Universal power law observed in an exponentially growing particle system

Misako Takayasu
The Graduate School of Science and Technology, Kobe University, Kobe 657, Japan
(Received 13 March 1992)

We generalize open random aggregating systems to the case of exponential growth of each aggregating unit. A statistically stationary state is confirmed under the uniform injection both numerically and theoretically. The steady-state mass distribution shows two different characteristic ranges: a small-mass range where the random aggregating process dominates, and a large-mass range where exponential growth rules the system. In both ranges, mass distributions follow power laws. In the large-mass range, the distribution converging asymptotically to $P(m) \propto m^{-1}$ is observed universally.

PACS number(s): 05.40.+j, 05.70.Ln

Recently, critical behaviors found in far-from-equilibrium open systems are attracting much attention on the point that they are realized without tuning any control parameter [1]. A successful attempt to model this behavior is done by Bak and Tang in the models of sandpile avalanches [2] and earthquakes [3]. In these models the phenomena are represented by the threshold dynamics based on the local conservation law and such models became widely known under the name of self-organized criticality.

On the other hand, open systems with nonthreshold dynamics, such as floating aerosols, are also known to realize critical behavior automatically and power-law distributions are commonly observed in such systems [4]. The basic processes of forming aerosols are diffusion and aggregation of particles. If the system is closed, the number of clusters decays monotonically in the irreversible aggregating process and the steady state will be a trivial configuration in which all particles gather into one large cluster. So the system is required to be open in order to reach nontrivial steady states.

Steady distributions are observed in the systems of random aggregation with continuous uniform injection of small-mass particles. The steady-state mass distribution follows a stable power law and it has been shown that the system recovers the same distribution regardless of any perturbation [5]. In that sense, we call this condition a "statistically" steady state. The steady state is supported by the balance of increasing the number of small particles by injection and decreasing it by aggregation process. In other words, if we single out a cluster of any size it grows larger and larger by repeating coalescence, but the number of small clusters of given size is kept constant by newly injected particles. Moreover, the cluster's aggregation rate is controlled by the injection rate so that the power law is always maintained.

Mean-field approaches to explain the steady distributions have been introduced by solving the Smoluchowski equation and have succeeded in explaining the power-law distributions [6]. A one-dimensional case is studied rigorously by Takayasu and co-workers by analyzing a model originally proposed by Scheidegger as a model of rivers [5]. They use the technique of characteristic functions to obtain the power-law mass distribution $P(>m) \propto m^{-1/3}$. Also the stability of the steady state with respect to an arbitrary initial condition and to any statistically uniform injection is mathematically proved.

Let us consider the situation when an aggregation with injection system has dissipation proportional to mass at each time step. In such a case the mass distribution shows a power law with an exponential decay at large mass due to the dissipation. For smaller dissipation the exponential decay becomes more gentle and for very small dissipation the distribution looks practically the same as in the case of nondissipated system. What will happen to the mass distribution when the system has an effect opposite to dissipation? Namely, we consider a mass growing effect, i.e., the particle grows proportional to its own mass at each time step. Does a stationary state still exist? In this paper, we discuss the surprising fact that a steady state exists even in such mass self-growing systems. Now, let us introduce the detail of the model in the case of one dimension.

We consider an aggregation process in one-dimensional discretized space-time with a periodic boundary. In one time step each particle jumps randomly either to the right or left neighbor. If two particles happen to jump on to the same site simultaneously, they immediately coalesce into a single particle with mass equal to the sum of the masses of the incident particles. In addition, we inject a unit mass particle onto each site. At the end of each time step operation, we multiply the total mass of each site by $1+\lambda$ ($\lambda > 1$).

A space-time configuration of particles' trajectories is shown in Fig. 1. The particles at the latest time step are at the bottom of this figure. The pattern is similar to branching trees covering the whole space-time. This figure is formed by the random-walk process, although both the structure of the branches and the mass self-growing process contribute to the mass of each particle.

Figure 2 shows the cumulative mass distributions for four different values of λ calculated by numerical simulations using the same random numbers. It is done on 1000 sites for 4000 time steps. The distribution has two characteristic ranges. In the small-mass range $(0 < m \leq 2^{10})$, the random aggregation process dominates

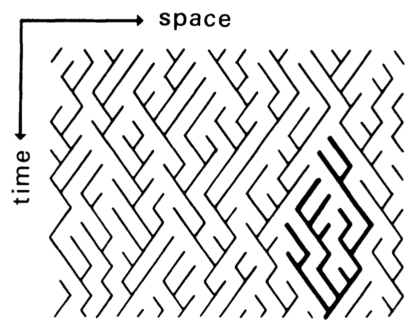

FIG. 1. A space-time configuration of particles' trajectories. The thick branched line shows the history of aggregation processes of a particle.

over the mass self-growing effect and a power-law distribution $P(>m) \propto m^{-1/3}$ appears to be independent of λ. In the large-mass range ($2^{10} \le m$, where m is dimensionless) the mass self-growing effect rules over the system and the distributions clearly depend on λ. Note that the tail of this range is extraordinarily long (about 10^{18} for $\lambda = 0.01$), and exponential decays caused by finite-size effect are observed only at the very end of the tails.

The dotted line in Fig. 3 shows the numerical result of the mass distribution for $\lambda = 0.01$. The data are obtained by taking average over ten realizations. It is clear that the functional form of the distribution does not depend on the number of maximum time steps. In this figure, we have more manifest appearance of the large-mass range. If we observe mass in relatively short scale, then it may be regarded as a part of a power law. However, we can tell that the tail is making a gradual curves on the log-log plot, i.e., the tail does not follow a rigorous power law.

A theoretical explanation for this tail is given as follows. Let us estimate the mass of a particle whose trajectory at time t is characterized by the height L with width

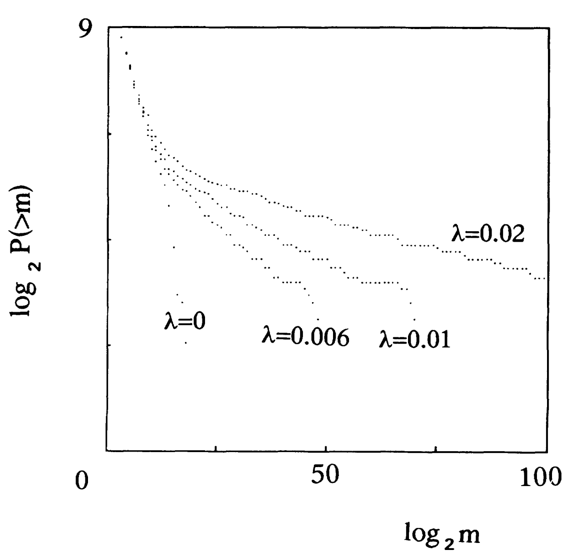

FIG. 2. Numerical results of the cumulative mass distribution for different λ's.

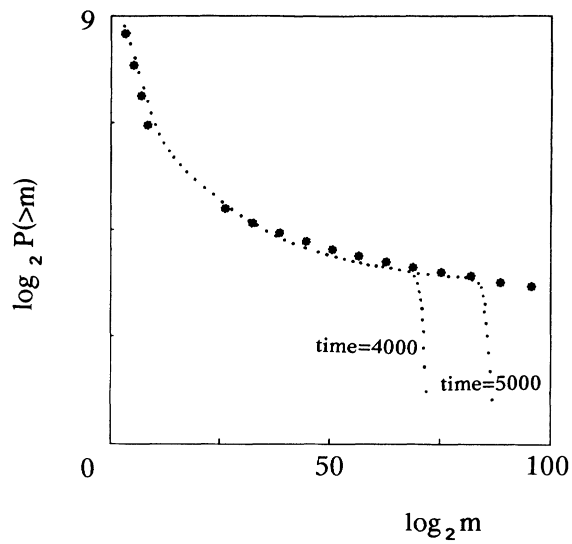

FIG. 3. Numerical results of the finite-time-step effects observed in the cumulative mass distribution in the case of $\lambda = 0.01$. The stars are the theoretical results for the steady state plotted as a comparison.

$w(t)$ as shown in Fig. 4. The mass m of the particle is given by

$$m = \sum_{t=0}^{L} w(t)(1+\lambda)^t = \sum_{t=0}^{L} w(t) e^{t \ln(1+\lambda)} . \tag{1}$$

When $L \gg 1/\lambda \gg 1$, Eq. (1) can be approximated as

$$m \simeq e^{\lambda L} \sum_{t=0}^{\infty} w(t) e^{-\lambda t} . \tag{2}$$

As the summation can be proved to remain finite the mass is estimated as

$$m \propto e^{\lambda L}, \quad L \gg 1/\lambda . \tag{3}$$

This implies that the mass self-growing effect becomes prominent for the particles having height L longer than $1/\lambda$.

As for the particles satisfying $L \le 1/\lambda$, we can treat them simply as the random aggregating particles with injection in one dimension. In this case the mass is given

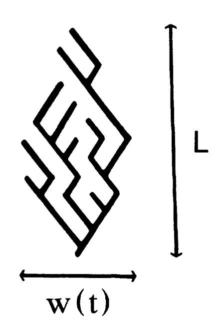

FIG. 4. Definition of the lifetime L and width $w(t)$ of a particle's trajectory.

by the summation of $w(t)$, which equals to the area of the branches in the space-time in Fig. 4. The relation between m and L is known from the problems of random walk. Thinking of the first encounter of the two random walkers starting from the same point, the area surrounded by the two trajectories in the space-time gives the area of the branches because the edges of the branches can be regarded as random walkers' trajectories [5]. As the width of the branches is nearly proportional to $L^{1/2}$, the mass for a particle with $L \le 1/\lambda$ is related to L as

$$m \propto L^{3/2}, \quad L \le 1/\lambda \ . \tag{4}$$

The inequality $L \le 1/\lambda$ corresponds to the small-mass range in Fig. 3 and $L \gg 1/\lambda$ corresponds to the large-mass range. Namely, $L = 1/\lambda$ gives the threshold of these two characteristic ranges.

Figure 5 shows the relation between m and L for numerical data of $\lambda = 0.01$ simulated on 1000 sites with 4000 time steps. We can confirm the validity of Eq. (3) for $L \gg 1/\lambda = 100$. The maximum value of L in the figure is 4000, which is restricted by the maximum number of steps of the simulation. Ten realizations are plotted on the figure yet we have practically no fluctuations. For $L \le 1/\lambda$, evidently the exponential relation between m and L is broken, which is consistent with our previous discussions. By plotting the same data in log-log scale, Eq. (4) has also been confirmed for small L.

Let us consider the size distribution of L. This distribution is equivalent to the distribution of two random walkers' first encounter time and is known to be given as [7]

$$P(>L) \propto L^{-1/2} \ . \tag{5}$$

Note that the patterns of branching trees are independent of the exponential growth, therefore L is independent of λ. Equation (5) is confirmed numerically in Fig. 6, which is obtained for the same situation as Fig. 5. It shows that

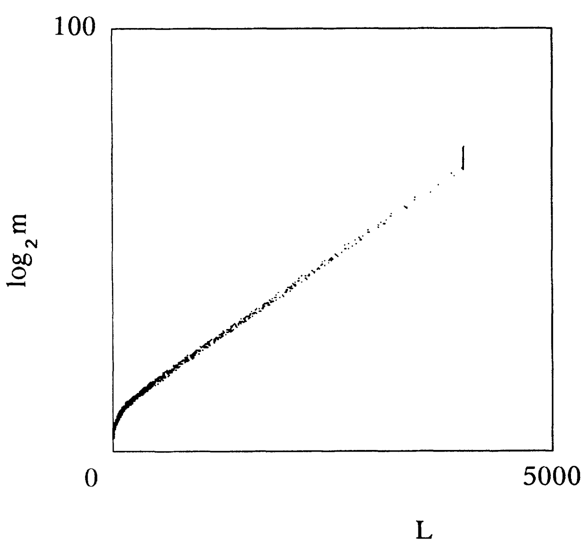

FIG. 5. Mass m vs lifetime L in semilogarithmic scale for one-dimensional case. The slope for large L fits nicely to the theoretical estimation $\lambda/\ln 2$ ($\lambda = 0.01$).

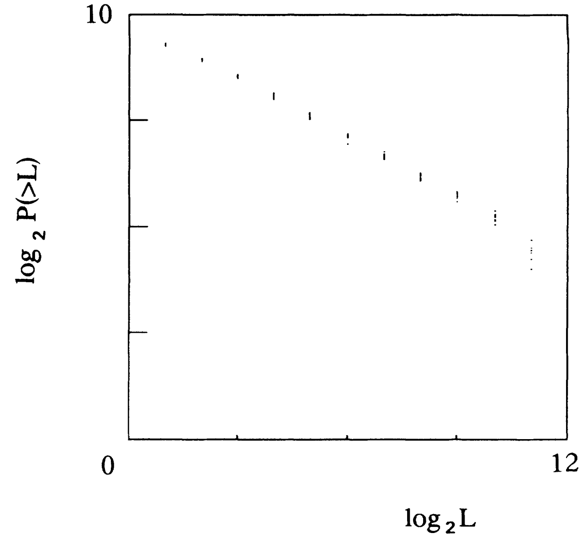

FIG. 6. The cumulative size distribution of L for one-dimensional case. The slope is close to the theoretical value $-\frac{1}{2}$.

Eq. (5) holds in the range $10 < L \le 4000$.

The distribution of m is obtained by assuming that L is uniquely determined for a given m by a monotonous function $L(m)$. Then we have the following relation:

$$P(>m) = P[>L(m)] \ . \tag{6}$$

From Eqs. (3), (5), and (6), and Eqs. (4), (5), and (6), we get, respectively,

$$P(>m) \propto \lambda^{1/2} (\ln m)^{-1/2}, \quad L \gg 1/\lambda \ , \tag{7}$$

$$P(>m) \propto m^{-1/3}, \quad L \le 1/\lambda \ . \tag{8}$$

Differentiating Eqs. (7) and (8) with respect to m, we obtain

$$P(m) \propto \frac{1}{m} (\ln m)^{-3/2} \lambda^{1/2}, \quad L \gg 1/\lambda \ , \tag{9}$$

$$P(m) \propto m^{-4/3}, \quad L \le 1/\lambda \ . \tag{10}$$

In the limit of $m \to \infty$, Eq. (9) behaves as $P(m) \propto 1/m$, which is often called the Zipf's law [8].

The stars plotted in Fig. 3 are the theoretical values obtained from Eqs. (7) and (8) with an appropriate choice of the proportional constants. It shows a good fit with numerical results.

Now we discuss the case of mean field. Let us figure the system with all sites connected to each other directly so that a particle on an arbitrary site can jump to any site, even to itself, in one time step. As in the case of one dimension, merged particles and injected unit particles on a site are combined to form a new particle with mass conserved at every time step. In the case without the mass self-growing process, the mass distribution is known to form a stable power law $P(<m) \propto m^{-1/2}$, like in the case of one dimension but with a different exponent. Then let us take into account the mass self-growing effect. Equa-

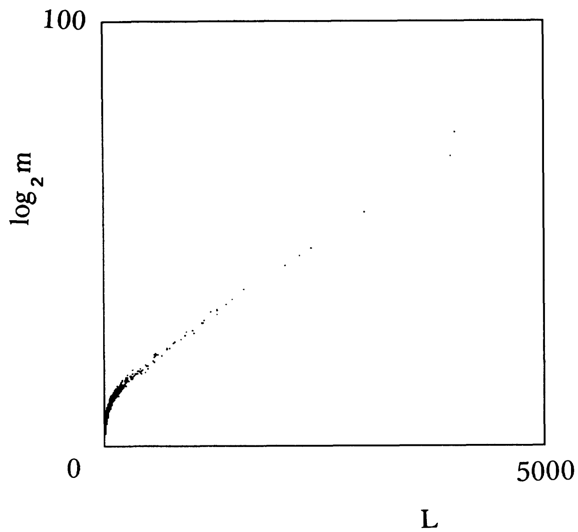

FIG. 7. Mass m vs lifetime L in the semilogarithmic scale for the mean-field case.

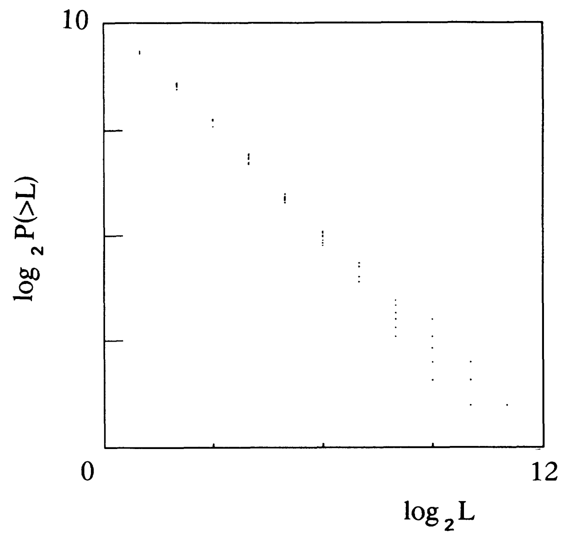

FIG. 8. The cumulative size distribution of L for the mean-field case. The slope is -1 as is estimated theoretically.

tion (3) holds again as we have seen in Fig. 7. The numerical results shown in Fig. 7 are obtained for the same condition as in Fig. 5; the system size is 1000, the maximum number of time steps is 4000, $\lambda=0.01$, and ten realizations are plotted without taking an average.

In the mean-field case we cannot apply the results from the random-walk problem but we can directly derive an equation for $P(>L)$ (see Fig. 8). The following are the processes contributing to the probability of having a particle with a lifetime longer than $L, P(>L)$ at step t. Consider a particle having a lifetime longer than $L-1$ at step $t-1$. Both in the case when it aggregates with particles having lifetimes smaller than $L-1$ and when it is not involved in any aggregation, it produces a particle of lifetime longer than L in the next time step. But if two or more particles having the lifetime longer than $L-1$ at time $t-1$ combine together to form a heavy particle at time t, then the probability $P(>L,t)$ decreases. Therefore the equation for the probability $P(>L,t)$ is given as

$$P(>L,t) \simeq P(>L-1,t-1)-P(>L-1,t-1)^2 , \quad (11)$$

where higher-order terms are neglected. The stationary solution of Eq. (11) is obtained by taking continuum limit with respect to L as

$$P(>L) \propto L^{-1} . \quad (12)$$

From Eqs. (3), (6), and (12), we obtain the probability density as

$$P(m) \propto \frac{1}{m}(\ln m)^{-2}\lambda, \quad L \gg 1/\lambda , \quad (13)$$

$$P(m) \propto m^{-3/2}, \quad L \leq 1/\lambda . \quad (14)$$

Again we have $P(m) \propto 1/m$ in the limit of $m \to \infty$.

Our results for one-dimension and for the mean-field case indicate that the asymptotic behavior of the mass distribution converging to $1/m$ may be universal in the exponentially growing particle system. The mechanism of having such a universality can be summarized as follows. Imagine an exponentially growing variable X; $X \propto \exp(\lambda Y)$, where the characterizing variable Y has a probability distribution in the form of power law $P(Y) \propto Y^{-\alpha}$. In the same way as we derive Eqs. (9) and (13), the probability density for X is shown to become

$$P(X) \propto \frac{1}{X}(\ln X)^{-\alpha}\lambda^{\alpha-1} . \quad (15)$$

This simple mechanism can be proposed as a model for the systems having long tails satisfying Zipf's law X^{-1} in various fields of science. Concerning the model introduced in this paper, the diffusion and aggregation processes of particles are not so essential to the universality of $P(X) \propto X^{-1}$, but the exponential growth and the power-law distribution of $P(Y)$ play an important role. An application of this model to ecological systems is now under consideration. In that case, we consider X as a population of a colony and Y as its lifetime.

The author would like to thank Professor H. Takayasu for valuable and stimulating discussions and gratefully acknowledges suggestions for Professor K. Ito and Dr. A. Yu. Tretyakov.

[1] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982); J. Feder, *Fractals* (Plenum, New York, 1988); T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989); H. Takayasu, *Fractals in the Physical Sciences* (Manchester University Press, Manchester, 1989).

[2] P. Bak, C. Tang, and K. Weisenfeld, Phys. Rev. Lett. **59**, 381 (1987).

[3] P. Bak and C. Tang, J. Geophys. Res. **94**, 635 (1989); K. Ito and M. Matsuzaki, *ibid.* **95**, 6835 (1990).
[4] S. K. Friedlander, *Smoke, Dust, and Haze* (Wiley-Interscience, New York, 1977); R. Jullien and R. Botet, *Aggregation and Fractal Aggregates* (World Scientific, Singapore, 1987).
[5] H. Takayasu, I. Nishikawa, and H. Tasaki, Phys. Rev. A **37**, 3110 (1988); H. Takayasu, M. Takayasu, A. Provata, and G. Huber, J. Stat. Phys. (to be published).
[6] W. H. White, J. Colloid Interface Sci. **87**, 204 (1982); E. M. Hendricks, R. M. Ziff, and M. H. Ernest, J. Stat. Phys. **31**, 519 (1983).
[7] W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1966), Vol. 2.
[8] G. K. Zipf, *Human Behavior and the Principle of Least Effort* (Addison-Wesley, Cambridge, MA, 1949).