic}}-v_{ip\,\text{scaled}})^2]\right\}^{1/2}\ , \tag{16}$$

where the subscript "ip scaled" represents the ith particle that has been perturbed and scaled. The relative size of the perturbation S, using the scaled system, is then

$$S=r\left\{k^2\sum_{i=1}^{N}[(X_{i\,\text{periodic}})^2+(V_{i\,\text{periodic}})^2]\right\}^{-1/2}\ . \tag{17}$$

VI. DATA AND RESULTS

For mode 1, two methods of perturbation were used. In a "compression" perturbation, only positive signs were used in Eqs. (14) and (15). Thus, when the perturbed positions and velocities were scaled to a total energy of $\frac{3}{4}$, the net effect was to compress the positions closer together and increase the velocities. For an "alternating" perturbation, the signs in each equation were simply alternated. One sign was not alternated in order to avoid symmetry.

The central idea was to find the maximum (critical) perturbation for a given orbit that would remain in the same stable segment of the phase space. Essentially a bisection procedure was carried out to obtain the result. Initial large and small perturbations of a mode-1 orbit were selected. These perturbed orbits were allowed to evolve until the Lyapunov characteristic number for each orbit converged. If the Lyapunov characteristic number (LCN) converged to zero for the smaller perturbation and to some positive number for the larger perturbation, then the average of the perturbations was used as input for a new larger perturbation. The process was repeated until we had two perturbations, fairly close together, that sandwiched the critical perturbation between them. This critical perturbation was not determined exactly, but was at most off by 1%. Figure 3 illustrates this process for a system of six particles with a compressed perturbation of the mode-1 orbit. For a perturbation of 9.59% the orbit remains stable, while a perturbation of 10.38% leads to instability. The critical perturbation is estimated to be 10.0% which is the average of the two.

Critical perturbations for both the compression and alternating modes are summarized in Table I. The data for alternating perturbation suggest that the critical perturbation decreases logarithmically with increasing N (see Fig. 4). However, this empirical relationship is valid only through $N=10$. Evidently, $N=11$ represents a "critical

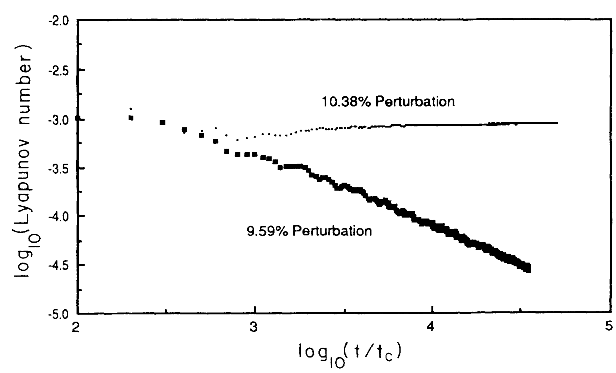

FIG. 3. Critical perturbation for six particles experiencing a compression perturbation in mode 1.

TABLE I. Critical perturbation as a function of system population for mode-1 periodic orbits using both compression and alternating perturbations.

System	Compression perturbation (%)			Alternating perturbation (%)		
N	Zero LCN	Positive LCN	Critical perturbation	Zero LCN	Positive LCN	Critical perturbation
3	79.29	79.36	79.3	136.0	None found	
4	30.27	30.71	30.5	27.76	29.75	28.8
5	16.33	17.22	16.8	17.44	19.07	18.3
6	9.59	10.38	10.0	9.74	11.87	10.8
7	6.26	7.4	6.8	5.60	7.21	6.4
8	Not taken			2.39	3.97	3.2
9	Not taken			0.80	2.42	1.6
10	3.22	3.66	3.4	0.40	1.61	1.0
11	None found	8×10^{-8}		None found	8×10^{-8}	
12	Not taken			None found	8×10^{-8}	
14	Not taken			None found	8×10^{-8}	
20	Not taken			None found	8×10^{-8}	

dimensionality." A perturbation $\approx 8\times10^{-8}\%$ resulted in a positive Lyapunov characteristic number. This perturbation also gave positive Lyapunov characteristic numbers for systems containing 11, 12, 14, and 20 particles, which are shown in Fig. 5.

Similar results were obtained for the compression perturbation. The critical dimensionality was $N=11$; however, the logarithmic relationship, although evident, was not quite as perfect for $N\leq 10$. Both the alternating and compression perturbations of $8\times10^{-8}\%$ converged to the same Lyapunov characteristic number of 0.031 for $N=11$. It is also of interest to note that no positive Lyapunov characteristic number was found for three particles that suffered an alternative perturbation as high as 136%. Apparently, this stable orbit occupies a very large region of the phase space.

For the breathing mode we examined system populations of $N=3$, 5, and 7 particles. The Lyapunov characteristic numbers for the exact orbits are 0.293, 0.420, and 0.475, respectively, which indicates that these orbits are very unstable. Alternate perturbations from the exact orbit resulted in smaller, positive Lyapunov characteristic numbers. For $N=5$ three perturbed orbits of $9.5\times10^{-8}\%$, 0.96%, and 31.6% were examined. These orbits converged to approximately the same Lyapunov characteristic number of 0.07. The exact results are 0.070, 0.072, and 0.067, respectively. The 31.6% perturbation was the maximum possible perturbation without changing the initial ordering of the particles. Similarly, for $N=7$ perturbations of $8.8\times10^{-8}\%$, 0.88% and 16.0% resulted in roughly the same Lyapunov characteristic number. These are 0.097, 0.076, and 0.081, respectively. Again, a perturbation greater than 16.0% destroys the initial ordering of the particles. In the case of three particles, convergence to a common Lyapunov characteristic number was not found for all perturbations examined. Lyapunov characteristic numbers of 0.029, 0.027, and 0.000 49 were obtained from respective perturbations of $5.8\times10^{-8}\%$, 0.58%, and 57.2%. A perturbation greater than 57.2% results in a reordering of the initial system.

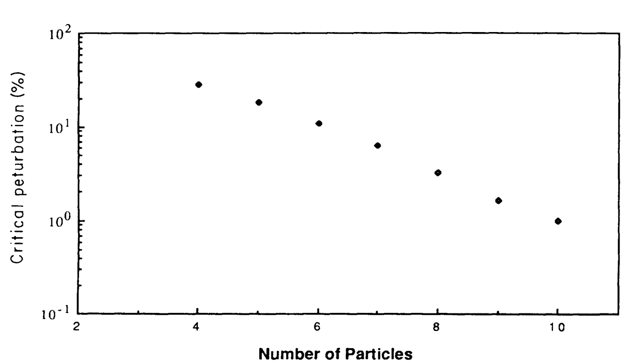

FIG. 4. Critical perturbation vs system population for alternating perturbations of mode 1.

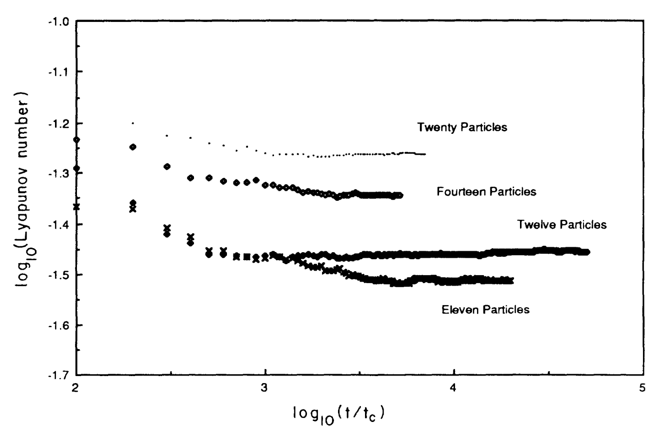

FIG. 5. Lyapunov characteristic numbers for $8\times10^{-8}\%$ alternate perturbations of mode 1 for $N=11$, 12, 14, 20.

VII. CONCLUSIONS

The combined results of research by FS [14], BFS [15], and WM [16] suggest that 1D systems with a population $6 \leq N \leq 10$ are ergodic and could reach the exact equilibrium distributions derived by Rybicki [7] after an evolution time $\approx 10^5 t_c$. However, these results are based on orbits that have been randomly initialized within some macroscopic constraints. In this research we have shown that in the neighborhood of a specific, periodic orbit (mode 1) stability is maintained for system populations of $3 \leq N \leq 10$. Evidently for $N \leq 10$, 1D systems are not ergodic. The size of the stable region in phase space appears to decrease logarithmically as a function of N and must be very small for $N=8$, 9, and 10. It should be mentioned that FS [14] did not rule out the *possibility* of very small stable regions that could not be detected by their method.

A critical dimensionality for the region of phase space surrounding the mode-1 periodic orbit is apparently reached at $N=11$. Any perturbation, regardless of how small, results in an unstable trajectory. Mode 1 for ten particles has a critical perturbation of about 1% while a perturbation of $8\times 10^{-8}\%$ leads to instability for 11 particles. This rapid change to stochastic behavior is surprising. In 1984, WM [16] studied the relationship between the encounter sequence and rate of divergence of proximally initialized pairs of trajectories. They found that trajectories which contain nearly multiple encounters diverge rapidly in phase space and the proportion of these encounters increases with N. Although there are no multiple encounters in mode 1, there are pairs of simultaneous encounters. The number of these pairs increases with an increase in N. As suggested by WM, the sudden onset of the "ergodic type" of behavior may be the result of near multiple encounters produced by the perturbations. It may be possible to test this multiple-encounter hypothesis by constructing a periodic orbit that contains no more than a single encounter (crossing) at any given time. If the multiple encounters are the source of stochasticity, then such a system might be stable for large N. We are presently examining this possibility. However, we can in no way infer from this research that 1D systems with $N>10$ are ergodic. Although the orbits in the vicinity of the mode-1 periodic orbit are unstable for $N>10$, the largest Lyapunov exponent obtained by perturbing this orbit is an order of magnitude less than the Lyapunov exponent computed for the breathing mode. Thus, the phase space remains, segmented for $N>10$ as well. For a few of these mode-1 orbits we have recently studied the effect of increasing the perturbation on the value of the Lyapunov exponent. We find that it is possible to enter a region with a Lyapunov exponent that is different from that associated with either periodic orbit. Thus, the general conclusion that the phase space remains segmented through $N=20$ is inescapable.

All orbits for the breathing mode were found to be unstable. This supports the idea of near multiple encounters as a source of ergodic behavior. Multiple encounters are fundamental to the breathing mode. Convergence to different Lyapunov characteristic numbers, depending on the size of the perturbation, for the case of three particles, lets us conclude that the phase space is segmented. An apparent convergence to common (but different) Lyapunov characteristic numbers for perturbations to systems of five and seven particles in the breathing mode suggests that all the perturbed orbits lie in the same stochastic region for this mode with $N \geq 5$. However, complete validation of this conclusion will require longer simulations with larger systems and of greater duration to ensure convergence to a common Lyapunov characteristic number for each case.

A recurring theme of this work is the identification of the physical source of chaos in the system. Froeshle and Sheidecker suggested that the encounters between the mass sheets are responsible for "ergodic" behavior. As discussed above, the work of Wright and Miller suggested a refinement of this idea, i.e., that only those trajectories associated with multiple encounters are unstable. The work reported here supports this observation but, in addition, shows that for $N>10$ trajectories with simultaneous pairwise encounters are also unstable.

In an interesting recent work, Kandrup [25] has used projection techniques introduced by Gurzadyan and Savvidy [26] to study the ergodic properties of the system. Besides utilizing the system dynamics directly, this approach introduces additional assumptions concerning the statistical distribution of likely trajectories in the phase space. In contrast with all other investigators, Kandrup concludes that the crossings provide a stabilizing influence on the dynamical evolution of the system, while the system motion between crossings is destabilizing. This seems especially surprising because, between encounters, the phase point is simply undergoing constant acceleration in the $2N$-dimensional phase space. For this simple force law it is easy to show that nearby orbits do not *exponentially* diverge in time, but rather separate at a rate less than t^1. Consequently, the Lyapunov exponent associated with a constant acceleration vanishes identically. Kandrup's analysis should not be dismissed because of this apparent discrepancy, as it strongly suggests that the phase space is predominantly mixing for $N>6$, a result which is consistent with the work of Henon's group and our earlier work. Rather, it symbolizes the fact that there are important questions concerning this apparently simple system which remain unanswered.

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[1] J. H. Oort, Bull. Astron. Inst. Neth. **6**, 289 (1932).
[2] G. Camm, Mon. Not. R. Astron. Soc. **110**, 305 (1950).
[3] S. Cuperman, A. Hartman, and M. Lecar, Astrophys. Space Sci. **13**, 397 (1971).
[4] O. C. Eldrige and M. Feix, Phys. Fluids **6**, 398 (1963).
[5] F. Hohl, Ph.D. dissertation, College of William and Mary, 1967.
[6] H. L. Wright, B. N. Miller, and W. E. Stein, Astrophys. Space Sci. **84**, 421 (1982).
[7] G. Rybicki, Astrophys. Space Sci. **14**, 56 (1971).
[8] M. Luwel, G. Severne, and P. J. Rousseeuw, Astrophys. Space Sci. **100**, 261 (1984).
[9] C. J. Reidl, Jr. and B. N. Miller, Astrophys. J. **318**, 248 (1987).
[10] C. J. Reidl, Jr. and B. N. Miller, Astrophys. J. **332**, 616 (1988).
[11] C. J. Reidl, Jr. and B. N. Miller, Astrophys. J. **371**, 260 (1991).
[12] B. N. Miller and C. J. Reidl, Jr., Astrophys. J. **348**, 203 (1990).
[13] V. I. Arnold and A. Avez, *Ergodic Properties of Classical Mechanics* (Benjamin, New York, 1968).
[14] C. Froeschle and J. Scheidecker, Phys. Rev. A **12**, 2137 (1975).
[15] G. Benettin, C. Froeschle and J. P. Scheidecker, Phys. Rev. A **19**, 2454 (1979).
[16] H. L. Wright and B. N. Miller, Phys. Rev. A **29**, 1411 (1984).
[17] Y. G. Sinai, *Introduction to Ergodic Theory* (Princeton University Press, Princeton, 1977).
[18] H. G. Helleman, in *Fundamental Problems in Statistical Mechanics,* edited by E. Cohen (North-Holland, Amsterdam, 1980), Vol. 5, p. 165.
[19] M. Henon and C. Heiles, Astron. J. **69**, 73 (1964).
[20] A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).
[21] G. Contopoulos and B. Barbanis, Astron. Astrophys. **222**, 329 (1989).
[22] G. Contopoulos, L. Galgani, and A. Giorgilli, Phys. Rev. A **18**, 1183 (1978).
[23] H. Lehtihet and B. N. Miller, Physica D **21**, 92 (1986).
[24] A. Matulich and B. N. Miller, Celestial Mech. **39**, 191 (1986).
[25] H. Kandrup, Phys. Rev. A **40**, 7265 (1989).
[26] V. G. Gurzadyan and G. K. Savvidy, Dokl. Akad. Nauk. SSSR **277**, 69 (1984) [Sov. Phys. Dokl. **29**, 520 (1984)]; Astron. Astrophys. **160**, 203 (1986).