PHYSICAL REVIEW A VOLUME 46, NUMBER 12 15 DECEMBER 1992

Instabilities and nonstatistical behavior in globally coupled systems

Gabriel Perez
International Centre for Theoretical Physics, P. O. Box 586, 34100 Trieste, Italy

Hilda A. Cerdeira
*International Centre for Theoretical Physics, P. O. Box 586, 34100 Trieste, Italy
and Universidade Estadual de Campinas, Instituto de Fisica, 13081 Campinas, São Paulo, Brazil*
(Received 17 August 1992)

The mean field in a globally coupled system of chaotic logistic maps does not obey the standard rules of statistics, even for systems of very large sizes. This indicates the existence of intrinsic instabilities in its evolution. Here these instabilities are related to the very nonsmooth behavior of mean values in a single logistic map, as a function of its parameter. Problems of this kind do not affect a similar system of coupled tent maps, where good statistical behavior has been found. We also explore the transition between these two regimes.

PACS number(s): 05.45.+b, 05.90.+m

I. INTRODUCTION

In recent times there has been a number of efforts to analyze the interplay between temporal chaos and space synchronization in globally coupled systems. These are systems of considerable importance in modeling phenomena as diverse as Josephson-junction arrays, multimode lasers, vortex dynamics in fluids, and even evolutionary dynamics, biological information processing, and neurodynamics [1]. There is a great wealth of phenomena in these systems, originating in the presence of two conflicting trends in their dynamics. On one side, the presence of a common driving factor, coming from some type of average over the system, introduces a partial synchronization in the evolution of its elements. On the other, the chaotic divergence between the evolution of any two different elements tends to destroy this coherence. There are, therefore, two limiting behaviors, one in which a large coupling forces the synchronization of a set of weakly chaotic elements, and another in which strongly chaotic but weakly coupled systems display incoherent behavior. This last situation is characterized as having exponential divergence of trajectories not only in time —positive Lyapunov exponents—but also in space, in the sense that if at any given time two different elements of the system have very close magnitudes, those magnitudes will diverge from each other exponentially fast. Notice that for strong coupling it is possible to have all the elements of the system converge into a single cluster, and at the same time to have this cluster move chaotically [2].

In fact, at first sight these weakly coupled systems do not look too different from a simple lattice of uncoupled identical chaotic elements, with maybe some shifts in their parameters. A more careful study reveals, however, that there is a detectable and nontrivial influence of the global coupling, which gives rise to some subtle coherent effects, spoiling the statistical properties of the system.

II. GLOBALLY COUPLED LOGISTIC MAPS

Here we consider some of these coherence effects through the particular example of a globally coupled lattice of logistic maps, obeying the equations

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \frac{\epsilon}{N}\sum_{j=1}^{N} f(x_n(j)), \qquad (1)$$

where i is the space index and n is the time index. Here $f(x)$ is the familiar logistic map, $f(x) = 1 - ax^2$, and the mean field h at time n appears in the last term of the equation above,

$$h_n \equiv \frac{1}{N}\sum_{j=1}^{N} f(x_n(j)). \qquad (2)$$

This is a simple prototype of globally coupled chaotic systems, and has been exhaustively explored in Refs. [2–4]. For large a and small ϵ the system settles in a "turbulent" regime, where, as mentioned before, all elements $x(i)$ evolve chaotically, without any obvious mutual synchronization.

In this regime, it is reasonable to expect the mean field to obey general statistical rules, since it is an average over quasirandom variables. In particular, it was expected that h should converge to a fixed value h^* as $N \to \infty$, with fluctuations around this limiting value normally distributed (central limit theorem), and with a dispersion that decays as $1/\sqrt{N}$ (law of large numbers). Surprisingly, it was found that this simple system failed to fulfill these expectations [3, 4]. This failure has also been verified in similar models [5], which suggests that this is a generic behavior. In particular, it was found that the dispersion of the mean field did not go to zero, as expected, but instead saturated to a fixed positive value for large N; broad peaks indicating a quasiperiodic component were found in the Fourier spectrum of the time sequence for the mean field; and the mutual information on the lattice also saturated to a nonzero value for large N.

To understand the relevance of these facts, we should notice that, if in effect the mean field converged to a fixed value, the system would decouple. Each and every one of its elements would behave like a single logistic map of the form

$$y_{n+1} = 1 - A(a, \epsilon, h^*) y_n^2 \tag{3}$$

with $A = a(1-\epsilon)(1-\epsilon+\epsilon h^*)$ and $y = x/(1-\epsilon+\epsilon h^*)$, where the value of h^* is obtained self-consistently. In fact, this assumption of convergence of h to a fixed limit has been used successfully in the study of a different globally coupled nonlinear system [6]. For logistic maps, this reduction of the dynamics of the (infinite) lattice to that of a single map does not happen, which clearly implies that the self-consistency equation for h^* is unstable around its fixed points.

III. STATIC MEAN-FIELD MAPPING

A. Definition

Let us consider h, for the time being, not as a dynamical variable but as a fixed input in the system, and call it h_{in}. Taking the $N \to \infty$ limit on a lattice of the type described by Eq. (1), we can define a system of equations that gives as a final result a static mean field h_{out}, in the following manner:

$$h_{\text{out}} = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(x_n(j)), \tag{4}$$

$$x_{n+1}(i) = (1-\epsilon) f(x_n(i)) + \epsilon h_{\text{in}}. \tag{5}$$

This gives us a function $h_{\text{out}}(h_{\text{in}}; a, \epsilon)$, which we will call the "static mapping." In this simplified problem we can check whether or not the self-consistency equation $h_{\text{out}} = h_{\text{in}}$ has a solution, and explore its stability. Notice that h_{out} is invariant because of the existence of an invariant distribution for x [and therefore for $f(x)$] [7], when the maps are in the chaotic regime. For cases where the maps are in some periodic regime (and even when they are in chaotic motion inside some periodic window, as in parts of the 3-window), the existence of an invariant distribution depends on the distribution of initial conditions. We will assume that in these cases all different phases of the relevant cycle are equally represented, so that an invariant distribution can be achieved.

It should be clear that this static mapping is not equivalent to the actual evolution of the mean field, $h_n = h_n(h_{n-1}, h_{n-2}, \ldots; a, \epsilon)$, also defined in the $N \to \infty$ limit. This "dynamic mapping" depends in principle on all previous values of h, although this dependence is negligible for very old h (i.e., for h_{n-m} when $m \gg 1$), and exhibits therefore a much richer behavior. What is important for us here is that they have the same fixed points. On the stability of these fixed points we propose the following hypothesis: *the dynamic mapping $h_n = h_n(h_{n-1}, h_{n-2}, \ldots)$ cannot be stable around its fixed point $h_i = h^*$, $i = n, n-1, \ldots$, if the static mapping is not.* Basically, we are assuming that if the process is unstable even in the very simplified form given by the static mapping, the complexities introduced by the dependence on all previous values of h cannot make its stability anything but worse. The numerical results verify this statement, as we will see next.

B. Numerical results

We have evaluated numerically the static mapping in the range of h_{in} that contains the fixed points $h_{\text{out}} = h_{\text{in}}$ for the parameters $a = 1.99$ and $\epsilon = 0.1$. The results are shown in Fig. 1. Although this is an extremely nonsmooth function, it has to be continuous, since for the different types of bifurcations present in the logistic map the average value of x changes continuously [8]. The fixed points in this graph give $h^* \approx 0.311$, not too different from the actual average of the mean field ($\langle h \rangle \approx 0.3063$), but different enough to imply that $\langle h \rangle$ does not fall on a fixed point. It is clear from the graph that none of these fixed points can be stable, since the absolute slopes $|\Delta h_{\text{out}}/\Delta h_{\text{in}}|$ obtained numerically are much larger than 1 almost everywhere. We should keep in mind that only 300 points have been calculated to get this figure, and therefore these slopes are defined only in a coarse-grained sense. In fact, the function $h_{\text{out}}(h_{\text{in}})$ has well-defined derivatives only inside its periodic windows. Therefore, even though this function cannot reveal all the complexity of the actual mapping $h_n = h_n(h_{n-1}, h_{n-2}, \ldots)$, its nonsmooth behavior is indicative of why h_n does not converge to an invariant value as $N \to \infty$.

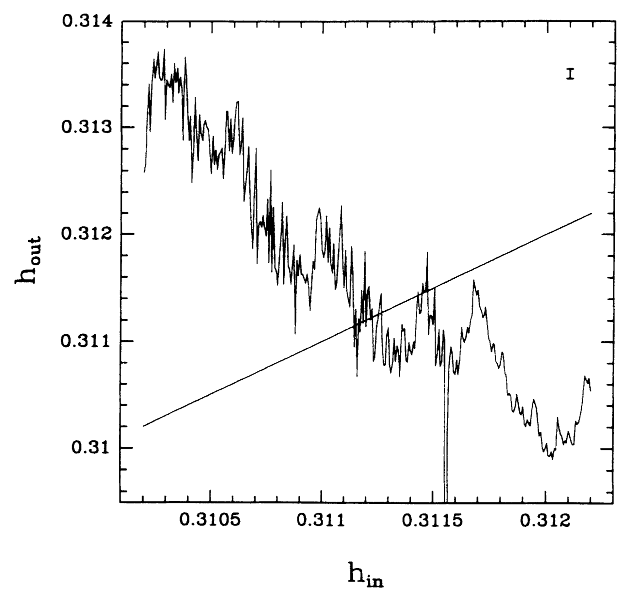

FIG. 1. Static mapping $h_{\text{out}}(h_{\text{in}})$ from Eq. (1). The line joins 300 points calculated over equally spaced h_{in} values. These points were obtained averaging over 1.5×10^8 iterations, after a transient of 4000 iterations. The straight line is the diagonal $h_{\text{out}} = h_{\text{in}}$. The typical error level is indicated with the error bar in the upper-right corner. Here $a = 1.99$, $\epsilon = 0.1$, and $f(x)$ is the logistic map.

The presence of the "well" visible in Fig. 1, and of which an enlarged view is given in Fig. 2, deserves some comment. The bottom of the well corresponds to a periodic 14-window that, as is common in the logistic map, begins in a tangent bifurcation and ends in an internal crisis. The infinite slope at the left end of the periodic window is due to the fact that at both sides of a tangent bifurcation in the logistic map the average value of x changes as $\langle x\rangle - \langle x\rangle_c \approx |a-a_c|^{1/2}$, where a_c is the critical parameter for the bifurcation. This is also true on the one-band side of an internal crisis [9], where the probability density spreads from the several bands at one side into the one on the other, also as $|a-a_c|^{1/2}$. This explains the infinite slope at the right end. These two facts, together with the continuity of $\langle x\rangle$ in period-doubling bifurcations, sustain our assertion that $h_{\rm out}(h_{\rm in})$ is continuous. These "wells" and their infinite-slope walls should not be isolated instances in the $h_{\rm out}$ versus $h_{\rm in}$ graph, since the periodic windows from where they arise are thought to be dense in the bifurcation diagram of the logistic map [7, 10]. This is what makes it impossible for the map to have a derivative except inside a periodic window.

As pointed out in Ref. [4], all these peculiar phenomena disappear if we change $f(x)$ in the set of equations (1) to a tent map, $f(x) = 1-a|x|$. For this system, the mean field h_n seems to converge to a limit, with fluctuations that decay as $1/\sqrt{N}$, as expected. A look to the bifurcation diagram for the tent map shows a complete absence of periodic windows, tangent bifurcations, or internal crisis, and suggests a smooth behavior of $\langle x\rangle$ as a function of a, which of course would imply a smooth behavior in $h_{\rm out}(h_{\rm in})$. This has been verified numerically, for $a = 1.99$ and $\epsilon = 0.1$. The results are shown in Fig. 3. The curve $h_{\rm out}$ versus $h_{\rm in}$ obtained here is extremely smooth, within our levels of error, and has a very small slope. The fixed point is $h_{\rm out} = h_{\rm in} = 0.1787$, in perfect agreement with the calculated value of $\langle h\rangle$. Therefore, the simplified

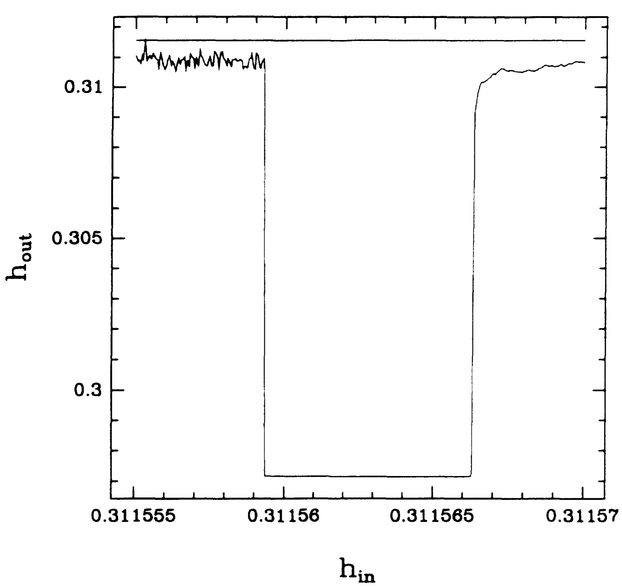

FIG. 2. Enlargement of the "well" visible in Fig. 1. The bottom corresponds to a periodic 14-window, the left wall is a tangent bifurcation, and the right wall is an internal crisis. The line joins 300 points calculated as averages over 7.5×10^7 iterations, after a transient of 4000 iterations. The straight line is the diagonal $h_{\rm out} = h_{\rm in}$. All other parameters are as in Fig. 1. Typical error bars are not significant at the scale of the figure.

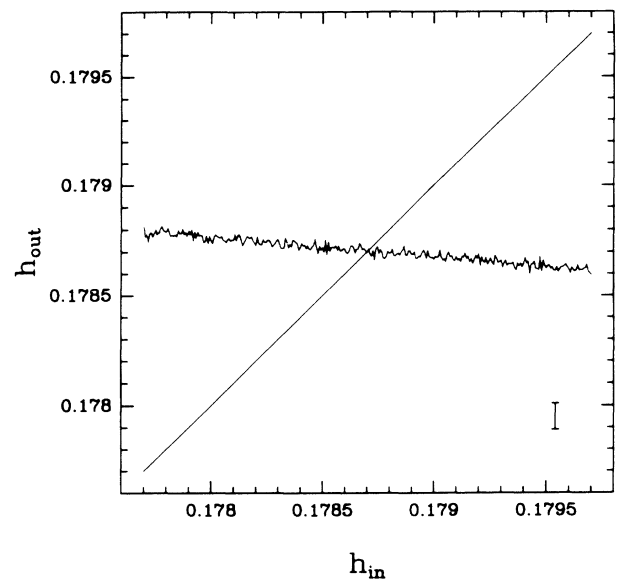

FIG. 3. Static mapping $h_{\rm out}(h_{\rm in})$ for the tent map. The line joins 300 points calculated as in Fig. 1. The straight line is the diagonal $h_{\rm out} = h_{\rm in}$, and a typical error bar is given in the lower-right corner.

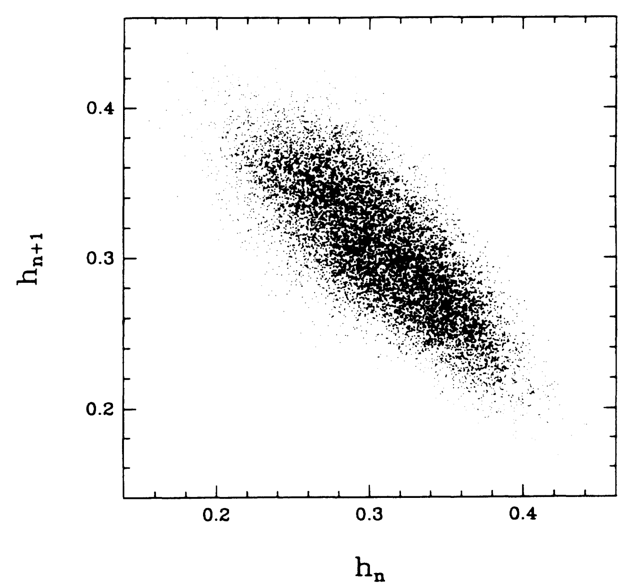

FIG. 4. Distributions of the values of the mean field h_{n+1} vs h_n in the dynamic mapping, where $f(x)$ is the logistic map. These results are for a lattice of size $N = 200\,000$. Here we have plotted 10 000 points, after a transient of 5000 iterations. Other parameters are as in Fig. 1.

static mapping does not suggest instabilities in the more complex dynamic mapping.

In summary, the strong instability of the static mapping $h_{\rm out}(h_{\rm in})$ is a good indicator of the lack of convergence of the mean field h to a fixed value as N grows. The convergence of h towards such a limit in the system with tent maps is accompanied by a smooth and almost flat $h_{\rm out}(h_{\rm in})$. However, we should not forget that this is only a static construction, and cannot represent the full dynamics of the problem. As a matter of fact, the plot of h_{n+1} versus h_n, obtained for a value of N such that the fluctuations have reached their saturation level, shows a very different behavior, as can be seen in Fig. 4. So we have to keep in mind that the static function tells us about the impossibility of achieving a fixed value for h in the $N \to \infty$ limit, but it does not say anything about the actual evolution of this quantity.

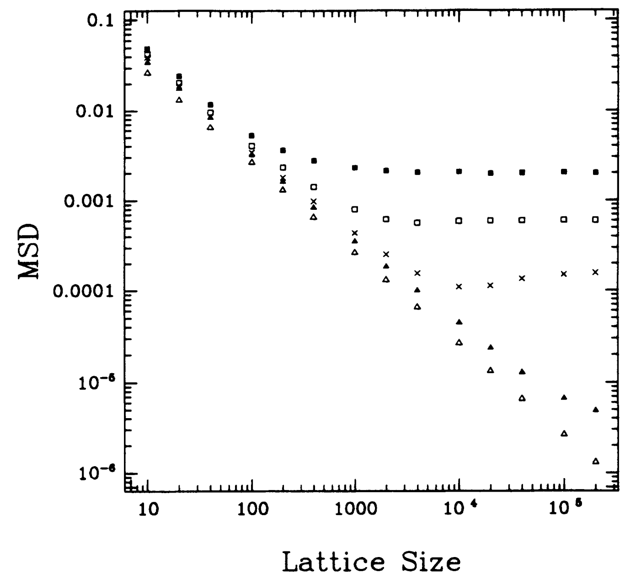

FIG. 5. Mean-square deviation for the mean field vs lattice size for several values of α, in the mixed map. For all points we have used a total of 102 400 iterations, with a transient of 5000 iterations. The values of α are (■) $\alpha = 1.0$ (logistic), (□) $\alpha = 0.95$, (×) $\alpha = 0.9$, (▲) $\alpha = 0.75$, (△) $\alpha = 0.0$ (tent). As before, $a = 1.99$ and $\epsilon = 0.1$.

IV. INTERMEDIATE CASES: MIXING TENT AND LOGISTIC MAPS

Given the fact that nonstatistical behavior is present in logistic but not in tent maps, it is natural to ask what happens for intermediate situations. For this we have considered a "logistic plus tent" map that interpolates between quadratic and linear behavior. It is given by

$$f_{lt}(x) = 1 - a[\alpha x^2 + (1-\alpha)|x|]. \tag{6}$$

It has as limits the tent map, when $\alpha = 0$, and the logistic map, when $\alpha = 1$. There are, of course, many other ways of interpolating between these two limits, a simple one being the power map $f_p(x) = 1 - a|x|^\gamma$, with $1 \le \gamma \le 2$. For concreteness, we will consider here only the function given in Eq. (6).

We have explored numerically the behavior of coupled lattices of these maps. The results for the mean-square deviation (MSD) of the mean field for α close to 1 show clear nonstatistical behavior, which seems to disappear monotonically with decreasing α. (See Fig. 5.) A very interesting feature here is the slight but consistent recovery of the values of the MSD for values of α less than 1, up to the value of saturation of the MSD. A similar phenomenon was found in Ref. [4], in a coupled lattice of logistic maps subject to the influence of static parametric fluctuations.

A much stronger evidence of coherence is found in the power spectrum of the mean field. As mentioned before, one of the signals of nonstatistical behavior in these systems is the appearance of broad peaks in the power spectrum, indicating a quasiperiodic component in the evolution of the system. As can be seen in Fig. 6, this quasiperiodicity is strongly accentuated in the case of maps with a small tent component ($1 - \alpha \approx 0.1$). The quasiperiodic behavior is strong enough as to be visible in the h_{n+1} versus h_n plot, as shown in Figs. 7 and 8. Obviously, as we make α even smaller this trend reverses and the power spectrum becomes almost flat.

This increase in the quasiperiodicity of the mean field has been encountered in two other cases: in the presence of a very small additive noise [11], and when the mean field is not global but includes only the $N/2$ nearest neighbors [12]. These three cases are similar in that all of them point to a connection between small smoothly distributed noise and an increase in quasiperiodicity. In our case, we could roughly consider the tent part of our map as a perturbation over the logistic part (for α close to 1), since one part is added to the other to obtain the total

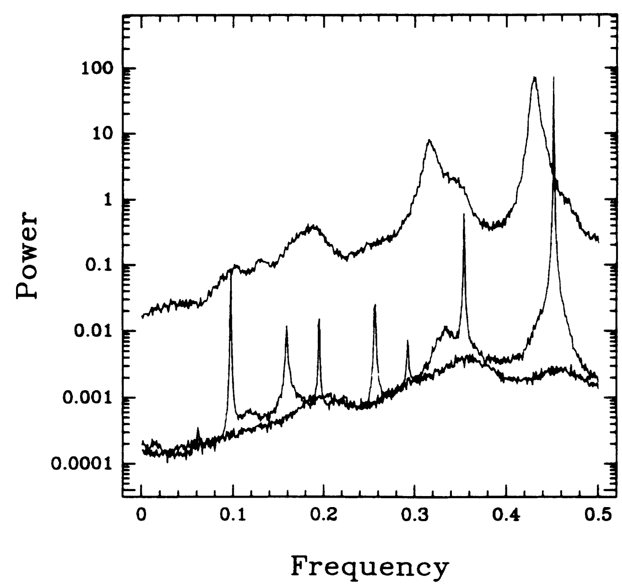

FIG. 6. Power spectra for the mean field for $\alpha = 1$ (upper line), $\alpha = 0.9$ (middle line), and $\alpha = 0.0$ (lower line). Here we are averaging over 100 runs of 1024 iterations each, after a transient of 5000 iterations. The parameters are $a = 1.99$ and $\epsilon = 0.1$.

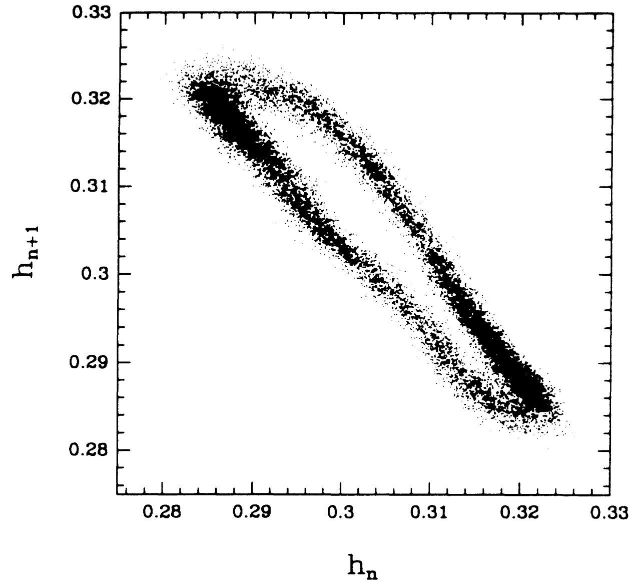

FIG. 7. Distributions of the values of the mean field h_{n+1} vs h_n in the dynamic mapping, where $f(x)$ is the mixed "logistic plus tent" map. Here $\alpha = 0.9$. All other parameters are as in Fig. 4.

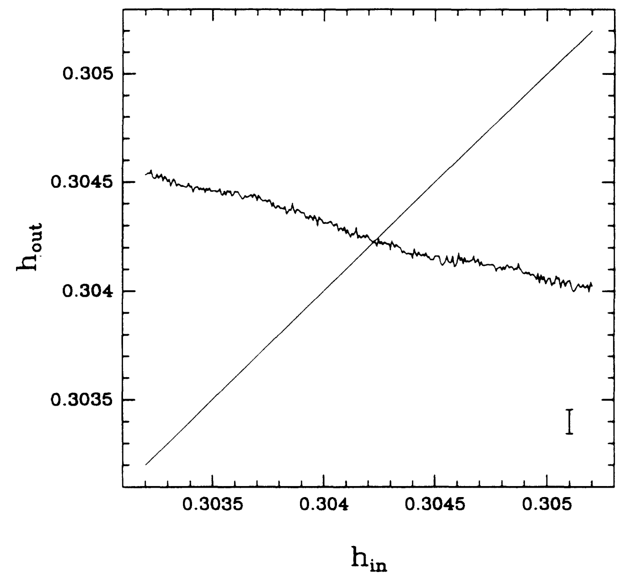

FIG. 9. Static mapping $h_{\text{out}}(h_{\text{in}})$ for the mixed "logistic plus tent" map. The line joins 300 points calculated as in Fig. 1. The straight line is the diagonal $h_{\text{out}} = h_{\text{in}}$, and a typical error bar is given in the lower-right corner.

mapping. Obviously, this is a highly correlated perturbation; however, taking into account that the tent map has a behavior closer to white noise (its invariant distributions for $\alpha \lesssim 2.0$ are almost flat) than that of the logistic map, the connection between these two processes is at least plausible. Notice that here we cannot invoke the influence of some periodic window for this increase in quasiperiodicity; first because periodic windows are almost nonexistent in the bifurcation diagram of $f_{lt}(x)$ for $\alpha = 0.9$, and second because this would make it difficult to explain why the total strength of the signal decreases, i.e., why the MSD goes down as we decrease α. A similar argument can be made for the semiglobally coupled map, in the sense that the influence of the elements of the lattice that are not directly affected by the—now local—mean field can be roughly considered as a smoothly distributed small noise.

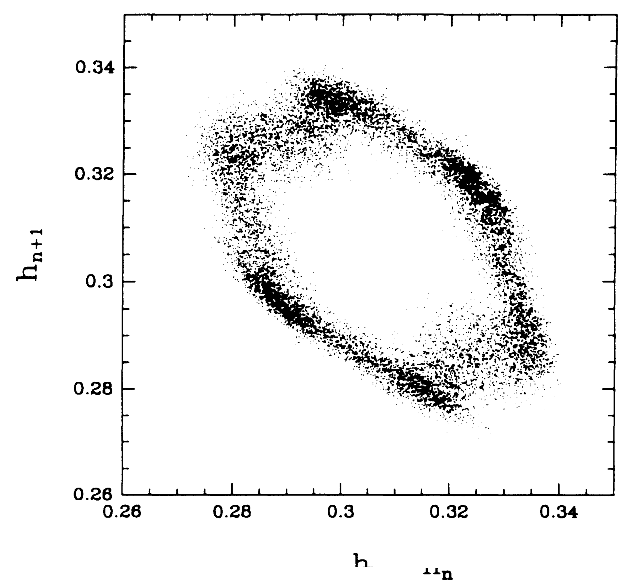

FIG. 8. Distributions of the values of the mean field h_{n+1} vs h_n in the dynamic mapping. Here $f(x)$ is the logistic map, as in Fig. 4, but we have added a uniformly distributed noise of amplitude $\sigma = 0.0045$. All other parameters are as in Fig. 4.

Finally, we have also checked, for these mixed maps, the behavior of the static mapping $h_{\text{out}}(h_{\text{in}})$. Result for $\alpha = 0.9$ can be seen in Fig. 9. Since the mixed map has a negative Schwarzian derivative except at $x = 0$, where derivatives are not defined, we expect to find only one attractor, and therefore a well-defined $\langle x \rangle$, independent of the initial value x_0. The behavior of the static mapping seems smooth and already (for this value of α) close to that of the tent map. Within our error levels, the curve still shows some structure. A careful look at the bifurcation diagram of this map shows that almost all the periodic windows have disappeared —this is due to the tent-like behavior of the map at its critical point—thus eliminating the multiple points of infinite slope in the h_{out} versus h_{in} graph. The coarse-grained slopes $|\Delta h_{\text{out}}/\Delta h_{\text{in}}|$ obtained here are much smaller than 1.

Therefore, the results for this case indicate that the stability of the static mapping (at least in the coarse-grained sense we have considered) is not sufficient to insure the stability of the actual dynamics. Our numerical results are of course insufficient to describe the behavior of the actual derivative $dh_{\text{out}}/dh_{\text{in}}$ (or, equivalently, $d\langle x \rangle/da$) in these maps, and may still allow for differentiability in the tent map and nondifferentiability in the mixed cases.

V. CONCLUSIONS

The behavior of the mean field in globally coupled chaotic systems contains a number of surprises. The nonstatistical behavior of this quantity indicates the existence of an intrinsic instability in the evolution of the system, when we consider its infinite-size limit. Here we have explored the relationship between this instability and the corresponding problem in a simplified mapping for the mean field, which assumes that the dynamics depends only on the last value of this quantity. This is a very crude approximation, since it assumes an infinitely fast relaxation of the probability densities of the process, but it still gives information about its fixed points and some idea about their stability.

The numerical results obtained here indicate that the stability of this static mapping may be a necessary but not sufficient condition for the stability of the actual dynamics, i.e., for a normal statistical behavior of the mean field on the system. This result should be taken only as a first step in the study of the behavior of this kind of problem. In principle, a complete program should be carried out through the analysis of the stability of the eigenmodes of the Perron-Frobenious equation of the system, a point that has been mentioned in Ref. [4].

Under the influence of the previously mentioned instabilities, the mean field develops a dynamics that is weakly quasiperiodic. This is already unexpected, and gives rise to some as yet unresolved questions, as, for instance, what is the mechanism that selects the dominant frequencies? Even more remarkable is the fact that several mechanisms have already been found to strongly increase this quasiperiodicity, and none of them can be considered a form of periodic driving. On the contrary, directly or indirectly all of them can be assimilated into the addition of a small white noise. Also, this increase in quasiperiodicity is accompanied by a reduction of the total strength of the signal.

Finally, we want to mention that there has been recent evidence showing that the phenomena we have explored here also appear in locally coupled systems. Periodicity and quasiperiodicity have been observed in some totalistic cellular automata in 3, 4, and 5 dimensions [13], in medium-range coupled one-dimensional lattice maps [12, 14], and in locally coupled high-dimensional lattice maps [15]. All of this wealth of evidence says that there should be a common and fairly robust mechanism that extracts periodic behavior out of coupled chaos. The precise nature of this mechanism is still unknown.

ACKNOWLEDGMENTS

G.P. wants to thank Professor S.-J. Chang for stimulating discussions and for valuable information. We thank the Istituto Nazionale de Fisica Nucleare (INFN) for computing facilities.

[1] J. Crutchfield and K. Kaneko, in *Directions in Chaos*, edited by B.-L. Hao (World Scientific, Singapore, 1987); P. Alstrom and R. K. Ritala, Phys. Rev. A **35**, 300 (1987); P. Hadley and K. Weisenfeld, Phys. Rev. Lett. **62**, 1335 (1989).

[2] K. Kaneko, Phys. Rev. Lett. **63**, 219 (1989); Physica **41D**, 137 (1990).

[3] K. Kaneko, Phys. Rev. Lett. **65**, 1391 (1990).

[4] K. Kaneko, Physica **55D**, 368 (1992).

[5] K. Kaneko, Physica **54D**, 5 (1991); G. Perez, C. Pando-Lambruschini, S. Sinha and H. A. Cerdeira, Phys. Rev. A **45**, 5469 (1992); S. Sinha, D. Biswas, M. Azam, and S. V. Lawande *ibid.* **46**, 3193 (1992).

[6] Y. Kuramoto and I. Nishikawa, J. Stat. Phys. **49**, 569 (1987); H. Daido, J. Stat. Phys. **60**, 753 (1990); S. H. Strogatz and R. E. Mirollo, J. Stat. Phys. **63**, 613 (1991).

[7] P. Collet and J. P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems* (Birkhaüser, Boston, 1980).

[8] S.-J. Chang (private communication).

[9] S.-J. Chang and J. Wright, Phys. Rev. A **23**, 1419 (1981).

[10] J. D. Farmer, Phys. Rev. Lett. **55**, 351 (1985).

[11] G. Perez, S. Sinha, and H. Cerdeira, Physica **D** (to be published).

[12] S. Sinha, D. Biswas, M. Azam, and S. V. Lawande, Phys. Rev. A **46**, 6242 (1992).

[13] H. Chaté and P. Manneville, Europhys. Lett. **14**, 409 (1991); J. Hemmingsson (unpublished); J. A. C. Gallas, P. Grassberger, H. J. Herrmann, and P. Ueberholz, Physica **180A**, 19 (1992); P.-M. Binder, B. Buck, and V. A. Macaulay, J. Stat. Phys. (to be published).

[14] P.-M. Binder and V. Privman (unpublished).

[15] H. Chaté and P. Manneville, Europhys. Lett. **17**, 291 (1992).