Tracking unstable orbits in experiments

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We present an alternative continuation method for tracking unstable periodic orbits by slowly varying an available system parameter. This is a predictor-corrector method for which we assume that initially the orbit is on a chaotic attractor. As we vary the parameter, the method can be used to track the orbit through regimes not necessarily chaotic. The method is designed for experimental situations in which we have no analytical knowledge of the system dynamics and only an experimental time series of the variables involved is available. We present numerical results of the method using the Hénon map.

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I. INTRODUCTION

When modeling a dynamical system, theoretical tools have been developed which allow the location of steady-state, periodic, and aperiodic phenomena. Furthermore, numerical methods have been developed which allow both stable and unstable phenomena to be followed as a function of parameters. In a parallel manner, recent progress in the theory of nonlinear dynamical systems has provided the experimentalist with a collection of new tools which have generated new areas of exploration from the measurement of a single time series. For example, if the experiment exhibits deterministic chaos, by measuring a single time series, the relevant dynamics may be constructed by using any one of a number of embedding techniques (see [1–3]). In addition, once the attractor is constructed, unstable periodic orbits contained in the attractor may be located using techniques such as those found in [4]. By making use of such information, new prediction improvements have been made, as well as new stabilizing methods. For example, if an unstable orbit contained in the chaotic attractor is desired, it is possible to control the orbit by performing small-amplitude fluctuations about some desired parameter value (see [5]).

In this paper, we extend these tools to include continuation methods for experimentalists. That is, we present a method which follows an unstable periodic orbit as a function of a single parameter. The method is based only on experimental data, and makes use of any small-amplitude control technique.

In what follows we consider a smooth, two-dimensional map:

$$\mathbf{x}_{n+1}=f(\mathbf{x}_n,p)\ ,\quad \mathbf{x}_n,\mathbf{x}_{n+1}\in\mathbb{R}^2\ , \tag{1.1}$$

depending on the parameter p. The map in Eq. (1.1) is generated by taking a Poincaré section of a chaotic attractor constructed from a time series. The problem we consider is to follow a given saddle orbit of this map as we vary the parameter p. For the initial value of $p=p_0$, we assume that the orbit lies on a chaotic attractor. This assumption allows us to find a good initial approximation for the orbit, by using the control method of Ott, Grebogi, and Yorke (OGY), for stabilizing an orbit on a chaotic attractor. Furthermore, we could assume that we know the orbit for a few initial values of p, in some interval in which there is a chaotic attractor. The number of these values depends on the version of the prediction method employed. In Sec. III we review a simple prediction-correction technique to solve this problem when the map is known. In Sec. II we recall the OGY control method, which is used to initialize our method, as well as in the correction step. In Sec. IV we present the two versions of our method, based on a prediction-correction method. In Sec. V we present numerical results using this method to track a period-1 orbit of the Hénon map. We end the paper with conclusions and future plans, which include the extension of the method to flows and its use for locating new attractors.

II. THE OGY CONTROL METHOD

The OGY method is meant for stabilizing a saddle orbit on a chaotic attractor, generated from a time series. This is done by making small perturbations of an available system parameter in a manner which will be briefly described below. Consider the two-dimensional discrete dynamical system (1.1) depending on a parameter p, which is allowed to fluctuate about some value p_0,

$$p_0+p_*>p>p_0-p_*\ .$$

Without loss of generality we can assume $p_0=0$. Denote $\mathbf{x}_F=\mathbf{f}(\mathbf{x}_F,p_0)$ as the fixed point one wants to stabilize. In what follows we denote $\xi_n=\mathbf{x}_n-\mathbf{x}_F$ and describe the algorithm for the map

$$\xi_{n+1}=\mathbf{P}(\xi_n,p)=\mathbf{f}(\mathbf{x}_n,p)-\mathbf{f}(\mathbf{x}_F,p)\ , \tag{2.1}$$

for which the fixed point is at $\xi=\mathbf{0}$. Due to the ergodicity of the chaotic attractor, the iterates will fall close to the fixed point, if we iterate (2.1) long enough. When this happens the control is activated and we change the parameter p in such a way that the next iterate will fall close to the stable manifold of the fixed point. To achieve this, one uses a linear approximation for p small enough:

$$\xi_{n+1} = \mathbf{P}(\xi_n, p) \cong \xi_F + A(\xi_n - \xi_F) \ . \tag{2.2}$$

We then approximate

$$\mathbf{g} \equiv \left. \frac{\partial \xi_F(p)}{\partial p} \right|_{p=0} \cong \frac{1}{\bar{p}} \xi_F(\bar{p}) \ ,$$

so that (2.2) becomes

$$\xi_{n+1} \cong p_n \mathbf{g} + [\lambda_u \mathbf{e}_u \mathbf{f}_u + \lambda_s \mathbf{e}_s \mathbf{f}_s] \cdot (\xi_n - p_n \mathbf{g}) \ . \tag{2.3}$$

In Eq. (2.3) $\mathbf{e}_s$, $\mathbf{e}_u$, λ_s, and λ_u are the eigenvectors and eigenvalues of the Jacobian of the map $\mathbf{P}$ and $\mathbf{f}_s$ and $\mathbf{f}_u$ are contravariant basis vectors defined by $\mathbf{f}_s \cdot \mathbf{e}_s = \mathbf{f}_u \cdot \mathbf{e}_u = 1$, $\mathbf{f}_s \cdot \mathbf{e}_u = \mathbf{f}_u \cdot \mathbf{e}_s = 0$. To choose p_n we proceed as follows: if ξ_n falls near the fixed point $\xi = \mathbf{0}$, we pick p_n so that ξ_{n+1} falls on the stable manifold of $\xi = \mathbf{0}$, otherwise $p_n = 0$. So the condition for control is

$$\mathbf{f}_u \cdot \xi_{n+1} = 0 \ .$$

Solving in the above for p_n, we get

$$p_n \equiv \frac{\lambda_u \xi_n \cdot \mathbf{f}_u}{(\lambda - 1)\mathbf{g} \cdot \mathbf{f}_u} \equiv \mathbf{C} \cdot \xi_n \ . \tag{2.4}$$

We assume $\mathbf{g} \cdot \mathbf{f}_u \neq 0$, and that

$$|p_n| < p_* \tag{2.4'}$$

holds, otherwise we set $p_n = 0$. Thus in the algorithm, the projection of ξ_n on the stable manifold of $\xi = \mathbf{0}$ has to satisfy

$$|\xi_n^u| < \xi_* \ ,$$

where by Eqs. (2.3) and (2.4′) $\xi_* = p_* |(1 - \lambda_u^{-1}) \mathbf{g} \cdot \mathbf{f}_u|$. A noise term $\epsilon \delta_n$ can be added to the right side of (2.3), where δ_n is a random variable and ϵ is the intensity of the noise. The quantities δ_n have zero mean ($\langle \delta_n \rangle = 0$), satisfy $\langle \delta_n \delta_m \rangle = 0$ for $m \neq n$, have unit mean-square value ($\langle \delta_n^1 \rangle = 1$), and have a probability density independent of n. If the noise is bounded,

$$|\delta_n^u \equiv \mathbf{f}_u \cdot \delta_u| < \delta_{\max} \ ,$$

then the control will be little affected by the noise, provided

$$\epsilon \delta_{\max} < \xi_* \ .$$

The algorithm above can be used in experimental situations where the dynamical system is not explicitly known, but where one can determine many experimental points form a chaotic time series:

$$\xi_1, \xi_2, \xi, \ldots, \xi_k \ ,$$

corresponding to $p = 0$. Again for simplicity let our orbit be a fixed point $\xi = \xi_F = \mathbf{0}$. The eigenvalues and eigenvectors can be determined experimentally also. To determine the vector $\mathbf{g}$ above we use the following approximation:

$$\mathbf{g} \equiv \frac{\partial \xi_F(p)}{\partial p} \cong \frac{1}{\bar{p}} \xi_F(\bar{p}) \ ,$$

in which $\xi_F(\bar{p})$ is available experimentally, for $\bar{p}$ close to $p = 0$. For these experimentally determined quantities, Eq. (2.2) will hold and the algorithm can proceed as before.

III. THE CONTINUATION METHOD WHEN THE MAP IS KNOWN

In this section we show how to track unstable orbits, as a function of parameter, when the equations of the map are known. We review only the most basic of these so-called continuation methods to motivate the algorithm. If the map $\mathbf{f}$ is known, locating the fixed points of $\mathbf{f}$ is equivalent to locating the zeros of $\mathbf{G}$, where $\mathbf{G}(\mathbf{x},p) = \mathbf{f}(\mathbf{x},p) - \mathbf{x}$. In general, continuation techniques are methods to find the solution field of an equation of the form

$$\mathbf{G}(\mathbf{x},p) = \mathbf{0} \ , \quad \mathbf{G}: R^{n+1} \to R^n \ . \tag{3.1}$$

We are interested in solving Eq. (3.1) for $\mathbf{x}$ regarded as a function of p. That is, we want to determine the curve $\mathbf{x}(p)$, starting at a given initial point $(\mathbf{x}_0, p_0)$, which satisfies $\mathbf{G}(\mathbf{x}(p),p) = \mathbf{0}$. For this we differentiate in Eq. (3.1) obtaining

$$D_{\mathbf{x}} \mathbf{G}(\mathbf{x},p) \mathbf{x}'(p) + D_p \mathbf{G}(\mathbf{x},p) = \mathbf{0} \ ,$$

$$\mathbf{x}(p_0) = \mathbf{x}_0 \ .$$

Suppose $(\mathbf{x}(p_0), p_0)$ is not a saddle-node bifurcation point. Then $D_{\mathbf{x}} \mathbf{G}(\mathbf{x}_0, p_0)$ is nonsingular, and there will be a unique solution $\mathbf{x}(\mathbf{p})$ satisfying

$$\mathbf{x}'(p) = -[D_{\mathbf{x}} \mathbf{G}(\mathbf{x},p)]^{-1} D_p \mathbf{G}(\mathbf{x},p) \ ,$$

$$\mathbf{x}(p_0) = \mathbf{x}_0 \ .$$

We will describe next a typical basic step in a continuation method. The method generates iteratively a sequence $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k, \ldots$ which represents an approximation of the curve $\mathbf{x}(p)$. This sequence is obtained as follows. Given $\mathbf{x}^k$, let $p^{k+1} = p^k + h^k$. In order to obtain $\mathbf{x}^{k+1}$, we first determine

$$\mathbf{T}(\mathbf{x}^k, p_k) = -[D_{\mathbf{x}} \mathbf{G}(\mathbf{x}^k, p_k)]^{-1} D_p \mathbf{G}(\mathbf{x}^k, p_k)$$

which is an approximation to the tangent to $\mathbf{x}(p)$. Then for a suitable step h_k along this tangent direction, the predicted point will be $\hat{\mathbf{x}}_{k+1} = \mathbf{x}_k + h_k \mathbf{T}(\mathbf{x}_k, p_k)$. Now a corrector step is applied, which produces a sequence of iterates converging to $\mathbf{x}^{k+1}$ on the continuation curve, in the n-dimensional hyperplane perpendicular to the curve and passing through $\hat{\mathbf{x}}_{k+1}$. In [6–8] a more general class of continuation methods is presented and analyzed in detail.

We now extend the prediction-correction scheme based on an Euler step and a convergent corrector to a chaotic time series.

IV. THE EXPERIMENTAL CONTINUATION METHOD

In this section, we will introduce the two versions of our method to track orbits of a map, which is going to be treated as a black box, the equations of this map not being known. The preceding section motivates a technique which is similar in that a prediction of a saddle at a new parameter is made, and then a correction must be made to bring the saddle back onto the branch of saddles. Our method is implemented first in R^2. Let the successive values of this map be denoted as in (1.1) in which $\mathbf{f}$ are not explicitly known. For simplicity we present the method when the orbit is a fixed point, i.e., we describe the algorithm for tracking $\mathbf{x}_F(p)$ as p varies, where $\mathbf{x}_F(p)=\mathbf{f}(\mathbf{x}_F,p)$.

Algorithm 1. All this method requires is to have the approximate values of the fixed point and corresponding eigenvectors and eigenvalues for some initial values of the parameter p, the subsequent values as we increase (or decrease) p being determined by a predictor-corrector method.

Initially we consider $p_0=0$ and we assume that for p close to zero the map has a chaotic attractor. With the $p=p_0$ fixed, we can iterate the map long enough so that, by the ergodicity of the chaotic attractor it will get close enough to the fixed point we want to track.

Once this happens, we can use OGY algorithm to get a good approximation of this fixed point. We assume that the corresponding eigenvectors and eigenvalues can be measured (see [4]). Our methods requires us to find in the same way a few other nearby points on chaotic attractors, for slightly different parameters, and use all these points for predicting the location of the fixed point as the parameter is further increased.

That is, for fixed nonfluctuating parameters $p_1,p_2,\ldots,p_m$, we use OGY to find approximate values $\mathbf{x}_F(p_1),\mathbf{x}_F(p_2),\ldots,\mathbf{x}_F(p_m)$. (Applying the method to the Hénon map required $m=4$, see Sec. V.)

We fit a line through these points, regarding this line as a function of the parameter p. We determine the best approximating line, when the error involved is the sum of the squares of the differences between the values on the approximating line and the given values. We then slightly increase the parameter p. The point on the line corresponding to this increased value will be our predicted value for the fixed point. We use exactly the same linear procedure to predict the eigenvalues of the fixed point, using the eigenvalues of the preceding points. To predict the eigenvectors we apply this linear prediction component wise.

When the parameter is increased, the parameter step size is chosen sufficiently small so that the unstable fixed point is controlled. (For example, a step Δp of at most 0.02 can be used for the Hénon map, when 5% noise is present, see Sec.V.) However, prediction without a correction of the control point will result in loss of control.

To correct the predicted value of the fixed point, we proceed similarly to the OGY algorithm. Suppose $\mathbf{x}_{\text{fix}}(p)$ is the predicted fixed point. Here $p=p_m+\Delta p$. To correct this value, we slightly change the parameter to some value $p+\delta p$, where δp is to be determined. The idea is to ensure that the next iterate $\mathbf{x}_{n+1}$ in (1.1) will fall on the stable manifold of $\mathbf{x}_{\text{fix}}(p)$.

Let $\hat{\mathbf{x}}_F(p)=\mathbf{x}_F(p_m)$ be the previously controlled fixed point. We now approximate locally:

$$\mathbf{x}_{n+1}-\mathbf{x}_{\text{fix}}(p)\cong\delta p\,\mathbf{g}+[\lambda_u\mathbf{e}_u\mathbf{f}_u+\lambda_s\mathbf{e}_s\mathbf{f}_s]\cdot[\hat{\mathbf{x}}_F(p)-\mathbf{x}_{\text{fix}}(p)-\delta p\,\mathbf{g}]\ , \tag{4.1}$$

where $\mathbf{g}$, $\mathbf{e}_u$, $\mathbf{e}_s$, λ_u, and λ_s are all obtained by prediction and have the same meaning as the vectors denoted in the same way in Sec. II. We now choose δp, in such a way that x_{n+1} falls on the stable manifold of the predicted fixed point. That means we must have

$$\mathbf{f}_u\cdot(\mathbf{x}_{n+1}-\mathbf{x}_{\text{fix}})=0\ . \tag{4.2}$$

From (4.2), using the approximation (4.1), we get

$$\delta p\equiv\frac{\lambda_u[\hat{\mathbf{x}}_F(p)-\mathbf{x}_{\text{fix}}(p)]\cdot\mathbf{f}_u}{(\lambda_u-1)\mathbf{g}\cdot\mathbf{f}_u}\ . \tag{4.3}$$

As in the OGY method we change p to $p+\delta p$ only if the fluctuation in the parameter is small, otherwise we take $\delta p=0$.

Once we have determined δp, we determine $\mathbf{x}_{n+1}$ from the black box. Our method takes $\mathbf{x}_{n+1}$ to be the corrected value of the fixed point, i.e., to be $\mathbf{x}_F(p)$. This correction gives a good relative error in our test problem. Still, this error can be slightly improved by repeating the correction step several times and taking the arithmetic mean of the corrected values as our final corrected value.

Summarizing, each step of this algorithm consists of increasing the parameter and making a linear prediction of the orbit, as well as of the associated eigenvalues and eigenvectors, based on the previous points on the continuation curve. This allows us to find the local linear approximation of the map given by Eq. (4.1), which we use to correct the predicted value of the orbit. This is done by adjusting the parameter, in such a way that the next iterate falls on the stable direction of the predicted fixed point. This iterate is then taken as the corrected value, i.e., as the new point on the continuation curve.

Thus we obtain a new point $\mathbf{x}_F(p)$ on the curve of fixed points. Next we increase the parameter further, use this point to update the fixed points necessary for prediction, and repeat the above procedure.

Algorithm 2. The prediction step can be done with two points only; i.e., we choose the initial two points on the chaotic attractor at two values of p, and use the line through these two points (as a function of p) to predict the next fixed point. In this case, though, the method would lose control initially, and then tracks the orbit but with large relative error, see Fig. 3. To get a good relative error when two points are used for prediction we add a new step to our correction procedure.

This is based on the following observation. Suppose we apply the OGY algorithm to stabilize a certain fixed point of the Hénon map. Then over a large number of iterates, say 100, we notice that the value of the parameter p oscillates around the mean value $\langle p \rangle \cong 0.01$ when 1% noise is present (see Fig. 1). In fact we notice that when no error is present in the fixed point the mean of the fluctuations of the parameter increases linearly with the noise level. Suppose now we introduce an error in the controlled fixed point. Let this error be 5%, for example, that is the distance between the controlled fixed point and the exact fixed point is 5%. Then we notice in Fig. 2 that the values of the parameter will oscillate around a different mean value $\langle p \rangle \cong -0.13$, and the error in the control point causes a decrease in the overall values of the $\mathbf{x}$ iterates.

The above suggests a way to improve our correction step.

We denote the error in the fixed point, introduced by the prediction step by $\xi = \mathbf{x}_F(p) - \mathbf{x}_{\text{fix}}(p)$. Here $\mathbf{x}_F(p)$ is the real fixed point and $\mathbf{x}_{\text{fix}}(p)$ is the predicted fixed point. In the correction step we allow the parameter to fluctuate about the value p, and the idea is to ensure that the mean of these fluctuations is zero, which is certainly the case when we are controlling about the real fixed point in the absence of noise. Thus we adjust the predicted fixed point until we notice that the mean of the fluctuations gets as close to zero as possible—an idea easy to implement in experiments.

When coding the method we get a simple relation between the error ξ in the predicted fixed point and the mean of the fluctuating parameter $\langle \delta p \rangle$. To get this relation we start from Eq. (4.3) which gives the value of δp, the fluctuating part of the parameter. We rewrite (4.3) as follows:

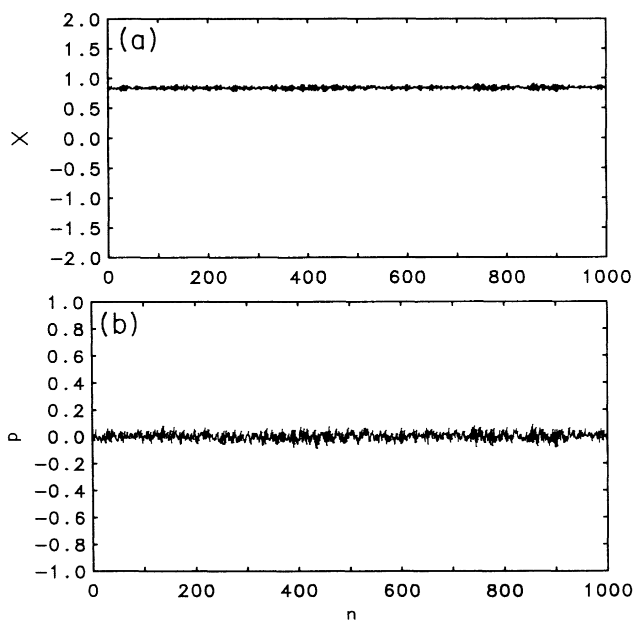

FIG. 1. (a) x_n vs n for the Hénon map, $A_0 = 1.29$, $B = 0.3$, for a noise of 1%, and no error in the fixed point. (b) p vs n for which $\langle p \rangle = 0.01$.

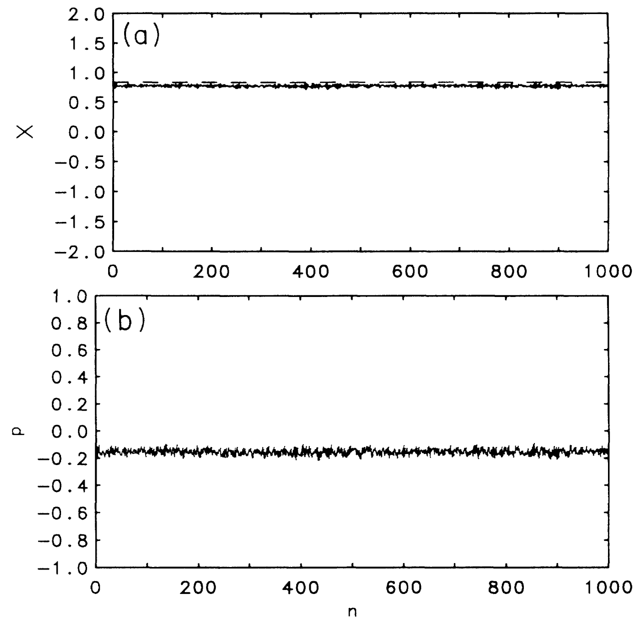

FIG. 2. (a) x_n vs n for the Hénon map, $A_0 = 1.29$, $B = 0.3$, for a noise of 1%,and error in the fixed point is 0.05. (b) p vs n for which $\langle p \rangle = -0.13$.

$$\delta p_n = \frac{\lambda_u (\mathbf{x}_n - \mathbf{x}_F)\cdot \mathbf{f}_u}{(\lambda_u - 1)\mathbf{g}\cdot \mathbf{f}_u} + \frac{\lambda_u \xi \cdot \mathbf{f}_u}{(\lambda_u - 1)\mathbf{g}\cdot \mathbf{f}u} , \tag{4.4}$$

where n stands for the number of times we repeated the correction step.

In the above the first term has mean value zero since $\mathbf{x}_f$ is the real fixed point and in the second term all quantities are known except the error vector ξ. So taking the mean in both sides of Eq. (4.4) over a large number of iterates we get

$$\langle \delta p_n \rangle = \frac{\lambda_u \xi \cdot \mathbf{f}_u}{(\lambda_u - 1)\mathbf{g}\cdot \mathbf{f}_u} , \tag{4.5}$$

which clearly shows the relationship between $\langle \delta p_n \rangle$ and ξ. The implication is that the control point may be moved in some small ball about the exact fixed point such that $\langle \delta p_n \rangle$ is minimized, which ensures, by (4.5), that ξ, the error in the fixed point, is minimized.

V. NUMERICAL RESULTS

We test our scheme using the Hénon map, which is given by the equations

$$x_{n+1} = A - x_n^2 + B y_n ,$$

$$y_{n+1} = x_n ,$$

where we take $B = 0.3$ and $A = A_0 + p$, where $A_0 = 1.29$ and p is the variable parameter. For this value of A_0, the attractor of the map is chaotic and contains an unstable period one (fixed point) orbit. The fixed point (x_F, y_F), is given by

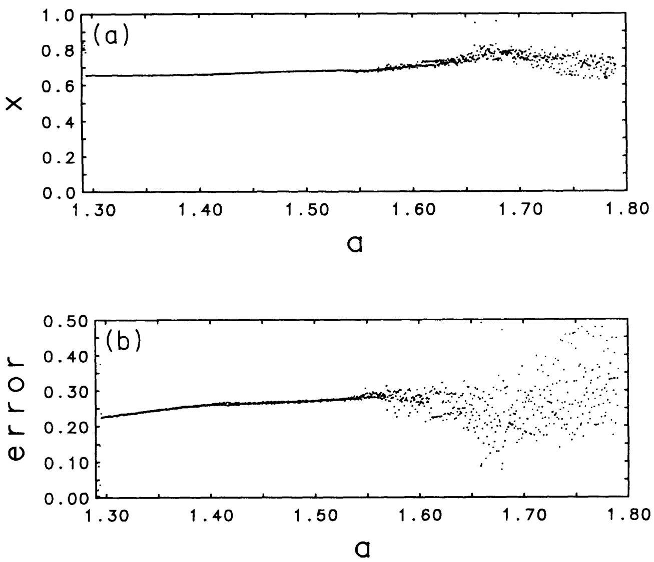

FIG. 3. (a) x vs p for the Hénon map, $A_0=1.29$, $B=0.3$. We use two points for prediction and no correction for the mean of the fluctuating part of the parameter. (b) The relative error vs p.

$$x_F = y_F = \tfrac{1}{2}\{(B-1)+[(B-1)^2+4A_0]^{1/2}\} .$$

The associated eigenvalues and eigenvectors can also be explicitly calculated (see [5]).

In a first experiment we try to track this fixed point using prediction with two points, and correction as in algorithm 1, i.e., we do not correct the mean of the fluctuating part of the parameter. As can be seen in Fig. 3, this method loses control.

In order to make the method work, one way is to add more points for prediction. We show in Fig. 4 the trajectory of the fixed point, when we use algorithm 1 with four

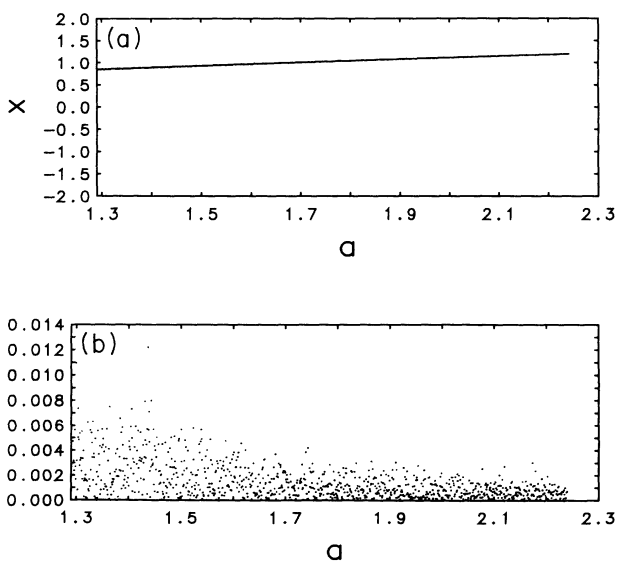

FIG. 4. (a) x vs p for the Hénon map, using algorithm 1, $A_0=1.29$, $B=0.3$, and 1% noise. (b) The relative error vs p.

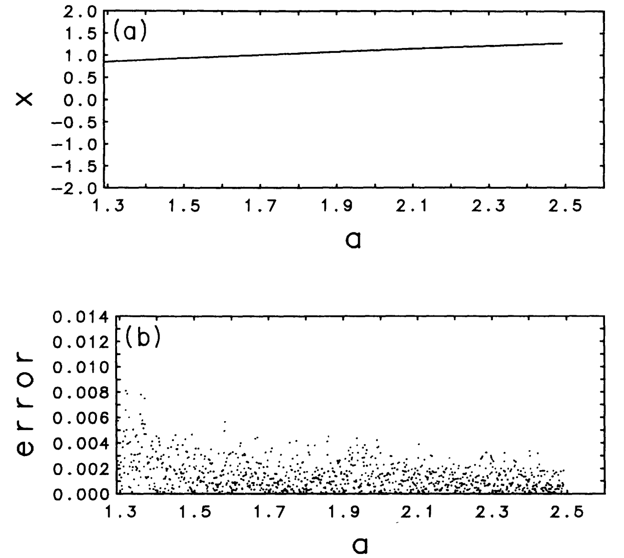

FIG. 5. (a) x vs p for the Hénon map, using algorithm 2, $A_0=1.29$, $B=0.3$, and 1% noise. (b) The relative error vs p.

points for linear prediction.

Next we apply algorithm 2 to this problem, i.e., we use two points for prediction and a more elaborate correction step; we notice in Fig. 5 that the scheme is able to track the orbit further.

We remark that for this map, all finite attractors disappear after $A=1.4$. It is clear from Figs. 3–5 that the tracking algorithms work in the absence of a chaotic attractor. The chaotic attractor is used only initially to locate an unstable periodic orbit.

For the Hénon map, we used linear prediction without any correction to determine the eigenvalues and eigenvectors used in the control part of the algorithms. This was possible since these quantities vary slowly with respect to the parameter. In a companion paper we will show a procedure to correct these values in a more general situation.

VI. CONCLUSIONS

We have introduced an efficient method to track orbits for both maps which are known, as well as experiments which generate dynamics as a one-dimensional time series. It was shown that this technique remains stable in the presence of added noise. Two versions of the method are available, algorithm 1 being essentially an averaging of the predicted fixed point, while algorithm 2 makes use of the error estimate (4.6) in the predicted fixed point in order to correct the prediction.

The novelty of the method consists in the fact that we do not need to know explicitly the equations of the map, which makes it especially useful to experimentalists and applicable to a wide variety of problems. The method also depends only on the application of a small-amplitude method of controlling unstable fixed points. By using other control methods, the continuation technique can be extended to higher period orbits [9], as well as aperiodic signals [10]. Since the development of the method here is for maps, it is also easily applied to flows by taking a Poincaŕe section of the flow.

Our technique now gives the experimentalist in dynamical systems an exploratory tool whereby new attractors can be located in a constructive manner. In a planned future paper, it will be shown how this method can be applied to such problems as accurate bifurcation location, branch switching between attractors, and the location of attractors having small basins of attraction.

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