COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review**. *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on "Numerical method for colored-noise generation and its application to a bistable system"

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It is argued that a recently proposed algorithm [K.Y.R. Billah and M. Shinozuka, Phys. Rev. A **42**, 7492 (1990)] for the integration of stochastic differential equations in the presence of correlated noise does not introduce a substantial improvement over existing schemes. We point out that although this scheme could be very interesting for noise characterized by particular correlation functions, for colored noise the present scheme is sensibly slower than existing algorithms. Some apparent discrepancies between the result of simulations carried out with the new algorithm and previous numerical work are explained.

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In a recent paper [1], a numerical algorithm for the solution of the prototype stochastic differential equation

$$x=f(x)+g(x)\xi(t) \tag{1}$$

has been proposed, where ξ is a zero-centered Gaussian noise of given intensity and correlation. The algorithm proposed, a direct implementation of the central-limit theorem, can be straightforwardly applied to a variety of correlation functions (hence, to different spectral densities) of the noise, and it has the putative advantage of overcoming some problems with the random number generator used for other algorithms. The authors apply their algorithm to study the escape rate in the bistable potential

$$f(x)=x-x^3\ ,\quad g(x)=1 \tag{2}$$

and, in presence of colored noise,

$$\langle \xi(t)\xi(t')\rangle = D/\tau \exp(-|t-t'|/\tau)\ , \tag{3}$$

and the results of their simulations are compared to some theoretical approaches.

Let us mention that there has been some controversy as to which is the best theoretical approach to describe the escape rate, although now the issue seems to have been settled [2]. Here, we are concerned with the comparison [1] between existing algorithms [3,4] and the algorithm proposed in Ref. [1]. The basic idea, in other approaches, is to write an auxiliary equation for ξ in the form

$$\dot{\xi}=-\xi/\tau+\sqrt{2D}\ /\tau\, h(t)\ , \tag{4}$$

where the random term $h(t)$ is a Gaussian white noise with a standard deviation of 1.

The algorithm of Ref. [1], on the other hand, generates the noise according to

$$\xi(t)=\sqrt{2}\sum_{n=1}^{N}[S(\omega_n)\Delta\omega]^{1/2}\cos(\omega_n t+\Phi_n)\ , \tag{5}$$

where $\omega_n=n\Delta\omega$, with $n=1,2,3,\ldots,N$ and $\Delta\omega=\omega_{\max}/N$. Φ_n is a random-phase term, uniformly distributed between 0 and 2π, $S(\omega)$ represents the power spectrum of the correlated noise

$$[S(\omega)=(D/\pi)/(1+\omega^2\tau^2)]\ ,$$

while $\omega_{\max}$ is the upper cutoff of the noise spectrum. The advantage of this method with respect to the solution of Eq. (4) would be the direct conversion of a *uniform* random set of variables (Φ_n) in a Gaussian nonwhite noise, thus avoiding possible problems related to poor efficiency of the Gaussian number generator used in Eq. (4). In practice, however, this can be avoided exherting the necessary care. On the other hand, the noise generated by Eq. (5) has a repetition period $T=2\pi/\Delta\omega$. This problem, along with the constraint $\omega_{\max}\tau\gg 1$, forces one to

use very large values of $N(\approx 10^5-10^6)$ with a considerable increase of the CPU time needed for the calculation of Eq. (5). In practice, $\xi(t)$ in Eq. (5) is generated via a fast-Fourier-transform (FFT) technique. It follows that the time taken to calculate one random $\xi(t)$ will vary as $\alpha \ln_2(M)+\beta$, where M is the number of random terms one wants to generate, α is the intrinsic time taken by the FFT, and β is the time necessary to generate one uniform random number Φ_n, and evaluate $\exp(i\Phi_n)$ in the FFT. On the other hand, the time necessary to generate $\xi(t)$ in Eq. (4) varies only as a constant (slightly larger than 1 [3], generally equivalent to two elementary multiplications plus one elementary addition) multiplied by the time necessary to generate a Gaussian random term, that we will quantify as $\gamma\beta$. For both methods, we have discarded overheads, although it is fair to say that on most computers for the algorithm of Eq. (5) this overhead is of the order of the time taken to generate a Gaussian random number from a uniform deviate via a Box-Müller algorithm. In fact, the evaluation of each of the M weights needed for the FFT is very expensive (at least three multiplications, one addition, one division, and one square root), and the necessity of using fairly large M's may induce considerable swapping, thus further increasing the CPU time. Typical figures on a CDC 7600 are $\alpha=2\mu$sec; $\beta=2.2$ μsec; $\gamma=2$. In the case of a bistable potential, with $D=0.1$ and $\tau=1.0$ (mean first-passage time $\sim$ 250), for reasonable integration time steps ($dh=0.1$) and for 100 averages, one needs approximately 2.5×10^5 random terms; in these conditions, the algorithm of Eq. (4) is expected to run about 11 times faster (on a CDC 7600) than the algorithm of Eq. (5). The comparison is even more favorable if more averages or a smaller integration time step are considered. The two algorithms are expected to have the same speed only if $M<4$, having disregarded overheads, although these figures may well depend on the actual computer used. For more complex spectral densities, also, it is possible that the algorithm of Eq. (5) could become more advantageous due to the necessary complication one must introduce for the (corresponding) Eq. (4).

Comparison of the numerical results [2a] obtained using existing algorithms and the algorithm of Ref. [1] seems to suggest a considerable difference between the results of the two methods. The authors of Ref. [1] do not explicitly comment on this discrepancy, although it seems clear from their previous discussion that this effect should be attributed to either the poor quality of the Gaussian number generator used, or a possible flaw in other algorithms.

In fact, this *apparent* discrepancy is due to an incorrect comparison of two *different* quantities. In their Fig. (3), authors of Ref. [1] plot the average time (divided by 2) needed to reach one minimum of the potential ($x=1$) starting from the other minimum ($x=-1$), as a function of the correlation time τ of the Gaussian noise. This quantity $T_{\rm bot}/2$ is then (erroneously) compared to the average transition time from $x=-1$ to $x=0$ ($T_{\rm top}$) from Ref. [2(a)].

These two quantities ($T_{\rm top}$ and $T_{\rm bot}/2$) are, in general, different, and the difference is far from artificial. A separatrix between the two attractors $\{x=\pm1\}$, not coincident with $\{x=0,\ y=any\}$, does survive in the limit $D\to0$ with τ being finite. Ramirez-Piscina *et al.* [5] have also clearly shown that the relation $T_{\rm top}=T_{\rm bot}/2$ is valid only in the limit $\tau\ll1$ (white noise); it is intuitive that in the opposite limit of strongly colored noise ($\tau\gg1$) $T_{\rm top}=T_{\rm bot}$. The transition between these two opposite limits is shown in Fig. 1. When the relevant results of the two methods are compared, the apparent discrepancy completely vanishes. Parenthetically, it is probably worth mentioning that the good agreement of the numerical results of Ref. [1] with the theoretical predictions of Masoliver, West, and Lindenberg [6] is coincidental; as a matter of fact, the theoretical approaches mentioned in Ref. [1] only make sense in the limit of vanishing intensity of the noise [2(b)], whereas $D=0.1$ is already far too large (the *qualitative*, as opposed to quantitative, behavior of the activation energy as function of τ is completely different from the theoretical one). Moreover, the analysis of Fox [7] has shown that the theoretical results of Ref. [6] are not correct, since they have been obtained through a wrong expansion of the relevant integral equation.

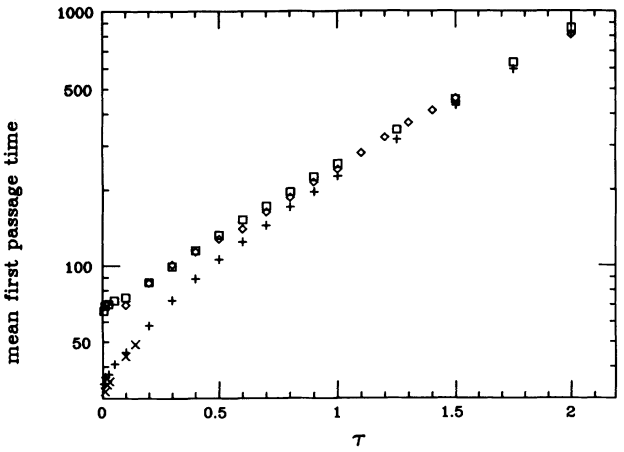

FIG. 1. Comparison between the mean first-passage times obtained with the different algorithms vs τ, noise correlation time. Squares, $T_{\rm bot}$ from Refs. [3] and [4]; diamond, $T_{\rm bot}$ from Ref. [1]; +, $T_{\rm top}$ from Refs. [3] and [4]; $\times$, $T_{\rm top}$ from Ref. [1].

Finally, as far as the error on the mean first-passage

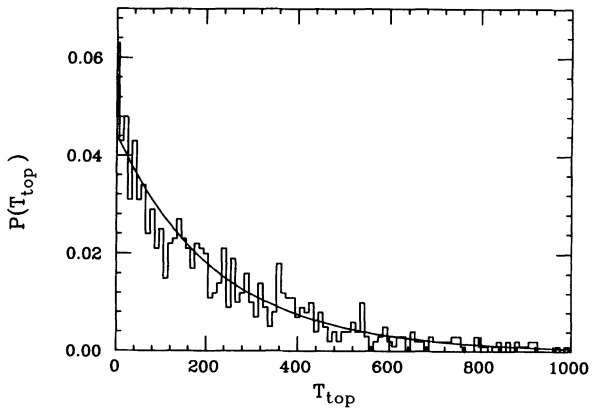

FIG. 2. Comparison between mean-first passage time distribution of $T_{\rm top}$ (histogram) and best-fitting exponential (solid line) for $D=0.1$ and $\tau=1.0$.

time is concerned, convergence to the "real" value can only be slow: In fact, the first-passage time-probability distribution seems to be described with a good approximation by an exponential function (not to mention theoretical evidence of this), for which the standard deviation coincides with the average. As such, the slow convergence must be attributed to the intrinsic features of the probability distribution, rather than to poor ergodicity. Figure 2 shows that, indeed, the mean first-passage times distribution for T_{top} varies as an exponential function. Furthermore, straightforward Monte Carlo simulation of an exponential function distributed with the same average and with an equal number of events, binned as the distribution shown, yields virtually the same χ^2 value (unnormalized 2.982 for the real distribution, 2.978 for the Monte Carlo simulation). It is difficult to understand how one could design an algorithm to overcome this intrinsic limit imposed by the statistics.

[1] K. Y. R. Billah and M. Shinozuka, Phys. Rev. A **42**, 7492 (1990).

[2] (a) R. Mannella and V. Palleschi, Phys. Lett. **129A**, 317 (1988); (b) R. Mannella, V. Palleschi, and P. Grigolini, Phys. Rev. A **42**, 5946 (1990).

[3] R. Mannella and V. Palleschi, Phys. Rev. A **40**, 3381 (1989).

[4] (a) M. Sancho, M. San Miguel, S. L. Katz, and J. Gunton, Phys. Rev. A **26**, 1589 (1982); (b) R. F. Fox, I. R. Gatland, R. Roy, and G. Vemuri, *ibid.* **38**, 5938 (1988); (c) R. Mannella, in *Noise in Nonlinear Dynamical Systems,* edited by F. Moss and P. V. E. McClintock (Cambridge University Press, Cambridge, England, 1989), Chap. 7, Vol. III.

[5] L. Ramirez-Piscina, J. M. Sancho, F. J. de la Rubia, K. Lindenberg, and G. P. Tsironis, Phys. Rev. A **40**, 2120 (1989).

[6] J. Masoliver, B. J. West, and K. Lindenberg, Phys. Rev. A **35**, 3086 (1987).

[7] R. F. Fox, Phys. Rev. A **37**, 911 (1988).