Stability of isotropic self-similar dynamics for scalar-wave collapse

M. Landman
BHP Melbourne Research Laboratories, P.O. Box 264, Clayton, Victoria 3168, Australia

G. C. Papanicolaou
Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, New York 10012

C. Sulem
Departement de Mathématique et d'Information, Ecole Normale Supérieure, Centre National de la Recherche Scientifique, 45 rue d'Ulm, 75230 Paris CEDEX 05, France and University of Toronto, Toronto, Ontario, Canada M5S 1A1

P. L. Sulem
Observatoire de la Côte d'Azur, Centre National de la Recherche Scientifique, Observatoire de Nice, Boîte Postale 229, 06304 Nice CEDEX 4, France and School of Mathematical Sciences, Tel-Aviv University, 69978 Tel-Aviv, Israel

X. P. Wang
Program in Applied Mathematics, Campus Box 526, University of Colorado, Boulder, Colorado 80309-0526
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The scalar model for collapse of Langmuir waves in plasmas is studied numerically in two and three dimensions, for both radially symmetric and anisotropic initial conditions. Using a dynamic rescaling method, singular solutions are shown to become isotropic and self-similar near collapse. In two dimensions, the self-similar profile is not universal. In the limit where the mass of the wave tends to its minimal value for collapse, the solution approaches a subsonic regime different from the generic singularity of the nonlinear Schrödinger equation.

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I. INTRODUCTION

The coupled system of scalar-wave equations,

$$i\psi_t+\Delta\psi-n\psi=0\ , \tag{1}$$

$$n_{tt}-\Delta n=\Delta|\psi|^2 \tag{2}$$

proposed in [1] is often used (see, for example, [2–9]) as a simplified description of Langmuir waves [10,11] when the vector character of the electric field can be neglected and is called the scalar Zakharov system. In this context, ψ denotes the envelope of the electric field and n the large-scale fluctuation of the ionic density. Other applications of these equations are mentioned in [2]. They include the electron-photon coupling in a solid-state plasma and the problem of strictive self-focusing in three-dimensional clusters of electromagnetic oscillations. In the subsonic limit where n_{tt} is negligible, the system reduces to the cubic nonlinear Schrödinger equation. Furthermore, Eq. (2), which describes the driving of the density waves by the ponderomotive force, can be rewritten in the form

$$n_t+\nabla\cdot\mathbf{v}=0\ , \tag{3}$$

$$\mathbf{v}_t+\nabla n=-\nabla|\psi|^2\ , \tag{4}$$

where $\mathbf{v}$ is the velocity of the medium.

The scalar Zakharov system is Hamiltonian with a Lagrangian density which is invariant under time and space translations, rotation, and phase shift. This leads to the conservation of total mass

$$\mathcal{M}=|\psi|^2_{L^2}\ ,$$

of linear and angular momenta and of energy [12]

$$\mathcal{H}=\int\{|\nabla\psi|^2+n|\psi|^2+\tfrac{1}{2}|\mathbf{v}|^2+\tfrac{1}{2}n^2\}d\mathbf{r}\ .$$

In one dimension, the scalar Zakharov system admits smooth solutions for all time. In higher dimensions, global existence requires small enough initial data [13]. In two dimensions this condition [14] reduces to $|\psi|^2_{L^2}<|R|^2_{L^2}$, where R denotes the positive solution of

$$\Delta R-R+R^3=0\ . \tag{5}$$

For large initial conditions, heuristic arguments and numerical simulations suggest a finite-time collapse both in two and three dimensions ([15–17] and references therein). Nevertheless, in contrast with the nonlinear Schrödinger equation, no rigorous results of collapse are available at present.

In the radially symmetric case, self-similar and asymptotically self-similar singular solutions have been predicted in two and three dimensions, respectively. In three dimensions, self-similar solutions can only exist asymptotically close to the collapse when the regime is strongly supersonic with the pressure term Δn negligible compared to the ion-inertia term in Eq. (2). Up to simple rescaling, these solutions have the universal form [1,4]

$$\psi(r,t)=\frac{1}{(t_*-t)}P\left[\frac{r}{(t_*-t)^{2/3}}\right]e^{i(t_0-t)^{-1/3}}, \quad (6)$$

$$n(r,t)=\frac{1}{(t_*-t)^{4/3}}N\left[\frac{r}{(t_*-t)^{2/3}}\right], \quad (7)$$

where $P(\eta)$ and $N(\eta)$ are isotropic scalar functions given by

$$\Delta P-\tfrac{1}{3}P-NP=0\ , \quad (8)$$

$$\tfrac{2}{9}(2\eta^2N_{\eta\eta}+13\eta N_\eta+14N)=\Delta P^2\ , \quad (9)$$

with

$$\Delta=\frac{d^2}{d\eta^2}+\frac{2}{\eta}\frac{d}{d\eta}.$$

In two dimensions, the self-similar collapse is no longer supersonic. The pressure term Δn in Eq. (2) remains important and

$$\psi(r,t)=\frac{1}{\alpha(t_*-t)}P\left[\frac{r}{\alpha(t_*-t)}\right]\times e^{i(t_*-t)^{-1}-i(t_*-t)^{-1}r^2/4},$$

$$n(r,t)=\frac{1}{[\alpha(t_*-t)]^2}N\left[\frac{r}{\alpha(t_*-t)}\right],$$

where $P(\eta),N(\eta)$ satisfy

$$\Delta P-P-NP=0\ , \quad (10)$$

$$\alpha^2(\eta^2N_{\eta\eta}+6\eta N_\eta+6N)-\Delta N=\Delta P^2\ , \quad (11)$$

with

$$\Delta=\frac{d^2}{d\eta^2}+\frac{1}{\eta}\frac{d}{d\eta}$$

and α^2 a free positive parameter.

There is no analytic proof of the existence of solutions for the ordinary differential equations (8),(9) and (9),(10). These equations were studied numerically in three dimensions in [4] and in two dimensions in [8]. In both cases, two pairs of localized solutions were computed, one of them (mode I) corresponding to a monotonic profile for both P and N.

In this paper we address the question of the dynamic stability of the self-similar solutions for both radially symmetric and anisotropic initial conditions in two or three dimensions. Which of them, if any, is approached by general singular solutions of the Zakharov equations near collapse? Is the parameter α universal in two dimensions or does it depend on the initial conditions? To answer these questions, Eqs. (1) and (2) are integrated numerically using the dynamic-rescaling method introduced in [18] and [19] for isotropic and anisotropic solutions of the nonlinear Schrödinger equation, respectively. A brief description of the method is given in Sec. II. In Sec. III two- and three-dimensional isotropic singular solutions are shown to become self-similar near collapse, with monotonic profiles. We note that in two dimensions the constant α in the equation for the profile tends to zero as the mass approaches its lowest value $|R|_{L^2}^2$ for collapse. Stability of this dynamics with respect to anisotropic initial conditions is demonstrated in Sec. IV. A few concluding remarks are presented in Sec. V.

II. DYNAMIC RESCALING

Equations (1) and (2) are first rewritten in the form

$$i\psi_t+\Delta\psi-n\psi=0\ , \quad (12)$$

$$n_t=w\ , \quad (13)$$

$$w_t-\Delta n=\Delta|\psi|^2\ . \quad (14)$$

Equations (13) and (14) are identical with Eq. (3) when $w=-\nabla\cdot\mathbf{v}$. The numerical method we use is very similar to the one we have developed for the nonlinear Schrödinger equation. We introduce a general change of dependent and independent variables of the form

$$\psi(\mathbf{x},t)=\frac{1}{L(t)}U(\xi,\tau)\ , \quad (15)$$

$$n(\mathbf{x},t)=\frac{1}{M(t)}V(\xi,\tau)\ , \quad (16)$$

$$w(\mathbf{x},t)=\frac{1}{H(t)}W(\xi,\tau)\ . \quad (17)$$

In (15)–(17)

$$\xi=D^{-1}(t)(\mathbf{x}-\mathbf{x}_0),\quad \tau=\int_0^t\frac{1}{\Omega^2(s)}ds\ , \quad (18)$$

where $\Omega(t)$ is a positive scalar function of time. Furthermore,

$$D(t)=O^T(t)\Lambda(t)\ , \quad (19)$$

with $O(t)$ an orthogonal matrix represented by the Euler angles and $\Lambda(t)$ a diagonal matrix whose diagonal elements are denoted by λ_i ($i=1,\dots,d$ with d the space dimension).

When solving the nonlinear Schrödinger equation, $\mathbf{x}_0(t)$ and $D(t)$ were prescribed by the constraints [19]

$$x_0^i=\frac{\int x_i|\psi|^{2p}d\mathbf{x}}{\int|\psi|^{2p}d\mathbf{x}}$$

and

$$\frac{\int\xi_i\xi_j|U|^{2p}d\xi}{\int|U|^{2p}d\xi}=\delta_{ij}\ ,$$

with $p=2$ or 3, in general. As shown in Sec. III, a more convenient choice for the Zakharov equations is

$$x_0^i=\frac{\int x_i|n|^{2p}d\mathbf{x}}{\int|n|^{2p}d\mathbf{x}}\ , \quad (20)$$

$$\frac{\int\xi_i\xi_j|V|^{2p}d\xi}{\int|V|^{2p}d\xi}=\delta_{ij}\ , \quad (21)$$

because this keeps W bounded away from zero near the collapse.

Substituting (15)–(18) into (12)–(14), we get the following evolution equations:

$$i[U_\tau - L^{-1}L_\tau U + \mathbf{f}\cdot\nabla U] + \mathcal{L}U - VU = 0 , \tag{22}$$

$$V_\tau - M^{-1}M_\tau V + \mathbf{f}\cdot\nabla V - \frac{M\Omega^2}{H}W = 0 , \tag{23}$$

$$W_\tau - H^{-1}H_\tau W + \mathbf{f}\cdot\nabla W - \frac{H}{M}\mathcal{L}V - \frac{H}{L^2}\mathcal{L}(|U|^2) = 0 , \tag{24}$$

$$\frac{d\lambda_i}{d\tau} = -a_{ii}\lambda_i\Omega^{2-d/2} , \tag{25}$$

$$\frac{d\mathbf{x}_0}{d\tau} = 2\Omega^{2-d/2}O^T\Lambda\beta , \tag{26}$$

$$\frac{dO}{d\tau} = -GO . \tag{27}$$

Here

$$\mathcal{L} = (\Omega^2\Lambda^{-2}:\nabla\nabla) ,$$

where $\nabla\nabla = (\partial^2/\partial\xi_i\partial\xi_j)$ $(i,j = 1,\ldots,d)$ and : denotes summation over both indices. The matrices $A = (a_{ij})$ and $G = (g_{ij})$ are defined by

$$a_{ij} = \frac{p\int(\delta_{ij} - \xi_i\xi_j)V^{2p-1}W\,d\xi}{\int|V|^{2p}d\xi} \tag{28}$$

and

$$g_{ii} = 0, \quad g_{ij} = \frac{2\lambda_i\lambda_j}{\lambda_i^2 - \lambda_j^2}a_{ij} \quad (i \neq j) .$$

Furthermore,

$$\mathbf{f} = \Omega^{2-d/2}(B\xi - 2\beta) ,$$

where $B = (b_{ij})$ and $\beta = (\beta_i)$ are

$$b_{ii} = a_{ii}, \quad b_{ij} = \frac{2\lambda_j^2}{\lambda_j^2 - \lambda_i^2}a_{ij} \quad (i \neq j) , \tag{29}$$

$$\beta_j = \frac{p\int\xi_j V^{2p-1}W\,d\xi}{\int|V|^{2p}d\xi} . \tag{30}$$

Note that (22)–(24) and (25) form a closed system for U, V, W and λ_i, while the translation $\mathbf{x}_0$ and the Euler angles are secondary quantities which can be determined separately from (26) and (27). The form of Eqs. (22)–(24) suggest that we take $L(t) = \Omega^{d/2}(t)$, $M(t) = \Omega^2(t)$, $H(t) = \Omega^{2+d/2}(t)$, and $\Omega^2(t) = d/\sum_i[1/\lambda_i^2(t)]$.

The coupled equations (22)–(27) and (28) are solved numerically by a finite-difference method with the approximate boundary conditions discussed in [19–21]. We use an Adams-Bashforth scheme for the time stepping and a second-order-difference scheme in space. The approximate boundary conditions allows a substantial reduction of the spatial integration domain.

III. COLLAPSE OF RADIALLY SYMMETRIC SOLUTIONS

For isotropic solutions, $\mathbf{x}_0 = \mathbf{0}$ and the matrix A is diagonal with $a_{ii} = a$ for $i = 1,\ldots,d$. Therefore, the scaling factors λ_i are also all equal and are denoted by λ. Consequently, $\Omega = \lambda$, $L = \lambda^{d/2}$, $M = \lambda^2$, $H = \lambda^{2+d/2}$, $\lambda^{-1}\lambda_\tau = -a\lambda$, and $\mathbf{f} = a\lambda\xi$. Equations (22)–(27) reduce to

$$i\left[U_\tau + a\lambda^{2-d/2}\left[\frac{d}{2}U + \xi U_\xi\right]\right] + \Delta U - VU = 0 , \tag{31}$$

$$V_\tau + a\lambda^{2-d/2}(2V + \xi V_\xi) - \lambda^{2-d/2}W = 0 , \tag{32}$$

$$W_\tau + a\lambda^{2-d/2}\left[\left[2 + \frac{d}{2}\right]W + \xi W_\xi\right] - \lambda^{d/2}\Delta V - \lambda^{2-d/2}\Delta|U|^2 = 0 , \tag{33}$$

$$\dot{\lambda} = -a\lambda^{3-d/2} , \tag{34}$$

where

$$a = \frac{p\int(1-\xi^2)V^{(2p-1)}W\xi^{d-1}d\xi}{\int V^{2p}\xi^{d-1}d\xi} . \tag{35}$$

A. Three-dimensional solutions

We first consider an initial condition $\psi_0(r) = 6e^{-r^2}$, $n_0(r) = -|\psi_0(r)|^2$, $\partial n_0/\partial t(r) = 0$. We observe that as τ increases, $|U|$, V, and W become stationary which indicates a self-similar collapse. In this limit, the phase of U at the origin is linear in τ. Furthermore, $\lambda(\tau) \to 0$ and $a(\tau)$ tends to a finite limit A [Fig. 1(a)]. It follows from (34) that

$$\lambda(t) = (t_* - t)^{2/3} ,$$

as predicted in [10] and displayed in Fig. 1(b). Substituting

$$U(\xi,\tau) = S(\xi)e^{+iC\tau}$$

in (31), we obtain, as $\tau \to \infty$

$$\Delta S - CS - VS = 0 ,$$

where S can be chosen to be real by a phase translation. In the equation for the density, the pressure term ΔV is asymptotically negligible, making the collapse supersonic. From (32) and (33), we obtain

$$\frac{A}{2}(2\xi^2 V_{\xi\xi} + 13\xi V_\xi + 14V) = \Delta S^2 .$$

Defining

$$P = \sqrt{1/A}\,S ,$$

$$N = \frac{1}{C}V ,$$

$$\eta = \sqrt{C}\,\xi ,$$

we get

$$\Delta P - P - NP = 0 , \tag{36}$$

$$\tfrac{1}{2}(2\eta^2 N_{\eta\eta} + 13\eta N_\eta + 14N) = \Delta P^2 , \qquad (37)$$

which are identical with Eqs. (2.1) and (2.2) of [4], up to a simple rescaling. Figure 2 shows the final profiles of $|U|$, V (after a rescaling). We observe that they fit the profiles P,N of mode I computed in [4].

We also consider two initial conditions for which ψ_0, n_0 are not peaked at the origin. The results for the initial condition $\psi_0(r)=6e^{-r^2}$, $n_0(r)=-r^2|\psi_0(r)|^2$, $\partial n_0/\partial t(r)=0$ are shown in Figs. 2(a) and 2(b). Results for the initial condition $\psi_0(r)=6r^2e^{-r^2}$, $n_0(r)=-r^2|\psi_0(r)|^2$, $\partial n_0/\partial t(r)=0$ are shown in Figs. 3(a) and (b). In both cases, the numerical results show that the profiles quickly become peaked at the origin and converge to the profiles P,N of mode I.

B. Two-dimensional solutions

Numerical integrations show that in this case also the solution displays a self-similar collapse where $a(\tau)\to A$. This is seen in Fig. 4(a) for initial conditions $\psi_0(r)=4e^{-r^2}$, $n_0(r)=-|\psi_0(r)|^2$, $\partial n_0/\partial t(r)=0$. This behavior corresponds to a scaling factor $\lambda(t)\approx(t_*-t)$. In contrast to the supersonic collapse observed in three dimensions, the pressure term must now be retained in the density equation. Furthermore, the phase of the self-similar solution can be calculated exactly. Writing

$$U(\xi,\tau)=S(\xi)\exp(-iC\tau)\exp[-(i/4)a\lambda\xi^2] ,$$

we get in the limit where λ tends to zero

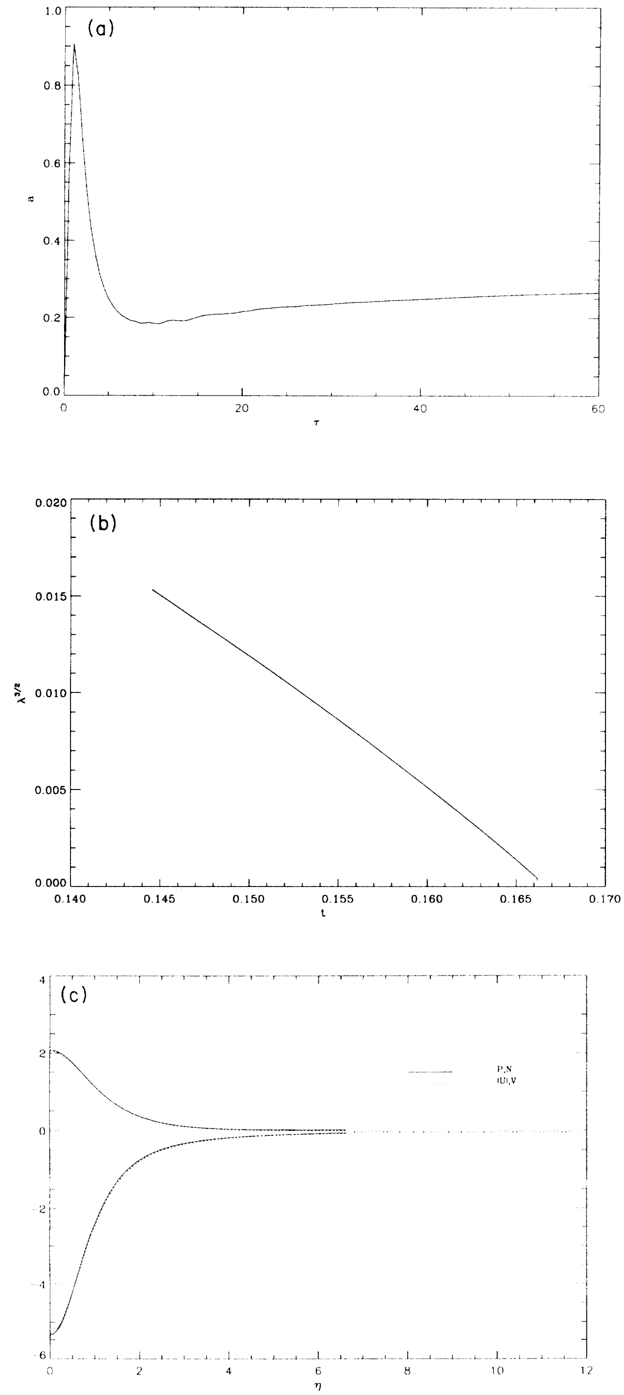

FIG. 1. Three-dimensional problem with isotropic initial conditions $\psi_0(r)=6e^{-r^2}$, $n_0(r)=-|\psi_0(r)|^2$, $\partial n_0/\partial t(r)=0$: (a) a vs τ; (b) $\lambda^{3/2}$ vs t; (c) asymptotic profiles compared to the mode I, computed in [4] (after rescaling).

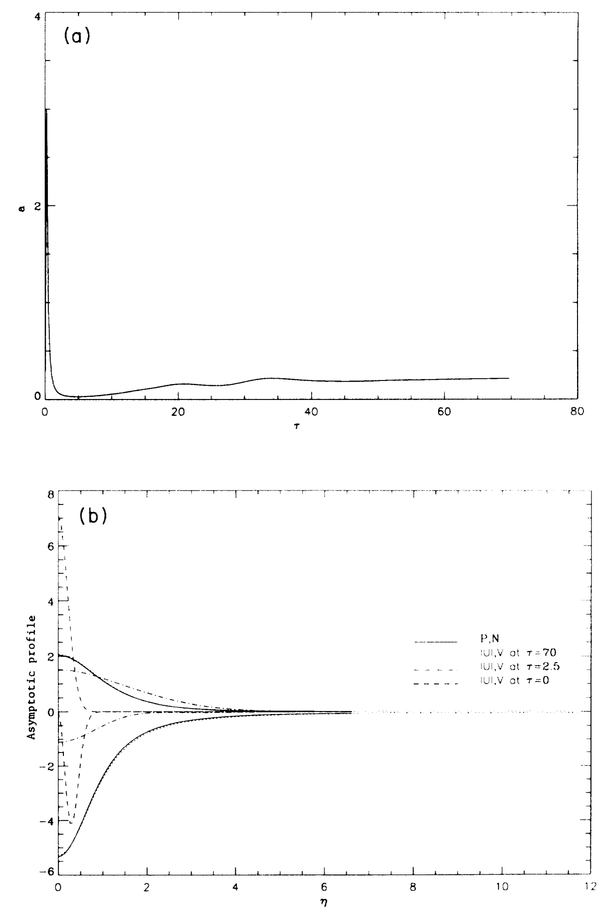

FIG. 2. Three-dimensional problem with isotropic initial conditions $\psi_0(r)=6e^{-r^2}$, $n_0(r)=-r^2|\psi_0(r)|^2$, $\partial n_0/\partial t(r)=0$: (a) a vs τ; (b) asymptotic profiles compared to the mode I, computed in [4] (after rescaling).

$$\Delta S - CS - VS = 0 \; ,$$

$$A^2(\xi^2 V_{\xi\xi} + 6\xi V_\xi + 6V) - \Delta V = \Delta S^2 \; .$$

With the normalization

$$P = \frac{1}{\sqrt{C}} S \; ,$$

$$N = \frac{1}{C} V \; ,$$

$$\eta = \sqrt{C}\, \xi \; ,$$

we have

$$\Delta P - P - NP = 0 \; , \tag{38}$$

$$\alpha^2(\eta^2 N_{\eta\eta} + 6\eta N_\eta + 6N) - \Delta N = \Delta P^2 \; , \tag{39}$$

Note that this system depends on the constant $\alpha^2 = A^2/C$. Figure 4(b) shows that, after rescaling, the profiles $|U|, V$ of the initial-value problem, fit the profiles P, N corresponding to this specific value of α^2.

The influence of the value of α on the profiles of the self-similar solutions is discussed in [8] where the collapse is defined as (moderately) subsonic if α is smaller than unity, and (moderately) supersonic if it is larger. The case $\alpha = 1$ is described as sonic. Our numerical simulations of the initial-value problem indicate that the constant α depends on the initial conditions. In order to make this statement more quantitative, we have considered two families of initial conditions with Gaussian and Lorentzian initial profiles $\psi_0(r) = ce^{-r^2}$ and $\psi_0(r) = (c/\sqrt{2})/(1+r^2)$, respectively, with $n_0 = -|\psi_0|^2$. For the same value of c, these two initial conditions have the same mass. As mentioned in the Introduction, the solutions of the two-dimensional Zakharov equations with a mass $|\psi|^2_{L^2} < |R|^2_{L^2}$ remain smooth for all time. For the above initial conditions, this corresponds to $c \le c_* = 2|R_0|_{L^2} \approx 3.724\,50$. Table I shows the values α_G^2 and α_L^2 for decreasing values of c, corresponding to the initially Gaussian and Lorentzian profiles, respectively. We see that in both cases $\alpha \to 0$ as c approaches c_* from

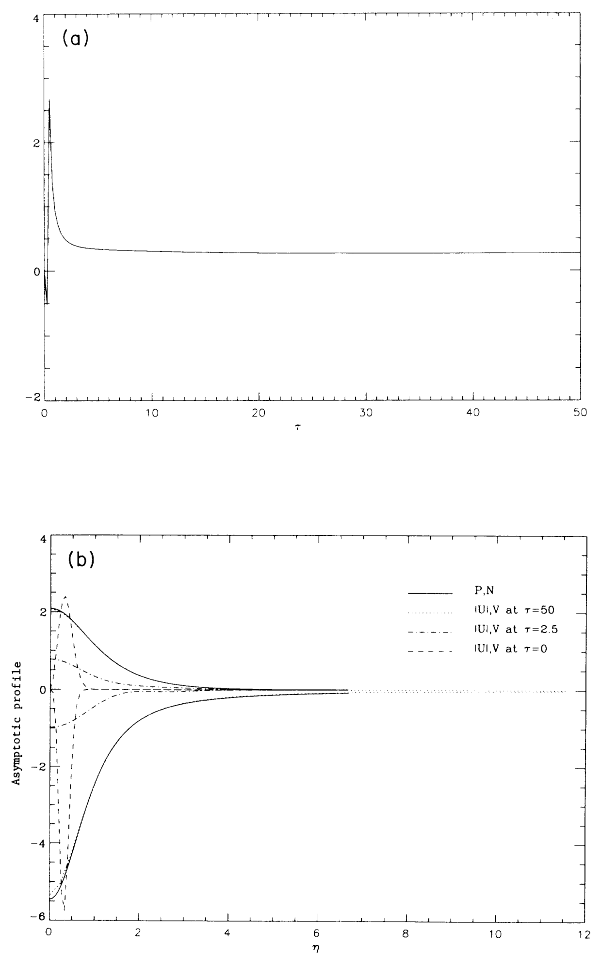

FIG. 3. Same as Fig. 2 for initial conditions $\psi_0(r) = 6r^2 e^{-r^2}$, $n_0(r) = -r^2|\psi_0(r)|^2$, $\partial n_0/\partial t(r) = 0$: (a) a vs τ; (b) asymptotic profiles compared to the mode I, computed in [4] (after rescaling).

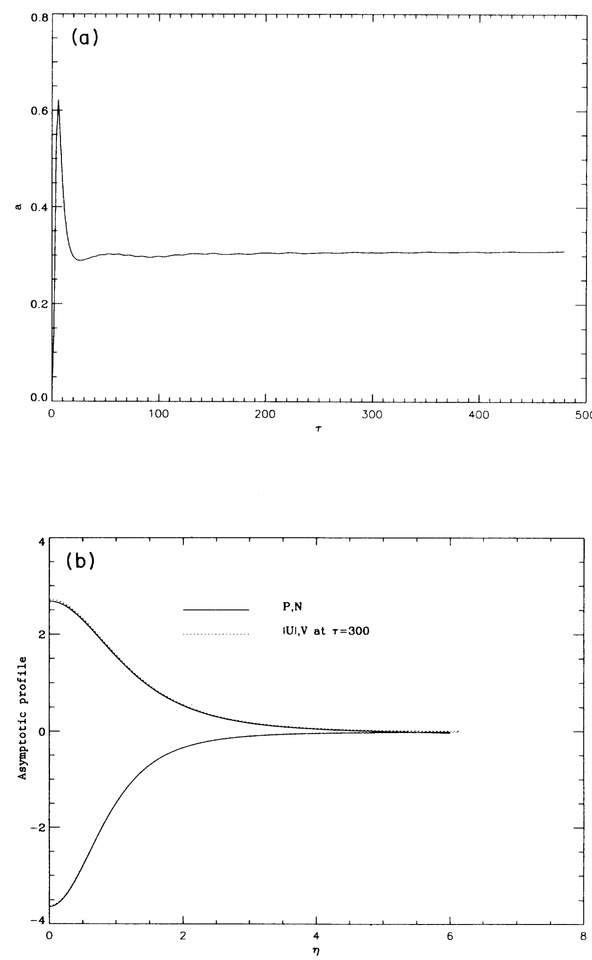

FIG. 4. Two-dimensional problem with isotropic initial conditions $\psi_0(r) = 4e^{-r^2}$, $n_0(r) = -|\psi_0(r)|^2$, $\partial n_0/\partial t(r) = 0$: (a) a vs τ; (b) asymptotic profiles compared to the mode I, computed in [8] (after rescaling).

TABLE I. Columns two and three list the dependence upon c of the Gaussian and Lorentzian profiles as discussed in the text.

c	α_G	α_L
4	1.02	0.86
3.8	0.88	0.76
3.6	0.74	0.66
3.4	0.59	0.53
3.2	0.41	0.38
3.0	0.31	0.29
2.9	0.25	0.20
2.85	0.18	0.175
2.8	0.10	0.09
2.75	0.001 068	0.001 003

above, although the numerical computation becomes delicate near the limit value.

As noticed in [8], when the parameter α tends to zero, the self-similar profile becomes (strongly) subsonic and P tends to the positive solution R of (5). In other words, the profile of the solution identifies with that of the nonlinear Schrödinger equation. Nevertheless, for $|\psi|_{L^2}$ arbitrarily close to $|R|_{L^2}$, the scaling factor $\lambda(t)$ varies asymptotically like $(t_* - t)$, while for the nonlinear Schrödinger, $\lambda(t) \approx (t_* - t)^{1/2} g(t_* - t)$, where g is a slowly varying function of the form $\ln\ln[1/(t_* - t)]^{-1/2}$ [22–26]. It follows that as $|\psi|_{L^2} \to |R|_{L^2}$, the singular solution of the Zakharov problem tends, up to a simple rescaling, to

$$\psi(r,t) \approx \frac{1}{t_* - t} R\left(\frac{r}{t_* - t}\right) \times \exp[i(t_* - t)^{-1} - i(t_* - t)^{-1} r^2/4] .$$

This function is an exact self-similar solution of the two-dimensional nonlinear Schrödinger equation. As a solution of the latter, it is generally unstable. A possible exception is when the mass is exactly critical. It was recently proved that in this case, it is the only singular solution [27,28].

C. Remarks on the profile equations

Equations (36), (37) and (38), (39) for the profile of the solution in two and three dimensions were solved numerically in [8] and [4], respectively. Note that $\eta = 0$ and $\eta = 1/\alpha$ are singular points for Eqs. (37) and (39). In order for the solutions to be smooth, proper conditions for N have to be prescribed.

In three dimensions, we solve N from (37) in terms of P in the form

$$N = \frac{P^2 - P_0^2}{\eta^2} - \frac{1}{2\eta^{7/2}} \int_0^{\eta} (P^2 - P_0^2) \eta^{1/2} d\eta , \tag{40}$$

where $P_0 = P(0)$. It is easy to see that

$$N \approx -\frac{2P_0^2}{3\eta^2}$$

as $\eta \to \infty$. Let $\eta \to 0$, we have

$$N(0) = \frac{6P(0)P''(0)}{7} . \tag{41}$$

It follows from Eq. (36) that

$$P''(0) = \tfrac{1}{3}[N(0) + 1]P(0) . \tag{42}$$

From (41) and (42) we have

$$N(0) = \frac{2P^2(0)}{7 - 2P^2(0)} .$$

Therefore, the value of N at the origin is not arbitrary but depends on P_0. When solving (36) and (39) by the shooting method, we only use P_0 as a shooting parameter chosen such that $P'(0) = 0$ and P decays rapidly at infinity.

The two-dimensional case is more difficult. We solve N from (39) in the form

$$N = \frac{P^2 - P_\alpha^2}{\alpha^2\eta^2 - 1} - \frac{\alpha^2}{(\alpha^2\eta^2 - 1)^{3/2}} \int_{1/\alpha}^{\eta} \frac{P^2 - P_\alpha^2}{(\alpha^2\eta^2 - 1)^{1/2}} \eta \, d\eta ,$$

where $P_\alpha = P(1/\alpha)$. When $\eta \to \infty$, we have

$$N \approx -\frac{C}{\eta^3} ,$$

where

$$C = \int_{1/\alpha}^{\infty} \frac{P^2}{(\alpha^2\eta^2 - 1)^{1/2}} \eta \, d\eta .$$

It follows from the integral formula and (38) that

$$N(1/\alpha) = \frac{2P_\alpha P'_\alpha}{3\alpha}$$

and

$$N'(1/\alpha) = \frac{4}{15\alpha} P_\alpha^3 P'_\alpha + \frac{2}{5\alpha} P'^2_\alpha - \tfrac{2}{5} P_\alpha P'_\alpha + \frac{2}{5\alpha} P_\alpha^2 .$$

Therefore the values of N at $\eta = 1/\alpha$ are determined by P_α, P'_α. We apply the shooting method at the sonic point $\eta = 1/\alpha$ with shooting parameters P_α, P'_α. The proper values of P_α, P'_α are chosen such that the solution P satisfies $P'(0) = 0$ and P decays rapidly at infinity.

IV. COLLAPSE OF INITIALLY ANISOTROPIC SOLUTIONS

We have integrated the full equations (31)–(33) with anisotropic initial conditions in two and three dimensions. In three dimensions, we start with $\psi_0(r) = 6e^{-x^2 - 2y^2 - 3z^2}$, $n_0 = |\psi_0|^2$, and $\partial n_0/\partial t(r) = 0$. We observe in Figs. 5(a) and 5(b) that the solutions $|U|, V$ tend to be isotropic; furthermore, the normalized rescaling factors $l_i = \Omega^2/3\lambda_i^2$ all tend to $\sqrt{1/3}$ and approach each other. This demonstrates that the solution in the primitive variables become isotropic near collapse. The profiles $|U|, V$ along the three coordinate planes fit the solutions P, N of mode I. Also, the phase at origin behaves like $C\tau$. Therefore we conclude that the solution converges to the isotropic singularity (3) and (4).

Note that the relaxation to isotropy is significantly slower than for the nonlinear Schrödinger equation with the same initial electric field. In the latter case, isotropy is approached around $\tau=12$, while in the above Zakharov case we have to run until about $\tau=250$ to see the convergence.

In two dimensions, we use the initial condition $\psi_0(r)=4e^{-(x^2+y^2/4)}$ with $n_0=|\psi_0|^2$ and $\partial n_0/\partial t(r)=0$. We also observe that the solution relaxes to isotropy. Normalized rescaling factors $l_i=\Omega^2/2\lambda_i^2$ are shown to approach each other in Fig. 6(a). We see that at the end of the run the profiles $|U|, V$ are isotropic and also fit the P,N profiles of mode I [Fig. 6(b)].

Collapsing solutions of the scalar Zakharov equations with anisotropic contraction rates were constructed in [29,30]. We did not observe such dynamics in our numerical simulations.

V. CONCLUSIONS

We have integrated the scalar Zakharov equations (1) and (2) with the dynamic rescaling method for both radially symmetric and anisotropic initial conditions in two and three dimensions. We observe that the isotropic self-similar singular solution of mode I is dynamically stable with respect to general anisotropic initial data that lead to a singularity at only one point. The size of the "caviton" scales like $(t_*-t)^{2/3}$ in three dimensions and like (t_*-t) in two dimensions as expected.

In the three dimensions, the collapse is supersonic. By contrast, in two dimensions the pressure term remains relevant near collapse. A specific subsonic regime is obtained in the limit where the mass of the waves tends to its lower bound for collapse. Note that the rate of blow up remains unchanged in this limit.

It is important to note that there are major differences between the scalar model and the full Zakharov equations [11,21], where ψ is replaced by a vector $\mathbf{E}$ and (1) is replaced by

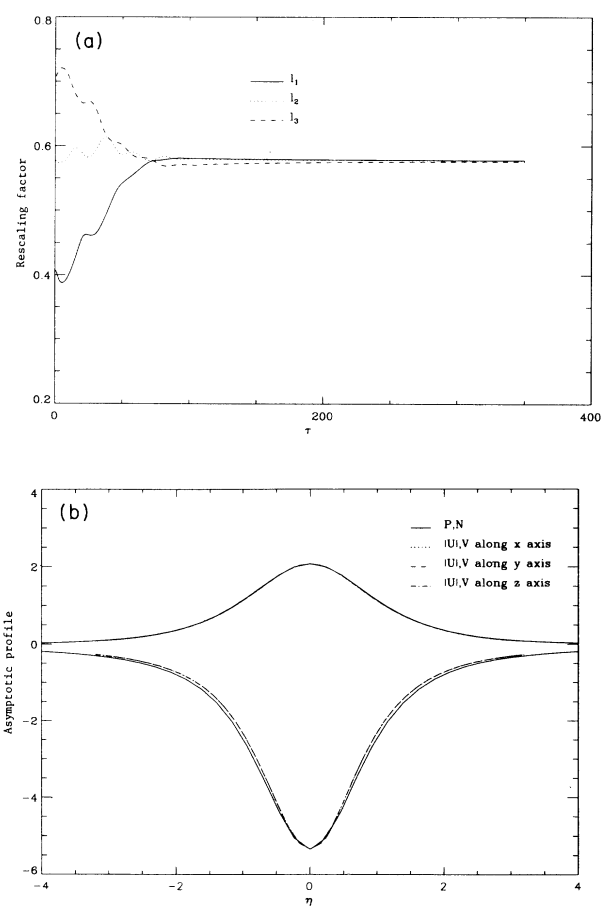

FIG. 5. Three-dimensional problem with anisotropic initial conditions $\psi_0=6e^{(-x^2-2y^2-3z^2)}$, $n_0(r)=-|\psi_0(r)|^2$, $\partial n_0/\partial t(r)=0$: (a) normalized rescaling factors vs τ; (b) asymptotic profiles along the coordinate axes (dashed lines and point line), compared to the mode I (solid line), computed in [4].

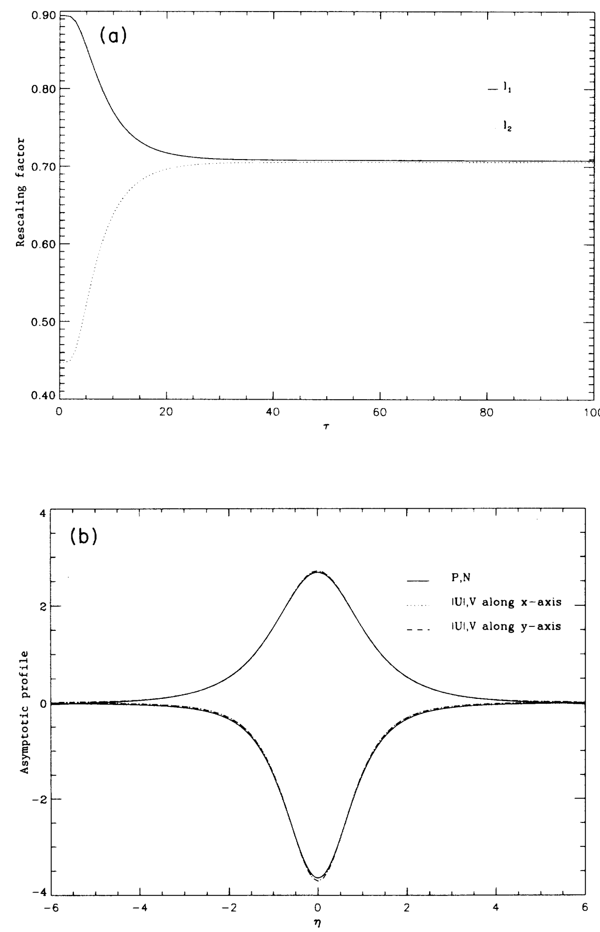

FIG. 6. Two-dimensional problem with anisotropic initial conditions $\psi_0=4e^{(-x^2-y^2/4)}$, $n_0(r)=-|\psi_0(r)|^2$, $\partial n_0/\partial t(r)=0$: (a) normalized rescaling factors vs τ; (b) asymptotic profiles along the coordinate axes (dashed lines and point line), compared to the mode I (solid line), computed in [8] (after rescaling).

$$i\mathbf{E}_t - \alpha\nabla\times(\nabla\times\mathbf{E}) + \nabla(\nabla\cdot\mathbf{E}) = n\mathbf{E} \ .$$

In [21] we showed numerically that, unlike the scalar case, singular solutions of the vector Zakharov equations are weakly anisotropic for a large class of initial data. Our observations support the argument given in [31] (see also [32] for the case of axially symmetric solutions), based on a spherical-harmonic decomposition.

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