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# Turbulent transport of a passive-scalar field by using a renormalization-group method

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A passive-scalar field is considered to evolve under the influence of a turbulent fluid governed by the Navier-Stokes equation. Turbulent-transport coefficients are calculated by small-scale elimination using a renormalization-group method. Turbulent processes couple both the viscosity and the diffusivity. In the absence of any correlation between the passive-scalar fluctuations and any component of the fluid velocity, the renormalized diffusivity is essentially the same as if the fluid velocity were frozen, although the renormalized equation does contain higher-order nonlinear terms involving viscosity. This arises due to the nonlinear interaction of the velocity with itself. In the presence of a finite correlation, the turbulent diffusivity becomes coupled with both the velocity field and the viscosity. There is then a dependence of the turbulent decay of the passive scalar on the turbulent Prandtl number.

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## I. INTRODUCTION

It has been well known for quite some time that the transport of fluctuations of vector and scalar fields by a turbulent fluid is greatly enhanced over the rate predicted by their natural molecular diffusivities. Traditionally, only phenomenological treatments were possible which use an *ad hoc* mixing assumption and a relevant mixing length. The estimation of transport coefficients then follows. These efforts fall under the general class of eddy transport (eddy viscosity or eddy diffusivity) and sub-grid modeling (used in large eddy simulations).

Recent advances in the renormalization-group (RG) method applied to fluid turbulence led to systematic studies of turbulent diffusion processes. Two distinct approaches have been taken: $\epsilon$-RG [1,2] and recursive RG [3–5]. In this work we take the latter approach. For a recent discussion on aspects of the two approaches, see Ref. [6]. The recursive RG method has been applied to the passive scalar advected by a random frozen (time-independently prescribed) fluid velocity [3] and to the Navier-Stokes equation [4,5]. Here we consider the evolution of a passive scalar by a fluid that is itself evolving by the Navier-Stokes equation. The passive scalar could be a model for temperature, chemical contaminant, a component of the magnetic field, etc. Here we concentrate on the nature of the coupling of the viscosity and the diffusivity.

We consider two cases: first, when there is no correlation between the passive-scalar field and the velocity, and second, when there is a correlation between them. If the passive-scalar field is the temperature, and the fluid evolves under gravity, then there is a correlation between the passive-scalar fluctuation and the vertical component of the fluid velocity, induced by buoyancy. In such a case, either the Navier-Stokes equation must be modified to take into account the gravity and the variation of density, or one must consider the Boussinesq or the full compressible-fluid equations. But here we look at the effect of the nonlinear terms only. Since the turbulent transport is essentially controlled by the nonlinear terms in the equation, our simple model might capture the essential features of the full problem.

In Sec. II we write the dynamical equations and introduce the scheme for eliminating small scales. In Secs. III and IV the elimination process is presented in detail. Self-similar properties are discussed in Sec. V, followed by a summary in Sec. VI.

## II. SMALL-SCALE ELIMINATION AND TURBULENT TRANSPORTS

We consider an incompressible fluid governed by the Navier-Stokes equation

$$\left(\frac{\partial}{\partial t}+\nu_0 k^2\right) u_\alpha(\mathbf{k},t) = \int d^3 j\, M_{\alpha\beta\gamma}(\mathbf{k}) u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) \ . \quad (1)$$

Here, $u_\alpha$'s are the Fourier coefficients of the $\alpha$th component of the fluid velocity and are a function of the wave number $\mathbf{k}$ and time $t$. The molecular viscosity is $\nu_0$. The coupling coefficients $M_{\alpha\beta\gamma}(\mathbf{k})$ are given by

$$M_{\alpha\beta\gamma}(\mathbf{k}) = [k_\beta D_{\alpha\gamma}(\mathbf{k}) + k_\gamma D_{\alpha\beta}(\mathbf{k})]/2i \ , \quad (2)$$

where

$$D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta / k^2 \ . \quad (3)$$

The coefficients $M$ in Eq. (2) reflect the incompressibility condition and the fact that, for such a flow, the pressure is a function of the velocity.

We consider a simultaneous equation for a passively advected scalar:

$$\left(\frac{\partial}{\partial t}+\kappa_0 k^2\right) \phi(\mathbf{k},t) = -ik_\alpha \int d^3 j\, u_\alpha(\mathbf{k}-\mathbf{j},t) \phi(\mathbf{j},t) \ , \quad (4)$$

where $\kappa_0$ is the molecular diffusivity for $\phi$.

 

If the dynamical equations (1) and (4) did not have the nonlinear terms on the right-hand side, then each Fourier mode of $u_\alpha$ or $\phi$ would decay linearly in time at a rate determined by their respective molecular transport coefficients $\nu_0$ and $\kappa_0$. But, because of the nonlinear couplings, fluctuations undergo spectral transfer, usually from large to small scales, where linear molecular dissipation is more efficient. This causes an enhancement of dissipation over the molecular rate. Much of the analytical effort in turbulence research goes into the calculation of these enhanced or turbulent-transport coefficients (also referred to as eddy-transport coefficients). To achieve such a goal, the wave-number space is divided into two parts, the large scales having $k \le k_1$ and the small scales having $k_1 < k \le k_0$. Here $k_1 = fk_0$, with $0<f<1$. The spectrum of fluctuations is taken to be zero for wave numbers larger than the dissipation wave number $k_0$. For a molecular Prandtl number of less than unity, it suffices to pick $k_0$ based on the molecular viscosity. The dynamical equation is written for the small scale, and then, with some approximation, an explicit expression is found for the small-scale field. Then this is substituted into the evolution equation for the large-scale field. An averaging over the small spatial scales is performed, and, with a suitably chosen closure scheme, the large-scale equation is closed with renormalized transport coefficients. The renormalized transport coefficients depend on the integral over the small-scale spectrum. The process is repeated by dividing the wave-number space again into a large-scale part $k \le k_2$ and a small-scale part $k_2 < k \le k_1$, with $k_2 = fk_1 = f^2 k_0$. This process of small-scale elimination continues up to a cutoff wave number $k_c$.

## III. REMOVAL OF THE FIRST SMALL-SCALE SHELL

Now we follow the above procedure for the elimination of the first small-scale shell in wave-number space in detail. Let us introduce the notation

$$u_\alpha(\mathbf{k},t) = u_\alpha^<(\mathbf{k},t), \quad k \le k_1 \tag{5}$$

$$u_\alpha(\mathbf{k},t) = u_\alpha^>(\mathbf{k},t), \quad k > k_1 \; . \tag{6}$$

Now, showing the explicit $k$ dependence, we rewrite Eqs. (1) and (4), first for the large scales and then for the small scales:

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^<(\mathbf{k},t) = \int d^3j\, M_{\alpha\beta\gamma}(\mathbf{k})[u_\beta^<(\mathbf{j},t) + u_\beta^>(\mathbf{j},t)][u_\gamma^<(\mathbf{k}-\mathbf{j},t) + u_\gamma^>(\mathbf{k}-\mathbf{j},t)] \; , \tag{7}$$

and

$$\left[\frac{\partial}{\partial t} + \kappa_0 k^2\right] \phi^<(\mathbf{k},t) = -ik_\alpha \int d^3j [u_\alpha^<(\mathbf{k}-\mathbf{j},t) + u_\alpha^>(\mathbf{k}-\mathbf{j},t)][\phi^<(\mathbf{j},t) + \phi^>(\mathbf{j},t)] \tag{8}$$

for the large scales, and

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^>(\mathbf{k},t) = \int d^3j\, M_{\alpha\beta\gamma}(\mathbf{k})[u_\beta^<(\mathbf{j},t) + u_\beta^>(\mathbf{j},t)][u_\gamma^<(\mathbf{k}-\mathbf{j},t) + u_\gamma^>(\mathbf{k}-\mathbf{j},t)] \; , \tag{9}$$

and

$$\left[\frac{\partial}{\partial t} + \kappa_0 k^2\right] \phi^>(\mathbf{k},t) = -ik_\alpha \int d^3j [u_\alpha^<(\mathbf{k}-\mathbf{j},t) + u_\alpha^>(\mathbf{k}-\mathbf{j},t)][\phi^<(\mathbf{j},t) + \phi^>(\mathbf{j},t)] \tag{10}$$

for the small scales. In Eqs. (7)–(10), the superscript on the field limits the range of its wave-number argument. For example, $\phi^<(\mathbf{k},t)$ indicates that $\mathbf{k}$ in the argument of $\phi^<$ is a large-scale wave number, so that $k \le k_1$. Notice that, by the same token, the range of integrals on the right-hand side of Eqs. (7)–(10) will be determined by which $u_\beta(\mathbf{j})$ or $\phi(\mathbf{j})$ is involved.

Now, we assume that $u_\alpha^>$ and $\phi^>$ evolve faster than their large-scale counterparts, so that the time derivatives in Eqs. (9)–(10) can be neglected [3,4]. (It has been shown that this approximation is equivalent to the stationarity approximation of the small-scale spectrum [7]. Thus, from Eqs. (9) and (10) we have

$$u_\beta^>(\mathbf{j},t) = \frac{1}{\nu_0 j^2} \int d^3j' M_{\beta\beta'\gamma'}(\mathbf{j})[u_{\beta'}^<(\mathbf{j}',t) + u_{\beta'}^>(\mathbf{j}',t)] \times u_{\gamma'}^<(\mathbf{j}-\mathbf{j}',t) + u_{\gamma'}^>(\mathbf{j}-\mathbf{j}',t)] \; , \tag{11}$$

and

$$\phi^>(\mathbf{j},t) = -\frac{i}{\kappa_0 j^2} j_{\beta'} \int d^3j' [u_{\beta'}^<(\mathbf{j}-\mathbf{j}',t) + u_{\beta'}^>(\mathbf{j}-\mathbf{j}',t)] \times [\phi^<(\mathbf{j}',t) + \phi^>(\mathbf{j}',t)] \; . \tag{12}$$

Now we substitute these small-scale fields into the equations for the large-scale fields; first Eq. (11) into Eq (7). We keep terms only up to second order in nonlinearity. On the right-hand side of Eq. (7) seven terms of the form $u^<u^<$, $u^<u^<u^<$, $u^<u^>u^<$, $u^<u^>u^>$, $u^>u^<u^<$, $u^>u^<u^>$, and $u^>u^>u^>$ are generated. We perform ensemble averages over the small scales, so that the following holds:

$$\langle u_\alpha^>(\mathbf{k},t)\rangle = 0 \; , \tag{13}$$

$$\langle u_\alpha^<(\mathbf{k},t)\rangle = u_\alpha^<(\mathbf{k},t) \; . \tag{14}$$

A typical example of the application of Eqs. (13) and (14), and of ensemble averages of a triple product, will be

$$\langle u_\alpha^<(\mathbf{k},t)u_\beta^<(\mathbf{j},t)u_\gamma^>(\mathbf{j}',t)\rangle=0 . \tag{15}$$

By this averaging process, the terms of the form $u^<u^>u^<$ and $u^>u^<u^<$ vanish. Moreover, the terms of the form $u^<u^>u^>$ vanish, since the latter two small-scale fields in the triple products are connected by the same vertex [4]. Terms of the form $u^>u^>u^>$ are dropped by a closure assumption. Therefore we are left with three terms. The first term $u^<u^<$ is just the large-scale quadratic nonlinearity in the exact form as the original Navier-Stokes equation (1). The second nonvanishing term is the $u^<u^<u^<$ term: a new triple nonlinearity generated by the small-scale elimination. We chose to keep this term. In some RG work this higher-order nonlinearity is neglected. The third nonvanishing term is the $u^>u^<u^>$ term, which after averaging takes the form $\langle u^>u^>\rangle u^<$. This is the enhancement of viscosity and is denoted by $\delta\nu_0$, which is taken to the left-hand side and added to $\nu_0$ after dividing by $k^2$. Thus

$$\left[\frac{\partial}{\partial t}+\nu_1 k^2\right]u_\alpha^<(\mathbf{k},t)=\int d^3j\, M_{\alpha\beta\gamma}(\mathbf{k})u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)$$
$$+2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, d^3j'\frac{1}{\nu_0 j^2}M_{\beta\beta'\gamma'}(\mathbf{j})u_{\beta'}^<(\mathbf{j}',t)u_{\gamma'}^<(\mathbf{j}-\mathbf{j}',t)u_\gamma^<(\mathbf{k}-\mathbf{j},t) , \tag{16}$$

where all wave-number arguments are less than $k_1$, and

$$\nu_1=\nu_0+\delta\nu_0 . \tag{17}$$

The enhancement of viscosity is given by

$$\delta\nu_0=2\int d^3j\frac{L_{kj}}{\nu_0 j^2k^2}Q(|\mathbf{k}-\mathbf{j}|) , \tag{18}$$

with geometric factor $L_{kj}$ defined as

$$L_{kj}=-2M_{\alpha\beta\gamma}M_{\beta\beta'\gamma'}(\mathbf{k})D_{\beta'\gamma'}(\mathbf{k}-\mathbf{j})D_{\gamma'\alpha}(\mathbf{k}) . \tag{19}$$

In (18), $k_1<j$, $|\mathbf{k}-\mathbf{j}|\le k_0$. Above, we have used the following relations for the small-scale two-point velocity correlation

$$\langle u_\alpha^>(\mathbf{k},t)u_\beta^>(\mathbf{k}',t),=D_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')Q(|\mathbf{k}|) . \tag{20}$$

Above, we have substituted the small-scale velocity field from Eq. (11) into Eq. (7), and then obtained a renormalized evolution equation (16) for the large-scale field. This equation has two features. First the viscosity is enhanced due to turbulence, and second, a new higher-order (triple nonlinear) term is generated. The $u$ equations up to this point are identical to those of our previous work [4].

Now we will substitute the small-scale velocity and the passive-scalar fields (11) and (12) into the large-scale passive-scalar equation (8). Again we keep terms only up to second order in nonlinearity. On the right-hand side we have 15 terms of the form $u^<\phi^<$, $u^<u^<\phi^<$, $u^<u^<\phi^>$, $u^<u^>\phi^<$, $u^<u^>\phi^>$, $\phi^<u^<u^<$, $\phi^<u^<u^<$, $\phi^<u^>u^>$, $u^>u^<\phi^<$, $u^>u^<\phi^>$, $u^>u^>\phi^<$, $u^>u^>\phi^>$, $\phi^>u^<u^<$, $\phi^>u^<u^>$, and $\phi^>u^>u^>$. We assume that the ensemble averages over small-scale $\phi$ are similar to those of $u$'s. Thus

$$\langle\phi^>(\mathbf{k},t)\rangle=0 , \tag{21}$$

$$\langle\phi^<(\mathbf{k},t)\rangle=\phi^<(\mathbf{k},t) . \tag{22}$$

Performing ensemble averages over the small scale makes the terms of the form $u^<u^<\phi^>$, $u^<u^>\phi^<$, $\phi^<u^<u^>$, $u^>u^<\phi^<$, and $\phi^>u^<u^<$ vanish. The terms of the form $u^<u^>\phi^>$, and $\phi^<u^>u^>$ vanish, since the last two of the triple product in each of these are connected by the same vertex. Terms of the form $u^>u^>\phi^>$ and $\phi^>u^>u^>$ are dropped by a closure assumption. Now we are left with six nonvanishing terms. We consider these terms in the large-scale $\phi$ equation for two circumstances.

### A. Uncorrelated $\phi$ and $u_\alpha$

If the scalar field $\phi$ is uncorrelated with the fluid velocity, then two terms of the form $u^>u^<\phi^>$ and $\phi^>u^>u^<$ vanish after averaging. The terms of the form $\langle u^>u^>\rangle\phi^<$ are the turbulent enhancement of the diffusivity. The other terms are kept on the right-hand side of the large-scale $\phi$ equation. Again we keep triple nonlinear terms generated by the small-scale elimination process. Thus

$$\left[\frac{\partial}{\partial t}+\kappa_1 k^2\right]\phi^<(\mathbf{k},t)=-ik_\alpha\int d^3j\, u_\alpha^<(\mathbf{k}-\mathbf{j},t)\phi^<(\mathbf{j},t)$$
$$-k_\alpha\int d^3j\, d^3j'\frac{j_\beta}{\kappa_0 j^2}u_\alpha^<(\mathbf{k}-\mathbf{j},t)u_\beta^<(\mathbf{j}-\mathbf{j}',t)\phi^<(\mathbf{j}')$$
$$-ik_\alpha\int d^3j\, d^3j'\frac{M_{\alpha\beta\gamma}(\mathbf{j})}{\nu_0 j^2}u_\beta^<(\mathbf{j}',t)u_\gamma^<(\mathbf{j}-\mathbf{j}',t)\phi^<(\mathbf{k}-\mathbf{j},t) . \tag{23}$$

Here all wave-number arguments are less than $k_1$, and the enhanced diffusivity $\kappa_1$ is given by

$$\kappa_1=\kappa_0+\delta\kappa_0 \ . \tag{24}$$

The increment of the diffusivity is

$$\delta\kappa_0=\frac{k_\alpha k_\beta}{k^2}\int d^3j \frac{1}{\kappa_0 j^2} D_{\alpha\beta}(\mathbf{k}-\mathbf{j})Q(|\mathbf{k}-\mathbf{j}|) \ . \tag{25}$$

Here $k_1<j$, $|\mathbf{k}-\mathbf{j}|\le k_0$.

If the fluid velocity is *prescribed* as treated by Rose [3], (as opposed to evolving by the Navier-Stokes equation), then the last term in (23) would not be there. Notice that this is the only term through which viscosity appears in the passive-scalar dynamics.

### B. Correlated $\phi$ and $u_\alpha$

If for some reason there is a correlation between the passive-scalar fluctuations and any component of the fluid velocity, (for example, if $\phi$ is the temperature, then correlation could be induced by gravity), then two more terms out of the 15 listed just before Eq. (21) would be nonzero. Now let us assume that $\phi$ is correlated with $u_\alpha$, so that

$$\langle \phi^{>}(\mathbf{k},t)u_\alpha^{>}(\mathbf{k}',t)\rangle=C_\alpha(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}') \ . \tag{26}$$

Then the large-scale passive-scalar equation becomes

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\kappa_1 k^2\right]\phi^{<}(\mathbf{k},t)+\Delta_\beta k^2 u_\beta^{<}(\mathbf{k},t)=&-ik_\alpha\int d^3j\, u_\alpha^{<}(\mathbf{k}-\mathbf{j},t)\phi^{<}(\mathbf{j},t)\\ &-k_\alpha\int d^3j\, d^3j' \frac{j_\beta}{\kappa_0 j^2}u_\alpha^{<}(\mathbf{k}-\mathbf{j},t)u_\beta^{<}(\mathbf{j}-\mathbf{j}',t)\phi^{<}(\mathbf{j}',t)\\ &-ik_\alpha\int d^3j\, d^3j' \frac{M_{\alpha\beta\gamma}(\mathbf{j})}{\nu_0 j^2}u_\beta^{<}(\mathbf{j}',t)u_\gamma^{<}(\mathbf{j}-\mathbf{j}',t)\phi^{<}(\mathbf{k}-\mathbf{j},t) \ .\end{aligned} \tag{27}$$

Here all the wave-number argument are less than $k_1$, and $\kappa_1$ is the same as in the uncorrelated case, i.e., given by Eqs. (24) and (25). The additional turbulent dissipation is given by

$$\Delta_\beta=\int d^3j \frac{k_\alpha}{j^2k^2}\left[\frac{j_\beta C_\alpha(\mathbf{k}-\mathbf{j})}{\kappa_0}+\frac{iM_{\alpha\beta\gamma}(\mathbf{j})C_\gamma(\mathbf{k}-\mathbf{j})}{\nu_0}\right] \ . \tag{28}$$

Here $k_1<j$, $|\mathbf{k}-\mathbf{j}|\le k_0$. Notice that the new turbulent-dissipation term is proportional to the fluid velocity and the coefficient $\Delta_\beta$ depends on both primitive transports $\nu_0$ and $\kappa_0$.

If we write (28) symbolically as

$$\Delta_\beta=a\,\delta\kappa+b\,\delta\nu \ , \tag{29}$$

then

$$\Delta_\beta=a\,\delta\kappa\left[1+\frac{b}{a}\frac{\delta\nu}{\delta\kappa}\right] \ . \tag{30}$$

The last term in (30) is a function of the turbulent Prandtl number. This indicates that the turbulent Prandtl number appears in the turbulent diffusivity, and the rate of diffusion is proportional to the amplitude of the velocity. For the uncorrelated case, the viscosity can only appear implicitly through the triple nonlinear terms. For the frozen velocity field, as treated by Rose [3], this triple nonlinear term [the last term in (23) and (27)] does not arise and thus the turbulent passive-scalar dynamics has no direct dependence on the viscosity.

## IV. REMOVAL OF THE *n*th SHELL

Now we take Eq. (16), drop the superscript $<$, follow the same steps as was done for getting (16) starting from (5), and remove the 2nd shell followed by the 3rd, 4th, . . . , and the $n$th shell. Now $u_\alpha^{<}(\mathbf{k},t)$ will indicate that $k\le k_n$ for the $n$th step. Each time we have an additional triple nonlinear term induced by the process of small-scale elimination. The viscosity increment is contributed by the quadratic nonlinear product and each of the $n-1$ triple nonlinear products. The calculation for the fluid velocity is identical to that of our previous work [4]. The result after the $n$th step is

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu_n k^2\right]u_\alpha^{<}(\mathbf{k},t)=&\int d^3j\, M_{\alpha\beta\gamma}(\mathbf{k})u_\beta^{<}(\mathbf{j},t)u_\gamma^{<}(\mathbf{k}-\mathbf{j},t)\\ &+2M_{\alpha\beta\gamma}(\mathbf{k})\sum_{r=1}^{n}\int d^3j\, d^3j' \frac{1}{\nu_{n-r}j^2}M_{\beta\beta'\gamma'}(\mathbf{j})u_{\beta'}^{<}(\mathbf{j}',t)u_{\gamma'}^{<}(\mathbf{j}-\mathbf{j}',t)u_\gamma^{<}(\mathbf{k}-\mathbf{j},t) \ .\end{aligned} \tag{31}$$

Here $k\le k_n$. For the first integral on the right-hand side, $j$, $|\mathbf{k}-\mathbf{j}|\le k_n$. For the $r$th term in the sum of the last integrals in (31), we have $j'$, $|\mathbf{j}-\mathbf{j}'|$, $|\mathbf{k}-\mathbf{j}|\le k_n$, while $k_{n-r+1}<j\le k_{n-r}$. The eddy viscosity is

$$\nu_n=\nu_{n-1}+\delta\nu_{n-1} \ , \tag{32}$$

where

$$\delta\nu_{n-1}=2\sum_{r=0}^{n-1}\int d^3j\frac{L_{kj}}{\nu_{n-1-r}j^2k^2}Q(|\mathbf{k}-\mathbf{j}|)\ . \tag{33}$$

The limits of the wave numbers are $k_{n-r}<j$, $|\mathbf{k}-\mathbf{j}|\le k_{n-1-r}$ for the $r$th term in the sum (33).

Now, for the passive-scalar field, we again treat the correlated and the uncorrelated cases separately. For the uncorrelated case:

$$\left[\frac{\partial}{\partial t}+\kappa_n k^2\right]\phi^<(\mathbf{k},t)=-ik_\alpha\int d^3j\, u_\alpha^<(\mathbf{k}-\mathbf{j},t)\phi^<(\mathbf{j},t)-k_\alpha\sum_{r=1}^{n}\int d^3j\, d^3j'\frac{j_\beta}{\kappa_{n-r}j^2}u_\alpha^<(\mathbf{k}-\mathbf{j},t)u_\beta^<(\mathbf{j}-\mathbf{j}',t)\phi^<(\mathbf{j}')$$
$$-ik_\alpha\sum_{r=1}^{n}\int d^3j\, d^3j'\frac{M_{\alpha\beta\gamma}(\mathbf{j})}{\nu_{n-r}j^2}u_\beta^<(\mathbf{j}',t)u_\gamma^<(\mathbf{j}-\mathbf{j}',t)\phi^<(\mathbf{k}-\mathbf{j},t)\ . \tag{34}$$

The limits are $k\le k_n$ and, for the first integral on the right-hand side, $j$, $|\mathbf{k}-\mathbf{j}|\le k_n$. For the $r$th term in the sum of the last two integrals in (34), we have $j'$, $|\mathbf{j}-\mathbf{j}'|$, $|\mathbf{k}-\mathbf{j}|\le k_n$, while $k_{n-r+1}<j\le k_{n-r}$.

Here, $\kappa_n$ is the enhanced diffusivity

$$\kappa_n=\kappa_{n-1}+\delta\kappa_{n-1}\ . \tag{35}$$

The increment of the diffusivity is

$$\delta\kappa_{n-1}=\frac{k_\alpha k_\beta}{k^2}\sum_{r=0}^{n-1}\int d^3j\frac{1}{\kappa_{n-1-r}j^2}\times D_{\alpha\beta}(\mathbf{k}-\mathbf{j})Q(|\mathbf{k}-\mathbf{j}|)\ . \tag{36}$$

The limits of the wave numbers are $k_{n-r}<j$, $|\mathbf{k}-\mathbf{j}|\le k_{n-1-r}$ for the $r$th term in the sum (36).

For the correlated case:

$$\left[\frac{\partial}{\partial t}+\kappa_n k^2\right]\phi^<(\mathbf{k},t)+\Delta_\beta^n k^2 u_\beta^<(\mathbf{k},t)=-ik_\alpha\int d^3j\, u_\alpha^<(\mathbf{k}-\mathbf{j},t)\phi^<(\mathbf{j},t)$$
$$-k_\alpha\sum_{r=1}^{n}\int d^3j\, d^3j'\frac{j_\beta}{\kappa_{n-r}j^2}u_\alpha^<(\mathbf{k}-\mathbf{j},t)u_\beta^<(\mathbf{j}-\mathbf{j}',t)\phi^<(\mathbf{j}')$$
$$-ik_\alpha\sum_{r=1}^{n}\int d^3j\, d^3j'\frac{M_{\alpha\beta\gamma}(\mathbf{j})}{\nu_{n-r}j^2}u_\beta^<(\mathbf{j}',t)u_\gamma^<(\mathbf{j}-\mathbf{j}',t)\phi^<(\mathbf{k}-\mathbf{j},t)\ . \tag{37}$$

The limits are $k\le k_n$ and, for the first integral on the right-hand side, $j$, $|\mathbf{k}-\mathbf{j}|\le k_n$. For the $r$th term in the sum of the last two integrals in (37), we have $j'$, $|\mathbf{j}-\mathbf{j}'|$, $|\mathbf{k}-\mathbf{j}|\le k_n$, while $k_{n-r+1}<j\le k_{n-r}$. The diffusivity $\kappa_n$ is the same as that for the uncorrelated case (35) and (36). The additional dissipation of the passive scalar is given by

$$\Delta_\beta^n=\sum_{r=0}^{n-1}\int d^3j\frac{k_\alpha}{j^2k^2}\left[\frac{j_\beta C_\alpha(\mathbf{k}-\mathbf{j})}{\kappa_{n-r}}+\frac{iM_{\alpha\beta\gamma}(\mathbf{j})C_\gamma(\mathbf{k}-\mathbf{j})}{\nu_{n-r}}\right]\ . \tag{38}$$

The limits of the wave numbers are $k_{n-r}<j$, $|\mathbf{k}-\mathbf{j}|\le k_{n-1-r}$ for the $r$th term in the sum (38).

Notice that, although there are formally $n$ terms in the sum (31), (33), (34), (36), (37), and (38), only a few (depending on the shell fraction $f$) can be nonzero, owing to the wave-number constraints (see Ref [3]).

## V. SELF-SIMILAR PROPERTIES OF THE RECURSION RELATION

The renormalized Navier-Stokes equation and the viscosity remain unaffected by the passive scalar. Therefore the properties of the Navier-Stokes equation remain the same as in Ref. [4]. With appropriate rescaling of the spectrum, wave number, and the viscosity, the latter approaches a fixed point as the number of shell removals goes to large values. See Eqs. (26)–(30) in Ref. [4]. See also Ref. [8] for a numerical evaluation of additional fixed points.

For the passive scalar, if $u_\alpha$ and $\phi$ are uncorrelated, then Rose's argument [3] exactly holds for Eqs. (35) and (36). With appropriate rescaling of the velocity spectrum, wave number, and the diffusivity, the latter approaches a fixed point as the number of the shell removal goes to large values. If we take the fixed-point values of the turbulent viscosity [4] and diffusivity [3], then the turbulent Prandtl number is of order 1. It is common to argue that the turbulent Prandtl number should be of order unity since the spectral transfer of momentum and the passive scalar should be similar. For example, in Ref. [9] the turbulent Prandtl number is set to unity in simulating compressible stellar convection.

For the correlated case it is more complicated, and the above argument does not carry over, in general, since the two transport coefficients, viscosity, and diffusivity, and the two correlations, $Q_{\alpha\beta}(k)$ and $c_\alpha(k)$, are all involved. For a special case where $Q_{\alpha\beta}(k)$ and $c_\alpha(k)$ scale similarly with $k$, then, under the assumption of similar scaling for both $\nu(k)$ and $\kappa(k)$, the recursion relation does go to a fixed point. The fixed-point value of $\Delta$ will then be a function of $f$, in particular,

$$\Delta_\beta^*(k) \approx f^{(m+1)/4} . \tag{39}$$

Here there is no singularity at $f=1$, unlike the fixed-point diffusivity (which is the same as in Rose [3]):

$$\kappa^*(k) \approx \frac{(1-f)f^{(m+1)/2}}{1-f^{(m+1)/2}} . \tag{40}$$

## VI. SUMMARY AND DISCUSSION

We have studied the turbulent transport of a passive scalar advected by an incompressible fluid governed by the Navier-Stokes equation using a recursive renormalization-group method. When the scalar field is uncorrelated with the fluid velocity, then the turbulent diffusivity is exactly the same as in the case of a frozen velocity field. But the triple nonlinear term generated in the dynamical equation for the passive scalar involves both the diffusivity and the viscosity. Thus, it is only through the triple nonlinear term that the turbulent Prandtl number can enter into the turbulent passive-scalar dynamics. This might indicate that the triple nonlinear terms generated in small-scale elimination processes cannot be irrelevant, contrary to some arguments given by Yakhot and Orszag [2] and Carati [10] for the Navier-Stokes case. The importance of triple nonlinear terms in the renormalized Navier-Stokes equation (with no passive scalar) has previously been demonstrated [11]. The nature of the interaction that brings Prandtl numbers into the dynamics involves a quartic of wave numbers. Two large-scale modes (one fluid velocity and one passive scalar) first influence a secondary small-scale passive-scalar mode, which then interacts with a large-scale fluid velocity to modify a large-scale passive-scalar field.

In the case where the passive scalar is correlated with a component of the fluid velocity, then there is an additional turbulent dissipation of the scalar field. Then, the viscosity and the diffusivity are also coupled through the renormalization transport coefficient. The turbulent Prandtl number in the case can enter into the problem without the triple nonlinear terms.

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[1] D. Forster, D. Nelson, and M. Stephen, Phys. Rev. A **16**, 732 (1977).
[2] V. Yakhot and S. A. Orszag, J. Sci. Comput. **1**, 3 (1987).
[3] H. A. Rose, J. Fluid Mech. **81**, 719 (1977).
[4] Y. Zhou, G. Vahala, and M. Hossain, Phys. Rev. A **37**, 2590 (1988).
[5] Y. Zhou, G. Vahala, and M. Hossain, Phys. Rev. A **40**, 5865 (1989).
[6] Y. Zhou and G. Vahala, Phys. Rev. A **46**, 1136 (1992).
[7] Y. Zhou and G. Vahala, Phys. Lett. A **147**, 43 (1990).
[8] Y. Zhou and M. Hossain, Phys. Lett. A **151**, 249 (1990).
[9] M. Hossain and D. J. Mullan, Astrophys. J. **380**, 631 (1991).
[10] D. Carati, Phys. Rev. A **44**, 6932 (1991).
[11] M. Hossain, Phys. Lett. A **161**, 277 (1991).