Dielectric permeability of quasi-two-dimensional one-component plasmas

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The classical method of moments is applied to determine the dispersion law of the quasi-two-dimensional Coulomb-system soft collective mode. An interpolation formula is found for the longitudinal dielectric permeability, satisfying all known exact relations and sum rules. The sum rules (the inverse-dielectric-permeability imaginary-part frequency moments) are calculated, taking into account the magnetic interaction between electrons.

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INTRODUCTION

An increasing interest in the investigation of quasi-two-dimensional (2D) electron systems has been observable in recent years. Quasi-two-dimensional here means that the electrons have quantized energy levels along one dimension, but are free to move in two dimensions. These systems are not indeed two-dimensional because the electromagnetic field of interacting electrons is not confined to a plane but spills out into the third dimension. In addition, the one-electron wave function has a finite extent in the third direction.

The best-known examples of 2D electron systems are electrons trapped on the liquid-helium surface or electrons confined in the vicinity of a junction between a semiconductor and insulators [in a metal-oxide-semiconductor field-effect transistor (MOSFET) structure] or between layers of different semiconductors (in heterojunctions). In these systems electrons are confined near an interface by an electrostatic field. Thus the electrons have quantized energy levels E_i $(i=0,1,\ldots)$ for the motion along the z direction transverse to the interface.

In inversion layers, the distance between the energy levels is about 100 K, whereas on the liquid-helium surface it is about 10 K. If temperature T and the Fermi energy $E_F=\pi n_S\hbar^2/m$ (n_S being the surface density of electrons, m the electron masses) are much smaller then the distance between the energy levels E_0 and E_1, the electrons form a quasi-two-dimensional electron gas with a fixed energy E_0 and wave function $\varphi(z)$ for the motion along the z direction.

It can be shown that within the one-electron approximation the wave function $\varphi(z)$ can be written as [1]

$$\varphi(z)=\begin{cases}2b^{3/2}z\exp(-bz), & z\geq 0\\ 0, & z<0\,,\end{cases} \tag{1}$$

where for the 2D system on the surface of liquid helium $b=me^2(\epsilon_{\rm He}-1)/4\pi(\epsilon_{\rm He}+1)\hbar^2$, with e being the bare electron charge, $\epsilon_{\rm He}=1.005\,72$ is the dielectric constant of liquid helium. For electron systems in inverse layers, the inverse "plasma thickness" b can be found by a variational method [2].

Though the electron motion is constrained in the third direction, the potential of electrostatic interaction is defined by a solution of the three-dimensional Poisson equation averaged over the wave function Eq. (1). A more detailed discussion of this problem is given in Sec. I. For the electron system on the surface of liquid helium, the interaction can be approximated by the Coulomb potential $V_\varphi(r)=\bar{e}^2/r$, with $\bar{e}==e/\epsilon^{1/2}$ being the renormalized electron charge and $\bar{\epsilon}=(\epsilon_{\rm He}+1)/2$ (as long as the 2D distance between the interacting electrons $r\gg b^{-1}$).

Thus we can define the coupling parameter of the one-component 2D plasma (2D OCP) as $\Gamma=\bar{e}^2/k_BTa$, where $a=(\pi n_S)^{-1/2}$ is the 2D Wigner-Seitz radius. Another dimensionless parameter characterizing the 2D plasma is the plasma parameter $\gamma=k_D^2/2\pi n_S$ (k_D^{-1} is the 2D Debye length, $k_D=2\pi e^2n_S/k_BT$).

The investigation of 2D plasma collective excitations began about 20 years ago. Platzman and Tzoar, by their random-phase-approximation (RPA) calculations, established the qualitative features of 2D plasmons [3]. It was pointed out that the quasi-two-dimensional Coulomb system collective mode is a soft one with a dispersion law

$$\omega_c(k\to 0)=\omega_p(k)\equiv(2\pi n_S\bar{e}^2k/m)^{1/2}\,. \tag{2}$$

The RPA approach, however, does not take into account correlational effects, which play a crucial role in strongly coupled plasmas.

Theoretical attempts to describe the 2D collective excitations for $\Gamma>1$ started with extending the RPA into the strong-coupling regime [3]. Studart and Hippolito [4] used the static mean-field theory to evaluate the static local-field corrections. Their theory, however, does not satisfy the conventional ω^{-4} sum rule. This latter defect does not appear in the dynamic mean-field-theory calculation by Golden and Lu [5].

The 2D OCP is known to crystallize at $\Gamma=137\pm15$

into a hexagonal Wigner lattice. The dispersion relation for the crystalline state has been calculated by Bonsall and Maradudin [6]. Using a harmonic approximation they obtained

$$\omega_c(k \to 0, \Gamma \to \infty) = \omega_p(k)(1 - 0.172ka) \ . \quad (3)$$

Another approach to calculate the dispersion law in strongly coupled plasmas was developed in [7] and [8]. This approach is based on the model of quasilocalized particles occupying randomly located sites and undergoing oscillations around them. The site positions themselves change, too. The crucial idea is to describe the rapid oscillations by averaging the equilibrium configurations in time. The calculations based on this formalism lead to the following expression for the 2D dielectric permeability:

$$\epsilon(\mathbf{k},\omega) = 1 - \frac{\omega_p^2(k)}{\omega^2 - \omega_p^2(k) D(\mathbf{k})} \ , \quad (4)$$

where

$$D(\mathbf{k}) = \frac{1}{A} \sum_{\mathbf{q}} \frac{(\mathbf{k}\cdot\mathbf{q})^2}{k^3 q} [S(\mathbf{k}-\mathbf{q}) - S(q)] \ ,$$

A is the system area, and $S(q)$ is the static structure factor. Expression (4) satisfies the conventional ω^{-4} sum rule at high Γ values such that $3ka/2\Gamma < D(k)$.

The aim of our previous discussion was to clarify the crucial role the sum rules play in different theories. In Ref. [9] there was found the high-frequency expansion of the 2D dielectric permeability up to ω^{-6}. In this paper the sum rules were, however, calculated taking into account only the electrostatic interaction between electrons.

Kalman and Genga [10] showed that even in a nonrelativistic classical plasma the influence of the transverse electromagnetic field induced by moving charged particles is important for the calculation of the high-frequency properties of plasmas. In Ref. [11] it was shown that the influence of magnetic interactions between electrons leads to a compensation of the Coulomb term in the ω^{-4} sum rule in the case of a 3D plasma. The physical reasons for this compensation were explained as follows.

Within the RPA, Bohm and Pines [12] showed that the magnetic interaction of two moving electrons with an average velocity v is characterized by the screening length $L_{BP} \sim cL_D/v$ ($L_D = k_B T/4\pi e^2 n$ being the 3D Debye length, n the 3D electron density) and is similar to the Debye interaction, but has an opposite sign and is smaller v^2/c^2 times. The average magnetic energy per particle is $(-ve^2/2cL_D)(v/c)^2$ and the interaction energy in the large "screening volume" L_{BP} can be estimated as $(-e^2 L_D^2 n/2)$. On the other hand, the energy of the screened Coulomb interaction in the smaller volume L_D is equal to $e^2 L_D^2 n/2$ and compensates the former. It should be emphasized that the compensation between Coulomb and magnetic interactions is independent of the degree of nonideality and the degeneracy of plasmas. Due to this fact, it is important to prove the influence of transverse electromagnetic interactions in the case of the 2D electron gas.

In Sec. I we will calculate the 2D ω^{-4} sum rule, taking into account the magnetic interaction between moving electrons. An interpolation formula for the longitudinal dielectric permeability, satisfying all known sum rules (the compressibility sum rule and the ω^{-2} and ω^{-4} sum rules), will be suggested in Sec. II. On the basis of this formula we will calculate the 2D dispersion relations.

I. THE SUM RULES FOR A QUASI-TWO-DIMENSIONAL PLASMA

The aim of this section is to find the coefficients of the high-frequency expansion of the 2D longitudinal dielectric permeability. To include the photon degrees of freedom (the photons are three-dimensional) we have to take into account the electron motion along the third dimension, transverse to the surface, described by the fixed wave function $\varphi(z)$.

The dielectric permeability connects the induced charge density to a weak external density perturbation:

$$\rho_{\text{ind}}(\mathbf{k},\omega) = [\epsilon^{-1}(\mathbf{k},\omega) - 1]\rho_{\text{ext}}(\mathbf{k},\omega) \ . \quad (5)$$

The linear response of the system to an external potential $U_{\text{ext}}(\mathbf{R},t)$ is described by the expression

$$\rho_{\text{ind}}(\mathbf{R},t) = \frac{1}{\hbar}\int_{-\infty}^{t} d^3R'dt' \langle [\rho(\mathbf{R}',t'),\rho(\mathbf{R},t)] \rangle \times U_{\text{ext}}(\mathbf{R}',t') \ . \quad (6)$$

Here $[A,B]$ is the commutator of operators A and B, and $\langle A \rangle$ is the equilibrium average of A,

$$\rho(\mathbf{R},t) = e\Psi^\dagger(\mathbf{R},t)\Psi(\mathbf{R},t) = e\Psi^\dagger(\mathbf{r},t)\Psi(\mathbf{r},t)|\varphi(z)|^2 = \rho(\mathbf{r},t)|\varphi(z)|^2 \ ,$$

where $\mathbf{R} = (\mathbf{r},z)$ is a three-dimensional vector.

To find the dielectric permeability, we will connect U_{ext} and ρ_{ext}. Due to the Poisson equation

$$\Delta U_{\text{ext}} = 4\pi\rho_{\text{ext}} \quad (7)$$

and the boundary conditions at the surface $z=0$,

$$U_{\text{ext}}(\mathbf{r},z=-0) = U_{\text{ext}}(\mathbf{r},z=+0) \ , \quad (8)$$

$$\epsilon_{\text{He}}\frac{\partial}{\partial z}U_{\text{ext}}(\mathbf{r},z=-0) = \frac{\partial}{\partial z}U_{\text{ext}}(\mathbf{r},z=+0) \ , \quad (9)$$

we obtain

$$U_{\text{ext}}(\mathbf{R},t) = \int d^3R'\rho_{\text{ext}}(\mathbf{R}')G(\mathbf{R}',\mathbf{R}) \quad (10)$$

with the Green's function

$$G(\mathbf{R}',\mathbf{R}) = \bar{\epsilon}^{-1}\left\{\frac{1}{|\mathbf{R}'-\mathbf{R}|} + \frac{\epsilon_{\text{He}}-1}{\epsilon_{\text{He}}+1}[(\mathbf{r}'-\mathbf{r})^2 + (z+z')^2]^{-1/2}\right\} . \quad (11)$$

Substituting Eq. (10) into Eq. (6), we find the following expression:

$$\rho_{\text{ind}}(\mathbf{r},t)=\frac{1}{\hbar}\int_{-\infty}^{t}\langle[\rho(\mathbf{r}',t'),\rho(\mathbf{r},t)]\rangle \times V_\varphi(|\mathbf{r}''-\mathbf{r}'|)d^2r'd^2r''dt' \ . \tag{12}$$

Here $V_\varphi(r)=\int dz\,dz'G(r,z-z')|\varphi(z)|^2|\varphi(z')|^2$ is the effective Coulomb interaction between two electrons.

Since the electron system is homogeneous at the surfaces $z=$const, the commutator in Eq. (12) is a function of $\mathbf{r}'-\mathbf{r}$ only and the Fourier transform reads

$$\rho_{\text{ind}}(\mathbf{k},\omega)=V_\varphi(k)X(\mathbf{k},\omega)\rho_{\text{ext}}(\mathbf{k},\omega) \ , \tag{13}$$

with

$$X(\mathbf{k},\omega)=\hbar^{-1}\int_{-\infty}^{t}\langle[\rho(\mathbf{r}',t'),\rho(\mathbf{r},t)]\rangle \times e^{i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})+i\omega(t-t')} \times d^2r'd^2r\,dt'dt \tag{14}$$

and $V_\varphi(k)=\int d^2re^{i\mathbf{k}\cdot\mathbf{r}}V_\varphi(r)$. With the wave function from Eq. (1) we get [2]

$$V_\varphi(k)=\frac{2\pi}{\bar{\epsilon}k}\left\{\frac{1}{16}(1+\epsilon_{\text{He}})\left[1+\frac{k}{b}\right]^{-3} \times\left[8+9\frac{k}{b}+3\left[\frac{k}{b}\right]^2\right] +\tfrac{1}{2}(1-\epsilon_{\text{He}})\left[1+\frac{k}{b}\right]^{-6}\right\} . \tag{15}$$

In the long-wavelength limiting case we then have

$$V_\varphi(k\to 0)=\frac{2\pi}{\bar{\epsilon}k}\left[1-\tfrac{3}{16}(21-11\epsilon_{\text{He}})\frac{k}{b}+O(k^2)\right] . \tag{16}$$

Comparing Eqs. (13) and (5), we find

$$\epsilon^{-1}(\mathbf{k},\omega)=1+V_\varphi(k)X(\mathbf{k},\omega) \ . \tag{17}$$

Let us consider the function $X(\mathbf{k},\omega)$. At high frequencies, the Kramers-Kronig relation

$$\text{Re}X(\mathbf{k},\omega)=\frac{1}{\pi}\text{P}\int_{-\infty}^{\infty}d\omega'\frac{\text{Im}X(\mathbf{k},\omega')}{\omega'-\omega} \tag{18}$$

(P denotes principal value) provides

$$\text{Re}X(\mathbf{k},\omega\to\infty)=\sum_{n=2}^{\infty}\frac{M_n(\mathbf{k})}{\omega^n} \ , \tag{19}$$

where the moments $M_n(k)$ are defined by the expressions

$$M_n(\mathbf{k})=-\frac{1}{\pi}\int_{-\infty}^{\infty}\omega'^{(n-1)}\text{Im}X(k,\omega')d\omega'=\frac{i^{(n-1)}}{\hbar}\langle[\rho_{\mathbf{k}}^{(n-1)}(0),\rho_{-\mathbf{k}}(0)]\rangle$$
$$=\frac{i^{n-1}}{\hbar}(-1)^{[(n-1)/2]}\langle[\rho_{\mathbf{k}}^{[n-1-(n-1)/2]}(0),\rho_{-\mathbf{k}}^{[(n-1)/2]}(0)]\rangle \ . \tag{20}$$

Here $\rho_{\mathbf{k}}(t)$ is the Fourier transform of the charge-density operator $\rho(\mathbf{r},t)$ and $[p]$ is the integer of a real number p. The time derivatives in Eq. (20) are defined by

$$\dot{\rho}_{\mathbf{k}}=\frac{i}{\hbar}[\hat{H},\rho_{\mathbf{k}}],\ \ \ddot{\rho}_{\mathbf{k}}=\frac{i}{\hbar}[\hat{H},\dot{\rho}_{\mathbf{k}}],\ldots, \tag{21}$$

where $\hat{H}$ is the system Hamiltonian in the absence of the external field. Taking into account the photon degrees of freedom, we have

$$\hat{H}=\hat{T}+\hat{U}+\hat{W}+\hat{H}_f \ , \tag{22}$$

where

$$\hat{T}=\frac{1}{2m}\int d^3R\left[i\hbar\nabla_{\mathbf{R}}+\frac{e}{c}\mathbf{A}^{\text{int}}(\mathbf{R})\right]\Psi^\dagger(\mathbf{R}) \times\left[-i\hbar\nabla_{\mathbf{R}}+\frac{e}{c}\mathbf{A}^{\text{int}}(\mathbf{R})\right]\Psi(\mathbf{R}) \tag{23}$$

is the kinetic energy of the electron system, $\mathbf{A}^{\text{int}}(\mathbf{R})$ is the internal electromagnetic field induced by moving electrons,

$$\hat{U}=\int d^3\Psi^\dagger(\mathbf{R})u(z)\Psi(\mathbf{R}) \ , \tag{24}$$

and $u(z)$ is the potential which confines the electrons near the surface.

In addition,

$$\hat{W}=\int d^3R_1d^3R_2\Psi^\dagger(\mathbf{R}_1)\Psi^\dagger(\mathbf{R}_2)G(\mathbf{R}_1,\mathbf{R}_2)\Psi(\mathbf{R}_1)\Psi(\mathbf{R}_2) \tag{25}$$

is the electrostatic interaction energy between electrons, and $\hat{H}_f=\sum_{\mathbf{Q},\lambda}\hbar\omega_{\mathbf{Q}}c^\dagger_{\mathbf{Q},\lambda}c_{\mathbf{Q},\lambda}$ is the energy of the virtual photon field. Here $c^\dagger_{\mathbf{Q}}$ and $c_{\mathbf{Q}}$ are the creation and annihilation operators of photons with a momentum $\hbar\mathbf{Q}$ and the polarization $\lambda=1,2$, $\omega_Q=Qc$. It should be stressed that $\mathbf{Q}=(\mathbf{q},q_z)$ is a three-dimensional vector.

We turn next to the calculation of the moments $M_n(k)$. Due to symmetry properties only the even moments differ from zero. The standard procedure provides

$$M_2(k)=\frac{i}{\hbar}[\dot{\rho}_{\mathbf{k}},\rho_{-\mathbf{k}}]=\hbar k^2e^2n_S/m \ . \tag{26}$$

Let us introduce the plasma frequency of a real 2D plasma by

$$\tilde{\omega}_p^2(k)=V_\varphi(k)M_2(k)\ , \tag{27}$$

with the long-wavelength behavior

$$\tilde{\omega}_p(k\to 0)=\omega_p(k)\left[1-\tfrac{3}{32}(21-11\epsilon_{\rm He})\frac{k}{b}+O(k^2)\right] \tag{28}$$

due to Eq. (16), ω_p, defined by Eq. (2), is the plasma frequency of an ideal thin 2D electron plasma ($b=\infty$).

From Eqs. (27), (17), and (19) we get the well-known high-frequency expansion for the inverse dielectric permeability

$$\epsilon^{-1}(k,\omega)=1+\frac{\tilde{\omega}_p^2(k)}{\omega^2}+O(\omega^{-4})\ . \tag{29}$$

To calculate the fourth moment

$$M_4(k)=\frac{i}{\hbar}[\ddot{\rho}_{\mathbf{k}},\dot{\rho}_{-\mathbf{k}}]\ , \tag{30}$$

it is necessary to take into account the internal electromagnetic field. At low temperature it is sufficient to regard only the nonrelativistic terms (which do not vanish if we set the velocity of light equal to infinite).

The nonrelativistic part of the fourth moment consists of three terms:

$$M_4^{\rm nonrel}=M_{4,\hat{T}}(k)+M_{4,\hat{W}}(k)+M_{4,\hat{H}_f}(k)\ . \tag{31}$$

For the kinetic term we have

$$M_{4,\hat{T}}(k)V_\varphi(k)=\tilde{\omega}_p^2\omega_p^2\left[\frac{3k\langle E_{\rm kin}\rangle}{2\pi n_S}+\frac{\hbar^2}{8\pi n_s\bar{e}^2m}k^3\right]\ , \tag{32}$$

where $\langle E_{\rm kin}\rangle=N^{-1}\sum_{\rm p}(\hbar^2p^2/2m)a_{\rm p}^\dagger a_{\rm p}$ is the average kinetic energy of an electron.

The Coulomb term reads

$$M_{4,\hat{W}}(k)V_\varphi(k)=\tilde{\omega}_p^2\omega_p^2N^{-1}\times\sum_{\mathbf{q}\,(\neq 0)}\frac{(\mathbf{k}\cdot\mathbf{q})^2}{2\pi k^3}V_\varphi(q)\times[S(\mathbf{k}+\mathbf{q})-S(q)]\ . \tag{33}$$

For the photon term we obtain

$$M_{4,\hat{H}_f}(k)V_\varphi(k)=\tilde{\omega}_p^2\omega_p^2N^{-1}\times\sum_{\mathbf{q}\,(\neq 0)}S(\mathbf{k}+\mathbf{q})\left[\frac{3b}{8k}-\frac{(\mathbf{k}\cdot\mathbf{q})^2}{2\pi k^3}V_\varphi^{(1)}(q)\right]\ . \tag{34}$$

To derive Eq. (34) we used the wave function from Eq. (1). In Eq. (34) the effective magnetic interaction between two electrons is

$$V_\varphi^{(1)}(q)=\frac{2\pi}{q}\left\{\frac{1}{8}\left[1+\frac{q}{b}\right]^{-3}\left[8+9\frac{q}{b}+3\left[\frac{q}{b}\right]^2\right]\right\}\ . \tag{35}$$

For the electron liquid on liquid helium we have $\bar{\epsilon}\approx 1$ and thus $V_\varphi^{(1)}(q)\approx V_\varphi(q)$. In this case the second term on the right-hand side (rhs) of Eq. (34) compensates the first term on the rhs of Eq. (33). This is a result of compensation of Coulomb and magnetic interaction, discussed in Sec. I.

For the correlation part of the moment we obtain after some calculation

$$[M_{4,\hat{W}}(k)+M_{4,\hat{H}_f}(k)]V_\varphi(k)=\tilde{\omega}_p^2\omega_p^2\left[\frac{3b}{8k}[1+\tfrac{1}{2}h(0)]+2N^{-1}\sum_{\mathbf{q}\,(\neq 0)}[S(q)-1]\times\left[\frac{3b}{8k}-\frac{q^2}{2\pi k}V_\varphi(q)\right]\right]\ . \tag{36}$$

In the long-wavelength limit the latter expression exhibits a peculiar $(1/k)$ type divergence. In Eq. (36) $h(0)=(N)^{-1}\sum_{\mathbf{q}\,(\neq 0)}[S(q)-1]$ is the pair-correlation function. If one uses the Debye-Hückel approximation for the 2D static structure factor

$$S(q)=\frac{q}{q+k_D}\ , \tag{37}$$

we get for the correlation part of the moment

$$[M_{4,\hat{W}}(k)+M_{4,\hat{H}_f}(k)]V_\varphi(k)=\tilde{\omega}_p^2\omega_p^2\left\{\frac{3b}{8k}[1+\tfrac{1}{2}h(0)]-\frac{b}{8k}\frac{bk_D}{n_S}\left[\ln\left|\frac{2b}{k_D}\right|\left[\frac{k_D}{b}\frac{(3-k_D/2b)}{(k_D/2b-1)^3}\right]+\frac{2}{(k_D/2b-1)^3}\right]\right\}\ . \tag{38}$$

The main result of this section is that the inclusion of the magnetic interaction between electrons leads to a singularity for the dimensionless expression of the fourth moment $M_4(k)/\omega_p^4$ in the long-wavelength limit $k\to 0$. This fact leads to an essential change of the dispersion law of the quasi-two-dimensional Coulomb system collective soft mode. This problem will be discussed in Sec. II.

II. THE DISPERSION LAW OF 2D PLASMONS

The Nevanlinna formula from the classical theory of moments provides an expression for the dynamic dielectric permeability satisfying all known sum rules via the function $q=q(\mathbf{k},\omega)$ analytic in the upper half-plane $\mathrm{Im}z>0$ and having a positive imaginary part there, and such that $[q(k,z)/z]\to 0$ for $z\to\infty$ within the angle $\vartheta<\arg z<\pi-\vartheta$ $(0<\vartheta<\pi)$ [13,14] (see the Appendix):

$$\epsilon(\mathbf{k},\omega) =1-\frac{\tilde{\omega}_p^2[\omega+q(\mathbf{k},\omega)]}{\omega[\omega^2-\Omega^2(k)]+q(\mathbf{k},\omega)\{\omega^2+\tilde{\omega}_p^2[1-\epsilon(k)]^{-1}\}} . \tag{39}$$

Here $\epsilon(k)=\epsilon(k,0)$ is the static dielectric permeability and $\Omega^2(k)=M_4(k)/M_2(k)-\tilde{\omega}_p^2(k)$.

If one takes into account only the kinetic and Coulomb parts of the system Hamiltonian we have

$$\Omega^2(k)=\frac{3ka}{2\Gamma}+\frac{\hbar^2k^3}{8\pi n_S\bar{e}^2m} +\frac{1}{A}\sum_{\mathbf{q}\,(\neq 0)}\frac{(\mathbf{k}\cdot\mathbf{q})^2}{k^3q}[S(\mathbf{k}-\mathbf{q})-S(q)] , \tag{40}$$

and expression (39) can be regarded as a generalization of Eq. (4) which follows from Eq. (39), setting $q=0$ and $\Gamma=\infty$.

In Sec. I we have shown that the inclusion of the magnetic interaction between the electrons leads to a significant divergence of $\Omega^2(k)/\omega_p^2(k)$ in the long-wavelength limit. Thus the expansion of Eq. (39) in terms of $\omega_p^2(k)/\Omega^2(k)$ provides as $k\to 0$ a simpler expression:

$$\epsilon(k\to 0,\omega) =1-\frac{\tilde{\omega}_p^2}{\omega^2-\omega q'(\mathbf{k},\omega)+\tilde{\omega}_p^2[1-\epsilon(k)]^{-1}}+O(k^2) . \tag{41}$$

Here $q'(\mathbf{k},\omega)$ is a function with the same properties as $q(\mathbf{k},\omega)$ has. There is not any phenomenological way to choose $q'(\mathbf{k},\omega)$. Nevertheless, to investigate the soft collective mode it is sufficient to take its static value at $\omega=0$ $q'(k,\omega=0)=i\tau^{-1}(k)$:

$$\epsilon(k\to 0,\omega)=1-\frac{\tilde{\omega}_p^2}{\omega^2-i\omega\tau^{-1}(k)+\tilde{\omega}_p^2[1-\epsilon(k)]^{-1}} . \tag{42}$$

For the real and imaginary parts we obtain

$$\mathrm{Re}\epsilon(k\to 0,\omega) =1-\frac{\tilde{\omega}_p^2\{\omega^2+\tilde{\omega}_p^2[1-\epsilon(k)]^{-1}\}}{\{\omega^2+\tilde{\omega}_p^2[1-\epsilon(k)]^{-1}\}^2+\omega^2\tau^{-2}(k)} , \tag{43}$$

$$\mathrm{Im}\epsilon(k\to 0,\omega)=\frac{\tilde{\omega}_p^2\omega\tau^{-1}(k)}{\{\omega^2+\tilde{\omega}_p^2[1-\epsilon(k)]^{-1}\}^2+\omega^2\tau^{-2}(k)} . \tag{44}$$

On the other hand, there exists a connection between the dielectric permeability and the internal longitudinal conductivity $\sigma(\mathbf{k},\omega)$ defined by Ohm's law:

$$\mathbf{k}\cdot\mathbf{j}(\mathbf{k},\omega)=\sigma(\mathbf{k},\omega)[\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega)] ,$$

where the 2D current density $\mathbf{j}(\mathbf{r},t)$ has a dimension of A/m. That is why

$$\epsilon(\mathbf{k},\omega)=1+\frac{4\pi i}{\omega}b\sigma(\mathbf{k},\omega) . \tag{45}$$

The evaluation of Eq. (44) at $k=0$ gives

$$\mathrm{Im}\epsilon(k=0,\omega)=\frac{\tilde{\omega}_p^2\tau^{-1}}{\omega(\omega^2+\tau^{-2})} . \tag{46}$$

since $\epsilon(k=0)=\infty$ and $\tilde{\omega}_p(k\to 0)=\omega_p$. The singularity in Eq. (46) at $\omega\to 0$ guarantees the convergence of the static conductivity $\sigma_0=\sigma(0,0)$. Combining Eqs. (45) and (46) at $\omega\to 0$ we can define $\tau(k\to 0)$ by

$$\mathrm{Im}\epsilon(k=0,\omega\to 0)=\frac{4\pi}{\omega}b\sigma_0=\frac{\tilde{\omega}_p^2\tau}{\omega} , \qquad \tau(k\to 0)=\frac{4\pi\sigma_0 b}{\omega_p^2} . \tag{47}$$

Since we regard the collective behavior of 2D plasmas (i.e., the $k\to 0$ limiting case) we can set $\tau(k)=\tau(k\to 0)$. Thus formula (42) with $\tau(k\to 0)$ from Eq. (47) interpolates between the exact value of the static 2D conductivity σ_0 and the asymptotic expansion of the inverse dielectric permeability at $|\omega|\to\infty$ within the long-wavelength limiting case.

The dispersion relation now can be obtained by the familiar

$$\epsilon(\mathbf{k},\omega)=0 \tag{48}$$

relation, and from Eqs. (42) and (47) we obtain at $k\to 0$

$$\omega^2-i\omega\omega_p^2(4\pi\sigma_0 b)^{-1}+\tilde{\omega}_p^2\epsilon(k)[1-\epsilon(k)]^{-1}=0 . \tag{49}$$

The exact expression for $\epsilon(k)$ at $k\to 0$ is dictated by the compressibility sum rule,

$$\epsilon(k\to 0)=1+\frac{k_0}{k} , \tag{50}$$

where $k_0=k_D[\beta(\partial P/\partial n_S)_\beta]^{-1}$, $\beta=(k_BT)^{-1}$. The pressure P is defined by the correlation energy density $U_S=\frac{1}{2}n_S\int d^2r\,h(r)V_\varphi(r)$ through the connection $P=\beta^{-1}n_S+\frac{1}{2}U_S$, which yields

$$k_0=k_D\left[1+\frac{1}{2}\frac{\beta U_S}{n_S}+\frac{n_S}{2}\left[\frac{\partial}{\partial n_S}\frac{\beta U_S}{n_S}\right]_\beta\right]^{-1} . \tag{51}$$

Substituting Eq. (50) into Eq. (49) we can find the following dispersion law:

$$\omega_c(k\to 0)=\omega_p\left[1+\frac{k}{2k_1}-i\left[\frac{k}{k_2}\right]^{1/2}\right] , \qquad k_1^{-1}=k_0^{-1}-\left[\frac{3}{16b}\right](21-11\epsilon_{\mathrm{He}})-k_2^{-1} , \qquad k_2^{-1}=n_S\bar{e}^2(32\pi\sigma_0^2b^2m)^{-1} . \tag{52}$$

The imaginary part of the rhs of Eq. (52) describes the damping of the collective plasmon mode.

For the sake of comparison we regard now an ideal thin 2D plasma (i.e., $b=\infty$) without damping. In this case $k_1=k_0$, and

$$\omega_c(k\to 0, b=\infty)=\omega_p\left[1+\frac{k}{2k_0}+O(k^2)\right] . \quad (52')$$

The explicit expression of the correlation energy density U_S is known in the weak- ($\gamma \ll 1$ (Ref [15])) and strong-coupling ($\Gamma \gg 1$ (Ref. [16])) regimes

$$\frac{\beta U_S}{n_S}=\begin{cases}\frac{\gamma}{2}[\ln(2\gamma)+0.1544], & \gamma \ll 1 \quad (53)\\ -1.12\Gamma+0.71\Gamma^{1/4}-0.38, & \sqrt{2}<\Gamma<50 . \quad (54)\end{cases}$$

Substituting Eqs. (53) and (54) into Eq. (51) we get

$$k_0=\begin{cases}k_D\left[1+\frac{\gamma}{2}[\ln(2\gamma)+0.6544\right]^{-1}, & \gamma \ll 1 \quad (55)\\ k_D(-0.84\Gamma+0.399\Gamma^{1/4}+0.81)^{-1} , & \sqrt{2}<\Gamma<50 . \quad (56)\end{cases}$$

In the weak-coupling regime we have now the following dispersion law:

$$\omega_c(k\to 0, b=\infty, \gamma \ll 1) =\omega_p\left[1+\frac{k}{k_D}\left[\frac{1}{2}+\frac{\gamma}{4}[\ln(2\gamma)+0.6544]\right]\right] . \quad (57)$$

In the strong-coupling regime we obtain

$$\omega_c(k\to 0, b=\infty, \Gamma\gg 1) =\omega_p[1-(0.21-0.10\Gamma^{-3/4}-0.20\Gamma^{-1})ka] . \quad (58)$$

Equation (58) provides in the $\Gamma\to\infty$ limiting case

$$\omega_c(k\to 0, b=\infty, \Gamma=\infty)=\omega_p(1-0.21ka) , \quad (59)$$

which is slightly different from the Bonsall and Maradudin relation Eq. (3). We believe that the reason for this difference is the inclusion of the magnetic interaction in our calculations.

The dispersion law Eq. (52′) with k_0 from Eq. (51) coincides (excluding a thermodynamic factor c_p/c_v) with Baus's dispersion relation [17]. In the strong-coupling limit $\Gamma\to\infty$ we have $c_p/c_v\to 1$ and (52′) will not differ from Baus's expression. On the other hand, at $\gamma \ll 1$, $c_p/c_v\approx 2$, and there will appear a departure of (52′) from the expression in [17].

CONCLUSION

In this paper we have considered the dynamic properties of quasi-two-dimensional plasmas consisting of electrons with a fixed wave function along the z direction: $\varphi(z)=2b^{3/2}ze^{-bz}$. The second and fourth frequency moments of the inverse dielectric permeability imaginary part were calculated, taking into account both the third dimension and the magnetic interaction between electrons. The former one leads to a more complicated k dependence of the Fourier transform of the potential of electrostatic interaction, and only in the $k\to 0$ limit do we reproduce the familiar $V_\varphi(k)=2\pi/k\bar{\epsilon}$ and the dispersion relation $\omega_c(k\to 0)=\omega_p\equiv(2\pi n_S\bar{e}^2k/m)^{1/2}$. The latter one leads to the divergence of the ω^{-4} frequency moment.

On the basis of these results we have constructed an expression for the dynamic dielectric permeability [see Eq. (42)] which in the long-wavelength limiting case provides the well-known dispersion law with corrections depending not only on the electrostatic interaction via the static permeability (which satisfies the compressibility sum rule) but also on geometric effects and on the static conductivity of a 2D plasma [Eq. (52)].

APPENDIX

Let us consider the construction of the function $X(\mathbf{k},\omega)$ satisfying the sum rules Eq. (20). The explicit expression is obtainable from the Nevanlinna formula of the classical problem of moments [13]. The latter is, in particular, to construct the function $\eta(\mathbf{k},z) =-[\,\mathrm{Im}X(\mathbf{k},z)/\pi z\,]$ (analytic in the upper half-plane $\mathrm{Im}z>0$ and having there a positive real part) according to its first $2n$ frequency moments. In our case n is $n\le 2$, since there exists the finite zeroth frequency moment

$$M_0(k)=\int_{-\infty}^{\infty}\eta(\mathbf{k},\omega)d\omega=-X(\mathbf{k},0) \quad (A1)$$

due to the Kramers-Kronig relation as well as the second and fourth frequency moments $M_2(k)$ and $M_4(k)$ [Eqs. (26) and (31) taking into account Eqs. (32) and (36)].

Denote by $\{D_n(\mathbf{k},\omega)\}$, $n=0,1,\dots$ the system of orthogonal polynomials with the weight function $\eta(\mathbf{k},\omega)$:

$$\int_{-\infty}^{\infty}D_n(\mathbf{k},\omega)D_m(\mathbf{k},\omega)\eta(\mathbf{k},\omega)d\omega=\delta_{nm}\|D_n(\mathbf{k},\omega)\|^2 , \quad n,m=0,1,\dots .$$

The family of conjugate polynomials $\{E_n(\mathbf{k},\omega)\}$, $n=0,1,\dots$ is determined by the formula

$$E_n(\mathbf{k},\omega)=\int_{-\infty}^{\infty}\frac{D_n(\mathbf{k},\omega)-D_n(\mathbf{k},\omega')}{\omega-\omega'}\eta(\mathbf{k},\omega')d\omega' .$$

The polynomials $D_n(\mathbf{k},z)$ [and $E_n(\mathbf{k};z)$] can be found in terms of first $2n$ moments as a result of the Schmidt orthogonalization procedure. If they are normalized so that the coefficients at the senior powers of z are unity [or $M_0(\mathbf{k})$], then

$$\begin{aligned}&D_2=z^2-\omega_1^2, \quad D_3=z(z^2-\omega_2^2) ,\\ &E_2=M_0(\mathbf{k})z, \quad E_3=M_0(\mathbf{k})[z^2+\omega_1^2-\omega_2^2] .\end{aligned} \quad (A2)$$

Here $\omega_1^2=M_2/M_0$, $\omega_2^2=M_4/M_2$.

Let $\mathcal{R}_n$ be the set of all nondecreasing functions of limited variation $s(\mathbf{k},\omega)$ such that

$$\int_{-\infty}^{\infty}\omega^r ds(\mathbf{k},\omega)=\int_{-\infty}^{\infty}\omega^r\eta(\mathbf{k},\omega)d\omega=M_r(\mathbf{k}) , \quad r=0,1,\dots,2n .$$

Nevanlinna showed that there is a univalent correspondence between the $\mathcal{R}_n$ functions and function $Q_n(\mathbf{k},z)$, analytic in the upper half-plane $\mathrm{Im}z>0$ and having a positive imaginary part there, and such that uniformly within the upper half-plane the ratio $[Q_n(\mathbf{k},z)/z]\to 0$ converges to zero as $z\to\infty$.

This correspondence is set up by the Nevanlinna formula [13]

$$\int_{-\infty}^{\infty}\frac{ds(\mathbf{k},\omega)}{z-\omega}=\frac{E_{n+1}(\mathbf{k},z)+Q_n(\mathbf{k},z)E_n(\mathbf{k},z)}{D_{n+1}(\mathbf{k},z)+Q_n(\mathbf{k},z)D_n(\mathbf{k},z)} . \tag{A3}$$

In particular, among the functions $Q_n(\mathbf{k},z)$ there is (for a given n) only one $q_n(\mathbf{k},z)$ satisfying the equality

$$\int_{-\infty}^{\infty}\frac{\eta(\mathbf{k},\omega)}{z-\omega}d\omega=\frac{E_{n+1}(\mathbf{k},z)+q_n(\mathbf{k},z)E_n(\mathbf{k},z)}{D_{n+1}(\mathbf{k},z)+q_n(\mathbf{k},z)D_n(\mathbf{k},z)} . \tag{A4}$$

Taking into account the Kramers-Kronig relation, we obtain

$$\begin{aligned}X(\mathbf{k},z)&=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}X(\mathbf{k},\omega)}{\omega}\frac{z-\omega-z}{z-\omega}\\&=M_0+z\frac{E_3+q_2E_2}{D_3+q_2D_2} .\end{aligned} \tag{A5}$$

Using Eq. (A2), one rewrites Eq. (A5) as

$$X(\mathbf{k},z)=M_2\frac{q_2+z}{z(z^2-\omega_2^2)+q_2(z^2-\omega_1^2)} . \tag{A6}$$

From Eq. (A6) we get finally expression Eq. (39) for the dynamic dielectric permeability of a 2D OCP $\epsilon(\mathbf{k},\omega)=[1+V_\varphi(k)X(\mathbf{k},\omega)]^{-1}$, where we substituted $q_2(\mathbf{k},\omega)=q(\mathbf{k},\omega)$.

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