Stochastic resonance in transient dynamics

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Stochastic resonance has been examined in a number of systems that have steady states. We examine a dynamical system that allows for the escape of a particle from a potential well, and that is subjected to the combination of a periodic forcing term and noise. We show that in such a system the addition of a small amount of noise can actually increase the average time for the particle to escape from the well. This effect depends on the frequency of the external field and in this sense is a form of stochastic resonance.

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I. INTRODUCTION

The competitive effects of noise and (generally periodic) time-varying fields on the qualitative behavior of dynamical systems has proved to be of considerable interest within the past decade, particularly effects that have been studied under the heading of stochastic resonance [1–9]. So far stochastic resonance effects have mainly been studied in the context of nonlinear systems which exhibit steady-state behavior, exemplified by motion in a bistable potential. The typical problem appearing in the literature concerns the analysis of properties of the Langevin equation

$$\dot{y} = -\frac{\partial U}{\partial y} + A \sin\omega\tau + 2Dn(\tau) , \tag{1}$$

in which $n(\tau)$ is zero-mean, δ-correlated Gaussian white noise and $U(y)$ is a time-independent potential. The prototypic bistable potential can be written as

$$U(y) = -\frac{y^2}{2} + \frac{y^4}{4} , \tag{2}$$

and a typical investigation studies the interaction between the nonlinear $U(y)$, the noise, and the sinusoidal forcing term. Some qualitative properties of Eq. (1) with the potential in Eq. (2) may be derived from comparatively simple considerations. When both the noise and the sinusoidal forcing term are absent ($A = D = 0$), the system will eventually reach the equilibrium point $y_+ = +1$, provided that the particle is initially in the interval $(0, \infty)$ and reaches $y_- = -1$ starting from any point in $(-\infty, 0)$. When one adds noise to this picture (but not the sinusoidal term), the system will make transitions between y_+ and y_- and vice versa. Qualitative changes in the behavior of the system occur on the addition of the periodic term, as shown in Eq. (1).

Most analyses of stochastic resonance focus on characterizing the sharp enhancement of the signal power spectrum that occurs within a small neighborhood of the forcing frequency. The phenomenon of stochastic resonance was first suggested as accounting for the periodicity of the Earth's ice ages [1–4], but has been applied to the analysis of a number of other physical systems as well [5–7]. A number of variations of the basic notion of stochastic resonance have been investigated using both mathematical analysis and simulations [8,9]. Most analyses of such systems consider limiting cases of either weak noise, a weak periodic forcing term, or are made under the assumption that the frequency is either large or small in some suitably defined sense. Such analyses are generally directed towards the calculation of resonance effects in correlation functions.

Another type of resonance effect due to a periodic forcing term has been discussed in the context of a different parameter, the trapping time of random walks on a finite line terminated at both ends by traps [10] in the presence of a periodic modulation of the transition probabilities. The mean trapping time of a random walker was shown to exhibit a minimum at a resonant frequency. A slightly different version of the same model was studied independently by Reichl [11], who also demonstrated the existence of a resonance effect for a diffusion system in the presence of reflecting boundaries. These, in contrast to the present work, were for linear systems. More recently Zhou, Moss, and Jung [12] discussed escape-time distributions for particles in bistable systems.

In the present paper we discuss a different form of resonance for a particular choice of $U(y)$ in Eq. (1). We will refer to this as transient stochastic resonance. This refers to a resonance effect in the escape time from an unstable, periodically modulated nonlinear system. As a prototype of this type of system we choose the particular potential

$$U(y) = -\frac{y^2}{2} + \frac{y^3}{3} , \tag{3}$$

which is characterized by having two equilibrium points, one stable and one unstable, as illustrated in Fig. 1. The presence of noise in the system eliminates the possibility of a steady state for the system, which is equivalent to saying that the particle always escapes from the potential

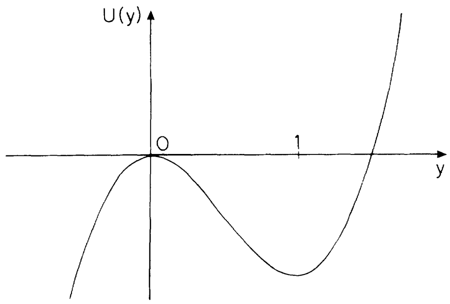

FIG. 1. A curve of the potential showing an unstable point at $y=0$ and a point of stable equilibrium at $y=1$.

well. We concentrate on the effect of the amplitude of the noise term D on the mean escape time in the presence of the periodic forcing term. No assumption will be made about the magnitude of A, in contrast to the analyses in Refs. [1–9].

In the absence of noise, i.e., $D=0$, a particle that moves according to Eq. (1) with the potential energy given by Eq. (3) will necessarily escape from the potential well and move to $y=-\infty$ provided that the amplitude A is sufficiently large. We will estimate a parameter related to the escape time from the potential well, in the presence of additive noise. One's initial intuitive feeling about such a system is that an increase in the amplitude of the noise should decrease the mean escape time because noise forces the particle to sample more of the available space than otherwise occurs without noise. Our simulation studies show that this is not necessarily the case and that at an appropriately chosen frequency the sinusoidal term can actually lead to an increase in the escape time when the amplitude of the noise term increases. This is the phenomenon that will be referred to as stochastic resonance in transient dynamics.

II. DETERMINISTIC MOTION

On substituting Eq. (3) into Eq. (1), we obtain the basic equation of motion

$$\dot{y}=y-y^2+A\cos(\omega\tau)+2Dn(\tau)\ , \tag{4}$$

in which A is a positive constant and $n(\tau)$ is zero-mean white noise defined by the properties

$$\langle n(\tau)\rangle=0,\quad \langle n(\tau)n(\tau')\rangle=\delta(\tau-\tau')\ . \tag{5}$$

The solution to Eq. (4) in the absence of both the noise and the external field ($A=D=0$) is

$$y(\tau)=\frac{y(0)}{y(0)-[y(0)-1]e^{-\tau}}\ , \tag{6}$$

which shows that a particle whose initial position is $y(0)$ will escape to infinity in a finite time provided that $y(0)$ is negative, but will not do so when $y(0)$ is positive. When the additive noise is allowed to contribute to the dynamics but not the oscillatory forcing term, it is clear without further calculation that the escape time will be finite for any initial position.

Before proceeding to an analysis of the full problem it is useful to examine some properties of Eq. (4) in the absence of noise but keeping the sinusoidal forcing term ($D=0$, $A\neq0$). While a full solution of this equation can be found in terms of Mathieu functions it is possible, by a more direct argument, to find relevant qualitative properties of the solution to the noise-free version of Eq. (4) to understand the role of frequency. We first observe that when A is sufficiently small, more precisely if $A<\frac{1}{4}$, the particle trajectory always remains bounded provided that the initial position satisfies

$$y(0)\geq\tfrac{1}{2}(1-\sqrt{1+4A}\,)\ . \tag{7}$$

This follows from the consideration that the bound on A ensures that for a fixed τ the equation

$$y-y^2+A\cos(\omega\tau)=0 \tag{8}$$

has two real roots, which is equivalent to the statement that the particle will remain trapped between the two values of y that correspond to these roots.

When the system is free of noise, but the sinusoidal forcing term is retained, we need to define what is meant by "escape" from the potential well, since there is no unambiguous escape in which $y(\tau)$ reaches $-\infty$ at a finite time similar to the effect given by the singularity in Eq. (6). However, in the present case, we will say that the particle has escaped from the well when $\lim_{\tau\to\infty}y(\tau)=-\infty$. We can derive a sufficient condition for this to occur by noting that in any cycle of the cosine term in which $y(\tau)<0$ there is a value of τ such that $y(\tau)$ reaches a minimum. Call the value of τ at which this occurs τ_m and the value of $y(\tau)$ at this point y_m. Escape will inevitably occur when the velocity at y_m remains negative even when the contribution from the sinusoidal term attains its maximum positive value, i.e., provided that

$$y_m-y_m^2+A<0 \tag{9}$$

or, equivalently,

$$-2y_m>\sqrt{1+4A}-1\ . \tag{10}$$

It is clear from both this relation and on intuitive grounds that the larger the value of the amplitude of the periodic term, the more negative must be the value of y_m in order for escape to occur. Figure 2 shows curves of $y(\tau)$ for several values of the frequency. The curves serve to illustrate the fact that escape can occur at the end of any cycle. They also illustrate the extreme sensitivity of results to changes in the frequency.

We next consider the question of whether the condition $A>\frac{1}{4}$ always guarantees that a particle will cross the barrier at $y=0$ and derive a criterion for the particle to reach this point without escaping from the well. Let $T=2\pi/\omega$ be the cycle time associated with the periodic term in Eq. (1). We need only consider the case in which the particle is initially within the well. It will be shown escape is impossible provided that $y(T)>y(0)$. For the purpose of demonstrating this result, define the difference function $\epsilon(\tau)=y(\tau+T)-y(\tau)$. We will show that the condition $\epsilon(0)>0$ implies that $\epsilon(nT)\geq0$ for $n=1,2,\ldots$.

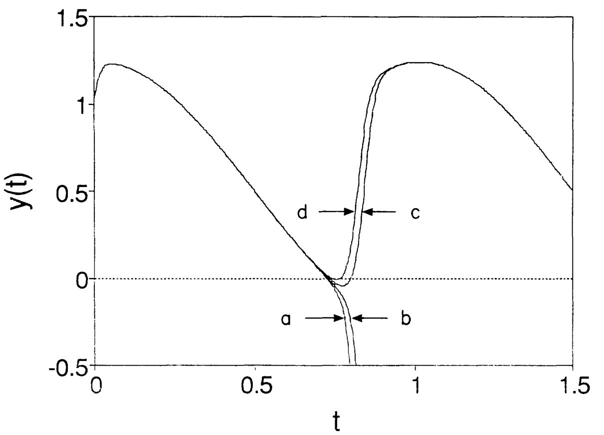

FIG. 2. Numerical solutions of Eq. (4) written in dimensionless form with the parameters $A=0.3$, $D=0$, and frequencies (a) $\omega=0.131\,96$, (b) $\omega=0.131\,97$, (c) $\omega=0.131\,98$, and (d) $\omega=0.131\,99$. Notice that particles can escape at different cycle numbers and that the escape cycle is very sensitive to frequency.

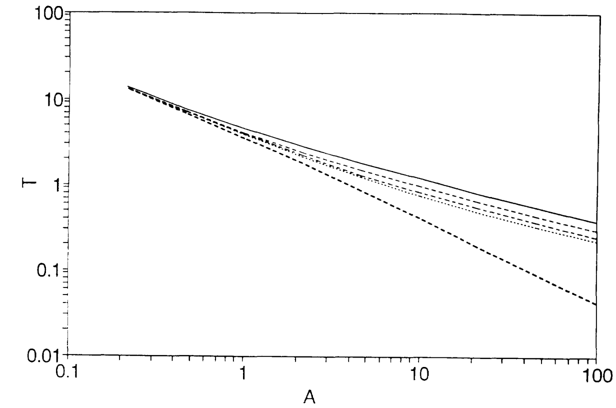

FIG. 3. Separatrices in the (A,T) plane for the noise-free case. The region above the uppermost curve corresponds to (A,T) pairs in which a particle "escapes" before the end of the first cycle. The region below the lowest curve corresponds to the parameters in which the particles never escape and the intermediate regions are for (A,T) pairs which lead to escape at a finite number of cycles.

This is equivalent to the assertion that the particle never escapes from the well.

The function $\epsilon(\tau)$ satisfies the equation

$$\frac{d\epsilon}{d\tau}=(1-2y)\epsilon-\epsilon^2 \tag{11}$$

with the solution

$$\epsilon(\tau)=\frac{\epsilon(0)f(\tau)}{\left[1+\epsilon(0)\int_0^\tau f(\xi)d\xi\right]}, \tag{12}$$

where the function $f(\tau)$ is defined by

$$f(\tau)=\exp\left[\tau-2\int_0^\tau y(\xi)d\xi\right]. \tag{13}$$

Since $f(\tau)$ is non-negative it follows from Eq. (12) that $\epsilon(\tau)$ has the same sign as $\epsilon(0)$. The implication of this analysis is that when $\epsilon(0)>0$ the particle cannot permanently leave the well. We cannot say what happens in general when $\epsilon(0)<0$, except in the special case in which $y(0)=0$, when one can show that escape is certain when $y(T)<0$. Qualitative aspects of the behavior of the escape time for different values of the amplitude A and T are shown in Fig. 3 in this special case. For all values of A and T located above the upper curve, escape occurs even before the end of the first cycle. The uppermost curve corresponds to systems in which the particle escapes at the end of the first cycle. The curve just below this corresponds to systems in which the particle escapes just at the end of the second cycle. Points lying in the region between these two curves correspond to systems in which the particle escapes at some time between the two cycles. The points below the lowest curve in Fig. 3 correspond to systems in which escape does not occur.

III. THE INFLUENCE OF NOISE

Let us next examine the effects of noise on this picture. When both A and D differ from zero in Eq. (4) the condition in Eq. (7) no longer suffices to guarantee return to the well since a return can theoretically occur from any value of y, although the larger the value of $-y$ the less likely this is to occur. We will be interested in the effect of the noise amplitude on the escape time. Since we present only the results of a simulation we cannot verify whether the particle, in fact, escapes to $-\infty$. On intuitive grounds one expects that an increase in the noise amplitude D should lead to a decrease in the quantity that we define as a measure of escape time. To study the effect of changing the noise amplitude we simulated the process whose mathematical expression is that in Eq. (4), introducing the noise term into the numerical algorithm suggested in Ref. [13]. For the values of A and D used in the numerical calculations we defined the escape time as the mean time for the particle to reach the value $y=-10$, since that appeared to guarantee that the particle never returned to the well. The initial condition for our simulations was set to $y(0)=1$, which corresponds to the parti-

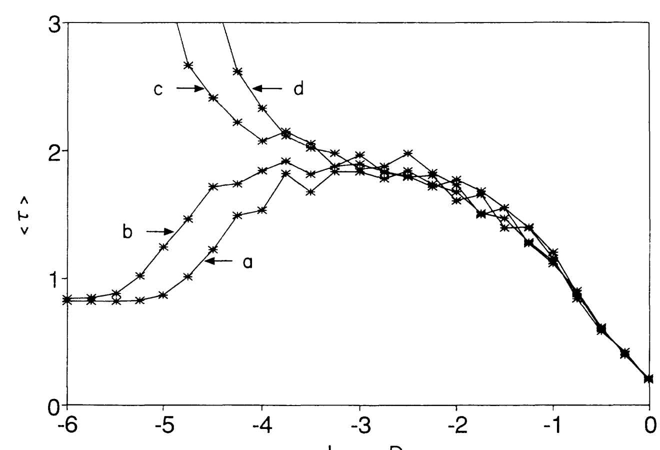

FIG. 4. The average escape time $\langle\tau\rangle$ plotted as a function of the logarithm of the noise amplitude for $A=0.3$, and the same frequencies as used in Fig. 2.

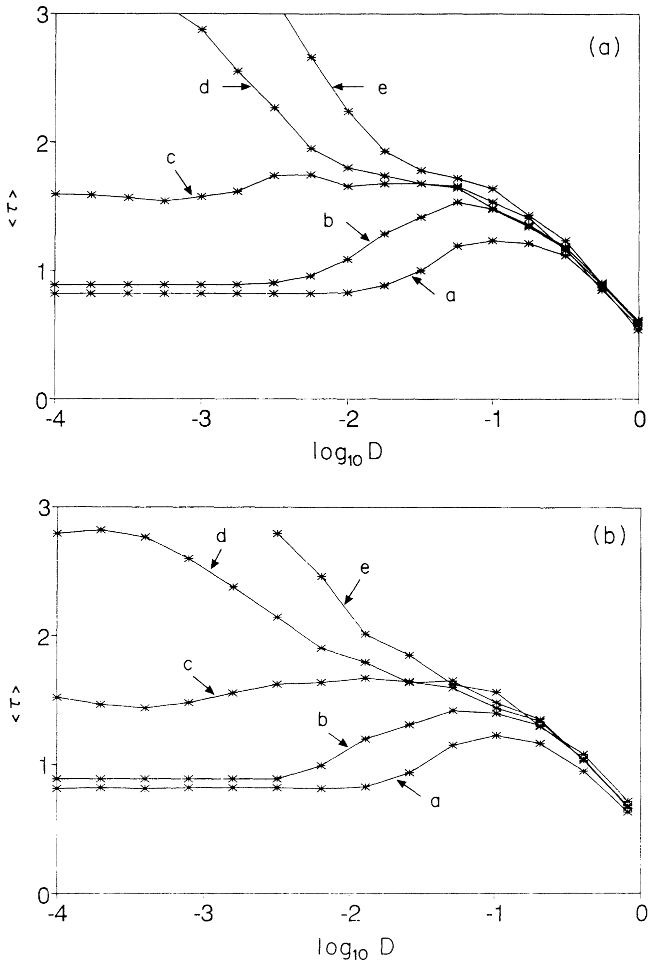

FIG. 5. A comparison of results obtained for the average escape time $\langle\tau\rangle$ with (a) $y(0)=1$ and (b) a Gaussian-distributed random variable with $\langle y(0)\rangle=1$ and $\sigma^2(y(0))=0.5$. The different curves correspond to different frequencies. These are (a) $\omega=0.510$, (b) $\omega=0.530$, (c) $\omega=0.540$, (d) $\omega=0.542$, and (e) $\omega=0.546$.

cle being initially at the well minimum, and then we calculated the average escape time $\langle\tau\rangle$ by averaging with respect to the results of 500 runs.

Some typical results obtained from our study are plotted in Fig. 4 as a function of the amplitude of the noise term. The frequencies correspond to those in Fig. 2. The most striking feature of the indicated results is that at some frequencies an increase in the noise amplitude induces an increase in $\langle\tau\rangle$. At sufficiently high amplitudes $\langle\tau\rangle$ will be determined entirely by the noise, leading to a coalescence of all of the curves. At the very lowest amplitudes the discrepancy between curves obtained roughly correspond to different escape times in the noise-free case. These are very sensitive to changes in frequency of the external field, as may be inferred from the curves in Figs. 2 and 3. If one focuses on particles that escape from the well during the first cycle, i.e., almost immedi-

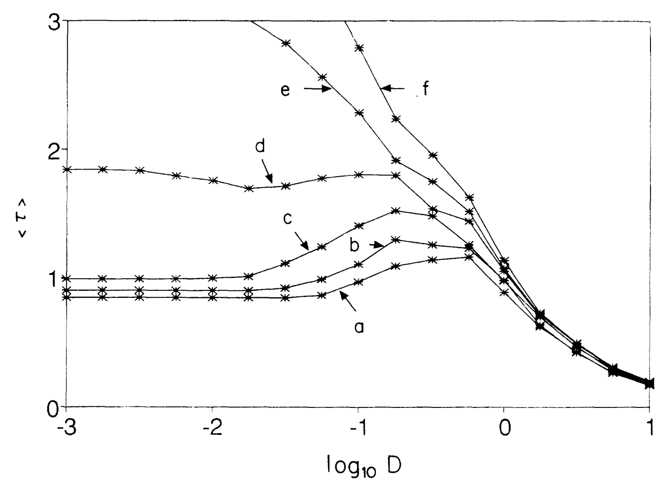

FIG. 6. A plot of the average escape time $\langle\tau\rangle$ for the larger amplitude $A=1$ and for the frequencies (a) $\omega=1.09$, (b) $\omega=1.14$, (c) $\omega=1.19$, (d) $\omega=1.24$, and (e) $\omega=1.34$.

ately, and ask how noise can affect the escape time, then one finds that the escape time must increase, since the noise can move a particle about to escape away from the top of the well, directing the motion towards the well. This is one route to stochastic resonance for the system under study.

A replacement of the initial condition $y(0)$ by a Gaussian distribution of this variable with $\langle y(0)\rangle=y(0)$ does not change the qualitative features of the resonant effect. This can be seen from the data shown in Fig. 5, which compares data obtained from runs with and without a distribution of initial positions. When the amplitude of the periodic term is increased, one needs a correspondingly larger value of D to change the process to an entirely noise-driven one and the maximum in $\langle\tau\rangle$ is also shifted to a higher value. This is illustrated by the curves in Fig. 6.

Let us mention some applications of our analysis. The kinetics of second-order phase transitions can be described by Eqs. (1) and (2). The competitive influence of a multiplicative periodic force and noise has been studied numerically in such systems [14] and experimentally [15]. Our system is different from the one studied in these references in that it describes the kinetics of a first-order phase transition under the combined influence of an additive periodic force and noise. Such a system could be studied experimentally by adding a periodic thermal pulse to a system consisting of supercooled water dispersed in oil which is undergoing solidification [16]. Equations of the type in Eq. (4) find a wide application in biology, chemistry, physics, and the social sciences. One such application outside the area of physics is the Verhulst-Pearl model equation for density-limited population growth [17]. In such an application the addition of a periodic force may, at some resonant condition, increase the time to reach population extinction, i.e., it may be useful for the survival of the species.

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