Horseshoes in a relativistic Hamiltonian system in 1+1 dimensions

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We study the classical motion of a relativistic two-body system, in 1+1 dimensions, with interaction described by a relativistic generalization of the well-known Duffing potential. The equations of motion are separable in hyperbolic coordinates and are solved in quadrature. The radial equation (in the invariant variable corresponding to the spacelike distance between the particles) has an effective potential depending on the separation constant for the hyperbolic "angular momentum," and analytic solutions are obtained for the separatrix motion. In the presence of weak driving and damping forces, the Melnikov criterion for the existence of homoclinic instability is applied, and it is shown that chaotic behavior is predicted for sufficiently strong driving forces (bounds are given).

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I. INTRODUCTION

It has been well known since the work of Poincaré [1] that under perturbation, the stable and unstable manifolds emanating from a hyperbolic fixed point are not identical. There may be an infinite number of transverse intersections. The motion then becomes complicated and very sensitive to initial conditions; it is characterized as chaotic. Smale and Moser [2] state that the presence of such orbits implies that some iterate of the Pioncaré map has an invariant hyperbolic set (a Smale horseshoe) containing a countable infinity of unstable periodic orbits, an uncountable set of nonperiodic, or chaotic, motions, and a dense orbit. This set can exert a strong influence on the behavior of orbits that pass close to the hyperbolic point. They display a very sensitive dependence upon initial conditions. A simple theoretical test function due to Melnikov [3–5] may be used to determine the presence of homoclinic instability. This method has recently been exploited for the study of the behavior of the *rf* superconducting quantum interference device [6] and in systems perturbed by multiplicative noise [7].

It is known, for instance, that quantum effects tend to suppress chaotic behavior in many cases [8]. This is primarily due to the fact that independent of the form of the Hamiltonian, the Schrödinger equation is a linear differential equation. It is natural to ask whether the effects of relativity could also suppress or modify the character of chaos. Among mechanisms that could be effective are the space contractions and time dilations occurring as a result of interaction, as discussed in a relativistic model of the standard map [9]. Moreover, in a covariant generalization of Hamiltonian dynamics, the four-force is given by $f^\mu = -\partial V/\partial x_\mu$ so that $f^i = -\partial V/\partial x_i$, but $f^0 = +\partial V/\partial t$; unless $\partial^2 V/\partial x_i \partial t = 0$ there is no Euclidean equivalent potential to represent the forces in such a way that the shape of the potential surface peaks and valleys make it possible to recognize stable and unstable fixed points. One might, therefore, expect a significant modification of the behavior of a chaotic system due to relativistic effects. There is a new topological dimension, time, which has great significance.

We have recently studied numerically the relativistic classical mechanics of a damped Duffing-like driven system in 1+1 dimensions [10], which is totally unstable in the timelike directions. Chaotic behavior was found in bounded nonperiodic orbits passing through the timelike regions, as well as evidence for a strange attractor in spacetime with strong damping. The problem has some similarity to that studied by Holmes and Marsden [11], in that it has two degrees of freedom (in this case, space and time). It differs in an important way due to the hyperbolic structure of the 1+1 Minkowski space.

In this paper, we study this system analytically by means of separation of variables in hyperbolic coordinates. The problem is somewhat simplified here by taking an infinite potential in the timelike region, thus excluding passage to that domain. We are then able to construct the Melnikov function and prove the existence of homoclinic instability. We shall find, in hyperbolic coordinates, that there is a local minimum in the effective potential as a function of the "radial" variable, after the separation of the hyperbolic angle. Motion in this locally stable region corresponds to the motion discussed in Ref. [10] in the neighborhood of the force-free hyperbola, which results, in the presence of dissipation, in what we shall call "limiting hyperbolic motion," the generalization of a limit cycle in the nonrelativistic case. We shall be studying, therefore, the possibility of homoclinic crossings of the "stable" orbits associated with this hyperbolic motion and the "unstable" orbits associated with a larger motion bounded by the light cone.

In Sec. II we review briefly the formulation of classical relativistic mechanics in $1+1$ dimensions. We introduce hyperbolic coordinates, as used previously in the treatment of relativistic quantum-mechanical scattering in $1+1$ dimensions [12], permitting the separation of variables in the equations of motion; the separation constant is the generalized "angular momentum" corresponding to the Lorentz boost function [13]. The relativistic-harmonic-oscillator problem is worked out both in rectangular and hyperbolic coordinates to illustrate the role of these variables.

In Sec. III we study the relativistic Duffing-like problem with motion restricted (by taking the potential $V\rightarrow\infty$ in the timelike region) to the spacelike sector. We give conditions for the existence of a separatrix determined by the value of the separation constant and study motion in the inner and outer wells. Solutions are obtained in closed form by quadratures. Motion in the outer well is not affected by conditions on the light cone and corresponds to some of the orbits found computationally in Ref. [10]. Motion in the inner well replaces the possibility explored in Ref. [10] of motion traversing the light cone to the second spacelike region, but the instability that exists in the neighborhood of the separatrix is of the same type.

In the presence of damping, the angular momentum goes to zero exponentially (independently of the driving force in the form that we shall consider). In this limit, the two equations describing the motion in space and time coalesce to a single equation that is identical to that of the one-dimensional nonrelativistic Duffing oscillator. In general the motion is completely stable or completely unstable, depending upon initial conditions and the damping coefficient (with our choice of infinite potential in the timelike region, the unstable configuration cannot occur). The existence of a strange attractor in the nonrelativistic case therefore implies the existence of a similar phenomenon in the relativistic case and was indeed found in our computer study [10]. The analytic investigation of this phenomenon will be discussed elsewhere.

In Sec. IV we discuss the introduction of driving and dissipative forces, and in Sec. V we compute the Melnikov functions, proving the existence of chaotic orbits analytically. Section VI contains a summary and discussion.

We recognize that the relativistic action-at-a-distance potential models we use have, at present, no direct basis as a low-energy limit of some more fundamental local theory (this is not to say there is not a proper nonrelativistic limit to our results). This question is under investigation. We use these models, however, which lend themselves easily to computation (as for the nonrelativistic theories), in order to study the properties of Poincaré-invariant dynamical structures that could, in principle, correspond to such a limit.

II. CLASSICAL RELATIVISTIC MECHANICS IN 1+1 DIMENSIONS

To achieve a manifestly covariant, consistent Hamiltonian form of classical relativistic dynamics, we follow the procedure of Stueckelberg [14] in defining a $2N$ (in $1+1$ dimensions) dimensional phase space consisting of the spacetime coordinates and energy momenta associated with the state of the system at each value of the invariant universal time τ. The motion of the system is generated by an invariant function K on this phase space, by means of the covariant Hamilton equations (we use metric $-+$ for the time and space components, respectively)

$$\frac{dx_i^\mu}{d\tau}=\frac{\partial K}{\partial p_{i\mu}},\quad \frac{dp_i^\mu}{d\tau}=-\frac{\partial K}{\partial x_{i\mu}}\ . \tag{2.1}$$

In the case of a set of free particles, the choice

$$K=\sum_i \frac{p_i^\mu p_{i\mu}}{2M_i} \tag{2.2}$$

results in

$$\frac{dx_i^\mu}{d\tau}=\frac{p_i^\mu}{M_i}\ ,\quad \frac{dp_i^\mu}{d\tau}=0\ , \tag{2.3}$$

so that

$$v_i=\frac{dx_i}{dt_i}=\frac{p_i}{E_i}\ , \tag{2.4}$$

in agreement with the usual Einstein kinematics. The variables (E_i,p_i) are assumed to be independent of each other; the masses squared $m_i^2=E_i^2-p_i^2$ are to be determined as solutions of the dynamical problem. Note that

$$-\left(\frac{dx_i^\mu}{d\tau}\right)\left(\frac{dx_{i\mu}}{d\tau}\right)=\frac{ds_i^2}{d\tau^2}=\frac{m_i^2}{M_i^2} \tag{2.5}$$

and is unity only for the "mass-shell" value $m_i^2=M_i^2$. Hence τ corresponds to the proper time of ideal free clocks on their respective mass shells.

On mass shell, the time component of Eq. (2.3) is

$$\frac{dt_i}{d\tau}=\frac{E_i}{M_i}=\frac{E_i}{\sqrt{E_i^2-p_i^2}}=\frac{1}{\sqrt{1-v_i^2}}\ ,$$

so that dt_i is precisely the time interval measured in the laboratory between two signals emitted by a source, traveling with velocity v_i, with interval $d\tau$, according to the Lorentz transformation. If the emitter is not on shell, there is a factor m_i/M_i. The observed time interval is therefore influenced by forces [15] that move the energy momentum off shell.

We shall study a model for the two-body interacting system for which the generator of the motion has the Poincaré invariant form

$$K=\frac{p_1^\mu p_{1\mu}}{2M_1}+\frac{p_2^\mu p_{2\mu}}{2M_2}+V(\rho^2)\ , \tag{2.6}$$

where $\rho^2=(x_1-x_2)^2-(t_1-t_2)^2$.

Since, according to (2.3),

$$\frac{dt_i}{d\tau}=\frac{E_i}{M_i}$$

(approximately) on shell, $E_i \to M_i$ in the nonrelativistic limit, and $dt_i/d\tau \to 1$ for every particle. All of the t_i can then be taken equal to τ, which then becomes the universal Newtonian time. The potential $V(\rho^2) \to V(x^2)$, since $t_1 = t_2$. We shall use this as a correspondence principle to construct the relativistic potential.

Since the Hamiltonian is quadratic in the energy momenta, we may separate the center-of-mass motion with a transformation of the same form as that used in nonrelativistic mechanics:

$$P^\mu = p_1^\mu + p_2^\mu \ , \quad p^\mu = \frac{M_2 p_1^\mu - M_1 p_2^\mu}{M_1 + M_2} \ ,$$

$$X^\mu = \frac{M_1 x_1^\mu + M_2 x_2^\mu}{M_1 + M_2} \ , \quad x^\mu = x_1^\mu - x_2^\mu \ , \tag{2.7}$$

and we shall call $M = M_1 + M_2$, $m = M_1 M_2 / (M_1 + M_2)$. Then Eq. (2.6) becomes

$$K = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p^\mu}{2m} + V(\rho^2) \ , \tag{2.8}$$

where now $\rho^2 = x^2 - t^2$. The total energy momentum of the two-body system is then a constant of the motion, determined by initial conditions, and we may therefore consider the reduced problem for the relative motion determined by

$$K_{\text{rel}} = \frac{p^\mu p_\mu}{2m} + V(\rho^2) \ . \tag{2.9}$$

We remark that the function

$$M^{01} = t_1 p_1 - x_1 E_1 + t_2 p_2 - x_2 E_2 \ ,$$

which generates the Lorentz boost (by a Poisson bracket), also decomposes to a sum over center of mass and relative parts

$$M^{01} = M^{01}_{\text{c.m.}} + M^{01}_{\text{rel}} \ .$$

We shall call

$$M^{01}_{\text{rel}} = tp - xE = m(t\dot{x} - x\dot{t}) = -\lambda \ , \tag{2.10}$$

where we have used the canonical equations (for $\dot{x}^\mu \equiv dx^\mu/d\tau$),

$$\dot{x} = \frac{\partial K_{\text{rel}}}{\partial p} = \frac{p}{m} \ , \quad \dot{t} = -\frac{\partial K}{\partial E} = \frac{E}{m} \ . \tag{2.11}$$

The solution of the reduced motion problem can be considered a description of a two-body problem when the center-of-mass motion is accounted for, or as the solution of the problem of a single particle moving in an external potential (with origin moving up the t axis with τ) when $M_2 \to \infty$ in such a way that $\epsilon_2 = E_2 - m_2$ is finite. In this approximate mass-shell limit,

$$\frac{p_2^\mu p_2^\mu}{2M_2} = \frac{p_2^2 - E_2^2}{2M_2} = \frac{p_2^2 - (\varepsilon_2 + M_2)^2}{2M_2} \sim -\varepsilon_2 - \frac{M_2}{2} \ , \tag{2.12}$$

so that

$$\frac{dx_2}{d\tau} = 0 \ , \quad \frac{dt_2}{d\tau} = 1 \ .$$

The relative momentum is $p^\mu = p_1^\mu$, so that

$$\begin{aligned} &x = x_1 \ , \quad p = p_1 \ , \\ &t = t_1 - \tau \ , \quad E = E_1 \ . \end{aligned} \tag{2.13}$$

The velocity of particle 1 is, therefore, in this case

$$v_1 = \frac{dx_1}{dt_1} = \frac{\dot{x}}{\dot{t} + 1} \ . \tag{2.14}$$

In order to obtain solutions of the equations of motion in quadrature, we introduce hyperbolic variables in the sectors $\mathrm{I}_\pm$, $\mathrm{II}_\pm$ (Fig. 1) according to

$$\begin{aligned} &\mathrm{I}_\pm\!: \ x = \pm\rho\cosh\beta \ , \quad t = \rho\sinh\beta \ , \\ &\mathrm{II}_\pm\!: \ x = \rho'\sinh\beta' \ , \quad t = \pm\rho'\cosh\beta' \ , \end{aligned} \tag{2.15}$$

$$\rho = \sqrt{x^2 - t^2} \ , \quad \rho' = \sqrt{t^2 - x^2} \ . \tag{2.16}$$

Then, in the four sectors

$$\begin{aligned} \mathrm{I}_\pm\!: \ &\dot{x} = \pm(\rho\sinh\beta\dot{\beta} + \cosh\beta\dot{\rho}) \ , \\ &\dot{t} = \rho\cosh\beta\dot{\beta} + \sinh\beta\dot{\rho} \ ; \end{aligned} \tag{2.17}$$

$$\begin{aligned} \mathrm{II}_\pm\!: \ &\dot{x} = \rho'\cosh\beta'\dot{\beta}' + \sinh\beta'\dot{\rho}' \ , \\ &\dot{t} = \pm(\rho'\sinh\beta'\dot{\beta}' + \cosh\beta'\dot{\rho}') \ ; \end{aligned} \tag{2.18}$$

and it follows from (2.10) that in

$$\begin{aligned} &\mathrm{I}_\pm\!: \ \lambda = \pm m\rho^2\dot{\beta} \ , \\ &\mathrm{II}_\pm\!: \ \lambda = \mp m\rho'^2\dot{\beta}' \ . \end{aligned} \tag{2.19}$$

It is clear from Eq. (2.19) and Fig. 1 that a counterclockwise motion can occur for $\lambda > 0$.

The second-order equations are

$$\begin{aligned} \mathrm{I}_\pm\!: \ &\ddot{x} = \pm[(\rho\dot{\beta}^2 + \ddot{\rho})\cosh\beta + (\rho\ddot{\beta} + 2\dot{\rho}\dot{\beta})\sinh\beta] \ , \\ &\ddot{t} = (\rho\dot{\beta}^2 + \ddot{\rho})\sinh\beta + (\rho\ddot{\beta} + 2\dot{\rho}\dot{\beta})\cosh\beta \end{aligned} \tag{2.20}$$

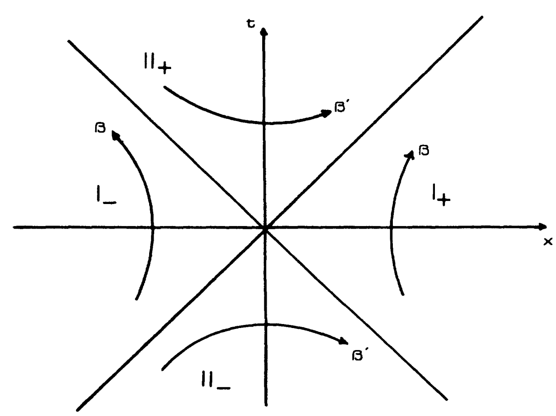

FIG. 1. The four sectors of the (x,t) plane, disjoint under proper Lorentz transformation. The sense of increasing hyperbolic angle is shown in each sector.

and

$$\text{II}_\pm:\ \ddot{x}=(\rho'\dot{\beta}'^{\,2}+\ddot{\rho}')\sinh\beta'+(\rho'\ddot{\beta}'+2\dot{\rho}'\dot{\beta}')\cosh\beta' ,$$

$$\ddot{t}=\pm[(\rho'\dot{\beta}'^2+\ddot{\rho}')\cosh\beta'+(\rho'\ddot{\beta}'+2\dot{\rho}'\dot{\beta}')\sinh\beta'] . \tag{2.21}$$

According to the second set of equations (2.1),

$$\ddot{x}=-\frac{1}{m}\frac{\partial V}{\partial x} ,\quad \ddot{t}=+\frac{1}{m}\frac{\partial V}{\partial t} . \tag{2.22}$$

We may choose V independently in the four sectors $V_{\text{I}_\pm}$, $V_{\text{II}_\pm}$, since no Lorentz transformation connects them. Using $\partial\rho/\partial x=\pm\cosh\beta$, $\partial\rho/\partial t=-\sinh\beta$ in $\text{I}_\pm$ and $\partial\rho'/\partial x=-\sinh\beta'$, $\partial\rho'/\partial t=\pm\cosh\beta'$ in $\text{II}_\pm$, we obtain the equations of motion

$$\text{I}_\pm:\ \left[\rho\dot{\beta}^2+\ddot{\rho}+\frac{1}{m}\frac{\partial V_{\text{I}_\pm}}{\partial\rho}\right]\cosh\beta+(\rho\ddot{\beta}+2\dot{\rho}\dot{\beta})\sinh\beta=0 , \tag{2.23}$$

$$\left[\rho\dot{\beta}^2+\ddot{\rho}+\frac{1}{m}\frac{\partial V_{\text{I}_\pm}}{\partial\rho}\right]\sinh\beta+(\rho\ddot{\beta}+2\dot{\rho}\dot{\beta})\cosh\beta=0$$

and

$$\text{II}_\pm:\ \left[\rho'\dot{\beta}'^2+\ddot{\rho}'-\frac{1}{m}\frac{\partial V_{\text{II}_\pm}}{\partial\rho'}\right]\sinh\beta' +(\rho'\ddot{\beta}'+2\dot{\rho}'\dot{\beta}')\cosh\beta'=0 , \tag{2.24}$$

$$\left[\rho'\dot{\beta}'^2+\ddot{\rho}'-\frac{1}{m}\frac{\partial V_{\text{II}_\pm}}{\partial\rho'}\right]\cosh\beta' +(\rho'\ddot{\beta}'+2\dot{\rho}'\dot{\beta}')\sinh\beta'=0 .$$

Eliminating the hyperbolic trigonometric functions in these pairs of equations, we obtain the separated equations

$$\text{I}_\pm:\ \ddot{\rho}+\rho\dot{\beta}^2=-\frac{1}{m}\frac{\partial V_{\text{I}_\pm}}{\partial\rho} ,\quad \rho\ddot{\beta}+2\dot{\rho}\dot{\beta}=0 . \tag{2.25}$$

The second of Eqs. (2.25) corresponds to the constant of motion λ in $\text{I}_\pm$. In the timelike sectors,

$$\text{II}_\pm:\ \ddot{\rho}'+\rho'\dot{\beta}'^2=\frac{1}{m}\frac{\partial V_{\text{II}_\pm}}{\partial\rho'} ,\quad \rho'\ddot{\beta}'+2\dot{\rho}'\dot{\beta}'=0 . \tag{2.26}$$

The second of Eqs. (2.26) corresponds to the constant λ in the timelike regions. With the help of Eqs. (2.19), these become

$$\text{I}_\pm:\ \ddot{\rho}+\frac{1}{m^2}\frac{\lambda^2}{\rho^3}=-\frac{1}{m}\frac{\partial V_{\text{I}_\pm}}{\partial\rho} ,$$
$$\text{II}_\pm:\ \ddot{\rho}'+\frac{1}{m^2}\frac{\lambda^2}{\rho'^3}=\frac{1}{m}\frac{\partial V_{\text{II}_\pm}}{\partial\rho'} . \tag{2.27}$$

In principle, these equations can be solved in each sector, and the solutions must be continued from $\text{I}_\pm$ to $\text{II}_\pm$ continuously in x, t and with conditions on $\dot{x}$, $\dot{t}$ that follow from the change in potential going across the light cone. These conditions are summarized in the Appendix for completeness.

We see from these equations that the form of the potential in $\text{I}_\pm$ and $\text{II}_\pm$ have a different meaning for $\ddot{\rho}$ and $\ddot{\rho}'$, due essentially to the difference in signature between the space and time components. In the separation variables ρ, β, however, one can interpret $V_{\text{I}_\pm}$ and $-V_{\text{II}_\pm}$ as potentials in the usual sense, e.g., in reading off minima as stable points and maxima as unstable points with respect to motion in ρ and ρ'. This is not possible for the interpretation of V in any sector, even locally, for the rectangular coordinates. The Euclidean curl of the forces represented by (2.22) is $-2\partial^2V/\partial x\partial t$, in general nonzero, and hence there is no corresponding potential functions for which the stable and unstable fixed points can be seen in the usual way [10]. The interpretation of the potential is made much simpler, as we see, in the separation variables, which yield equations very similar in form to the nonrelativistic problem (the attractive nature of the centripetal terms is due to the simple hyperbolic structure of 1+1 dimensions; see Ref. [16] for a discussion of the 3+1 problem in a classical context, and Ref. [17] in a quantum-mechanical context).

In terms of the hyperbolic variables, the Hamiltonian takes the form

$$\text{I}_\pm:\ K_{\text{rel}}=\frac{m}{2}(\dot{\rho}^2-\rho^2\dot{\beta}^2)+V_{\text{I}_\pm}(\rho^2) , \tag{2.28}$$

$$\text{II}_\pm:\ K_{\text{rel}}=\frac{m}{2}(-\dot{\rho}'^2+\rho'^2\dot{\beta}'^2)+V_{\text{II}_\pm}(\rho'^2) . \tag{2.29}$$

Using the conservation laws (2.19), these are

$$\text{I}_\pm:\ K_{\text{rel}}=\frac{m}{2}\dot{\rho}^2-\frac{\lambda^2}{2m\rho^2}+V_{\text{II}_\pm}(\rho^2) ,$$
$$\text{II}_\pm:\ K_{\text{rel}}=-\frac{m}{2}\dot{\rho}'^2+\frac{\lambda^2}{2m\rho'^2}+V_{\text{II}_\pm}(\rho'^2) . \tag{2.30}$$

These formulas and (2.19) enable us to solve for $\rho(\tau)$, $\beta(\tau)$ in quadrature,

$$\text{I}_\pm:\ \tau=(\pm)\sqrt{m/2}\int d\rho\left[K_{\text{rel}}+\frac{\lambda^2}{2m\rho^2}-V_{\text{I}_\pm}(\rho^2)\right]^{-1/2} , \tag{2.31}$$

$$\beta=\pm\int\frac{\lambda}{m}\frac{d\tau}{\rho^2} , \tag{2.32}$$

$$\text{II}_\pm:\ \tau=(\pm)\sqrt{m/2}\int d\rho'\left[-K_{\text{rel}}+\frac{\lambda^2}{2m\rho'^2} +V_{\text{II}_\pm}(\rho^2)\right]^{1/2} , \tag{2.33}$$

$$\beta'=\mp\int\frac{\lambda}{m}\frac{d\tau}{\rho'^2} , \tag{2.34}$$

where the signs $(\pm)$ are uncorrelated with the sector designations. They must be assigned according to initial conditions and the occurrence of turning points (such that $d\tau>0$ along the motion).

We note that since

$$\dot{\rho}'^2=\frac{2}{m}[V_{\text{II}_\pm}(\rho'^2)-K_{\text{rel}}]+\frac{\lambda^2}{m^2\rho'^2} ,$$

motion in the timelike region is forbidden for

$$\rho'^2>\frac{\lambda^2}{2m}\frac{1}{K_{\text{rel}}-V_{\text{II}_\pm}(\rho'^2)} . \tag{2.35}$$

For $\lambda\neq 0$, there is no choice of $V_{\text{II}_\pm}$ that prevents some penetration into the timelike region except for $V_{\text{II}\pm}=-\infty$. We shall make this simplifying assumption in the later sections.

To conclude this section with an illustrative example, we study the harmonic oscillator in $1+1$ dimensions in order to acquire some familiarity with the hyperbolic coordinates. For the oscillator, we take (in all sectors)

$$V(\rho^2)=\tfrac{1}{2}m\omega^2\rho^2 . \tag{2.36}$$

The canonical equations in rectangular coordinates are

$$\ddot{x}+\omega^2x=0 , \quad \ddot{t}+\omega^2t=0 , \tag{2.37}$$

and hence we may take solutions, for simplicity, of the form

$$x(\tau)=A\cos\omega\tau , \quad t(\tau)=A\sin\omega\tau . \tag{2.38}$$

In the sector I_+, these are, in hyperbolic coordinates,

$$A\cos\omega\tau=\rho\cosh\beta , \quad A\sin\omega\tau=\rho\sinh\beta , \tag{2.39}$$

so that

$$\rho^2=A^2\cos2\omega\tau , \quad \tanh\beta=\tan\omega\tau . \tag{2.40}$$

In I_+, $\omega\tau$ is bounded by $\pm\pi/4$, so $|\tanh\beta|\leq 1$, i.e., the mapping is well defined. We now demonstrate that ρ, β defined by (2.40) satisfy the equations of motion (2.25). By direct computation,

$$\ddot{\rho}=-\frac{A\omega^2}{(\cos2\omega\tau)^{3/2}}(1+\cos^2 2\omega\tau) . \tag{2.41}$$

Differentiating Eq. (2.40) with respect to τ, one obtains

$$\dot{\beta}=\omega\frac{\cosh^2\beta}{\cos^2\omega\tau} ;$$

using the relation $\cosh^2\beta=(1-\tanh^2\beta)^{-1}=(1-\tan^2\omega\tau)^{-1}$, it follows that

$$\dot{\beta}=\frac{\omega}{\cos2\omega\tau} , \tag{2.42}$$

well defined in I_+ (these solutions can be continued to the other quadrants; see the Appendix). Then,

$$\rho\dot{\beta}^2=\frac{A\omega^2}{(\cos2\omega\tau)^{3/2}} . \tag{2.43}$$

The addition of $\rho\dot{\beta}^2$ to $\ddot{\rho}$ cancels the first term of Eq. (2.41), and we obtain the first of Eq. (2.25). Both $\ddot{\rho}$ and $\rho\dot{\beta}^2$ are singular at $\omega\tau\to\pm\pi/4$ (on the light cone). These singularities cancel. For the generalized angular momentum, we see that

$$-\lambda=M^{01}_{\text{rel}}=m(t\dot{x}-x\dot{t})=A^2m\omega=-m\rho^2\dot{\beta} . \tag{2.44}$$

Finally, although the relative coordinates pass through the light cone, for $M_2\to\infty$, according to Eq. (2.14), there is a range $A\omega<1/\sqrt{2}$ for which $v_1<1$. In this range, the amplitude and frequency are sufficiently low, so that the upward motion of the whole system along the t axis prevents a transition of the motion of particle 1 through the light cone (with origin on $t=\tau$).

III. RELATIVISTIC DUFFING-LIKE PROBLEM

We shall now study the Duffing-like potential, with

$$V_{\text{I}_\pm}(\rho^2)=\frac{b}{4}\rho^4-\frac{a}{2}\rho^2 \quad (a,b>0) \tag{3.1}$$

and

$$V_{\text{II}_\pm}(\rho^2)=-\infty . \tag{3.2}$$

The motion is entirely confined to region I_+ for $\lambda\neq 0$. For $\lambda=0$, the trajectory can pass between I_+ and I_-. In this case, it follows from Eq. (2.10) that

$$t=\alpha x \tag{3.3}$$

for some constant $\alpha<1$ in $\text{I}_\pm$. As we have pointed out in Ref. [10], the relativistic equations (2.22) then coincide and become identical to the nonrelativistic Duffing oscillator (for both x and t). We shall return to this point when we introduce driving and dissipation. Dissipation implies convergence to such a line.

In I_+, for $\lambda=m\rho^2\dot{\beta}\neq 0$, Eq. (2.25) becomes

$$\ddot{\rho}+\frac{\lambda^2}{m^2\rho^3}+b\rho^3-a\rho=0 . \tag{3.4}$$

The first integral is given by the reduced motion Hamiltonian

$$K_{\text{rel}}=\frac{m}{2}\dot{\rho}^2+V_{\text{eff}}(\rho^2) , \tag{3.5}$$

where

$$V_{\text{eff}}(\rho^2)=\frac{b}{4}\rho^4-\frac{a}{2}\rho^2-\frac{\lambda^2}{2m\rho^2} . \tag{3.6}$$

The existence of neighboring stable and unstable orbits, a necessary condition for the existence of homoclinic crossings associated with the possible onset of chaos in the perturbed motion (horseshoes) is assured by the presence of a local minimum in the function (in the following, we take $\rho^2=\xi$)

$$f(\xi)=\frac{b}{4}\xi^2-\frac{a}{2}\xi-\frac{\lambda^2}{2m\xi} \tag{3.7}$$

for $\xi>0$. The condition $f'(\xi)=0$ is

$$\xi^3-\frac{a}{b}\xi^2+\frac{\lambda^2}{mb}=0 . \tag{3.8}$$

Following the standard procedure for obtaining the roots of a cubic equation [18], we have the condition for three real roots (the alternative is for one real root and two complex roots, which we cannot accept)

$$\lambda^2 \leq \frac{4m}{27}\frac{a^3}{b^2} \; . \tag{3.9}$$

For $a,b \neq 0$ and λ sufficiently small, we are assured the existence of a dip in the function $V_{\rm eff}(\rho^2)$ [for $b=0$, Eq. (3.8) degenerates to a quadratic with a single root for $\xi>0$].

The effective potential then appears as in Fig. 2; ρ_1 is the position of the local maximum. We designate the second turning point at the separatrix ρ_2.

According to Eq. (2.31),

$$\tau=(\pm)\sqrt{m/2}\int \frac{d\rho}{\left[K_{\rm rel}-\frac{b}{4}\rho^4+\frac{a}{2}\rho^2+\frac{\lambda^2}{2m\rho^2}\right]^{1/2}} \tag{3.10}$$

$$=(\pm)\tfrac{1}{2}\sqrt{m/2}\int \frac{d\xi}{\left[\xi K_{\rm rel}-\frac{b}{4}\xi^3+\frac{a}{2}\xi^2+\frac{\lambda^2}{2m}\right]^{1/2}} \; . \tag{3.11}$$

For $K_{\rm rel}=K_A$, there are three real roots for the function

$$K_{\rm rel}-\frac{b}{4}\xi^2+\frac{a}{2}\xi+\frac{\lambda^2}{2m\xi}=0 \; . \tag{3.12}$$

In the exterior well, the motion oscillates between the two turning points ($\dot\rho=0$) (Fig. 3). In the interior well, it runs between a turning point and the light cone where $\dot\rho$ changes (bounces) abruptly (Fig. 4). For $K_{\rm rel}=K_s$, the separatrix value, the motion in the exterior well is between a turning point and an asymptotic approach to the local maximum at ρ_1 (Fig. 5). In the interior well (Fig. 6)

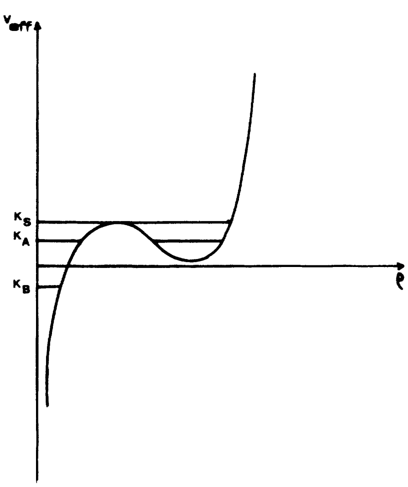

FIG. 2. The effective potential, including hyperbolic angular momentum contribution for the relativistic Duffing-like problem. K_s is the value (mass) of the relative Hamiltonian at the separatrix; the unstable fixed point is at ρ_1, and turning point at ρ_2. The value K_A has two separated regions of allowed motion, and K_β just one, between a hyperbolic curve where $V_{\rm eff}=K_A$ and $\rho=0$ (the light cone).

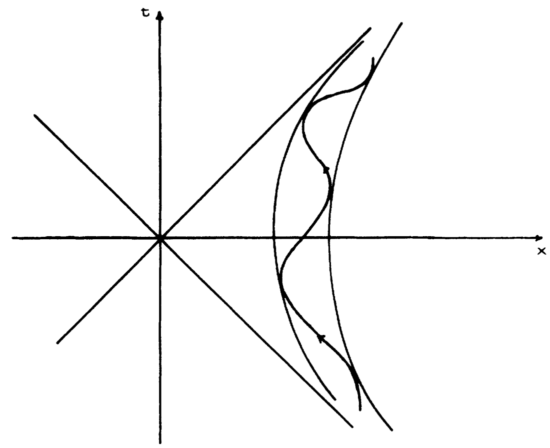

FIG. 3. A representative trajectory for $\lambda>0$ in the exterior well of Fig. 2 (for a choice of type K_A).

it goes between a bounce on the light cone to an asymptotic approach to the local maximum at ρ_1. In all of these cases, since $\lambda=m\rho^2\dot\beta$ is constant, and positive, $\dot\beta>0$ and the trajectories must move so that a radial line to the system point rotates counterclockwise.

The construction of the Melnikov function [3] requires an analytic solution at the separatrix value $K_{\rm rel}=K_s$. At the separatrix, the turning point condition at ρ_1 (true at ρ_2 also) is

$$K_s - V(\rho_1^2)_{\rm eff}=0 \; , \tag{3.13}$$

and the conditions for a local maximum are

$$\frac{d}{d\rho}V(\rho^2)_{\rm eff}\big|_{\rho_1}=0 \; , \tag{3.14}$$

$$\frac{d^2}{d\rho^2}V(\rho^2)_{\rm eff}\big|_{\rho_1}<0 \; . \tag{3.15}$$

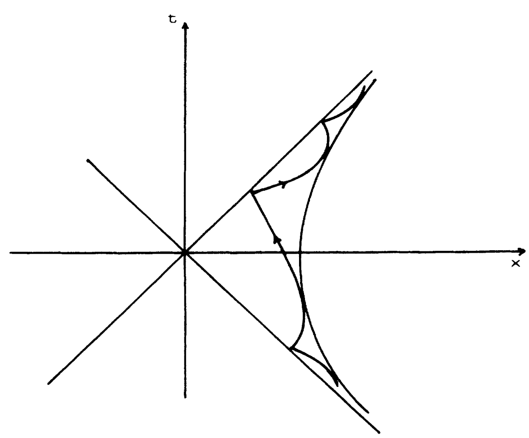

FIG. 4. A representative trajectory for $\lambda>0$ in the inner well of Fig. 2 (for a choice of type K_A or K_B).

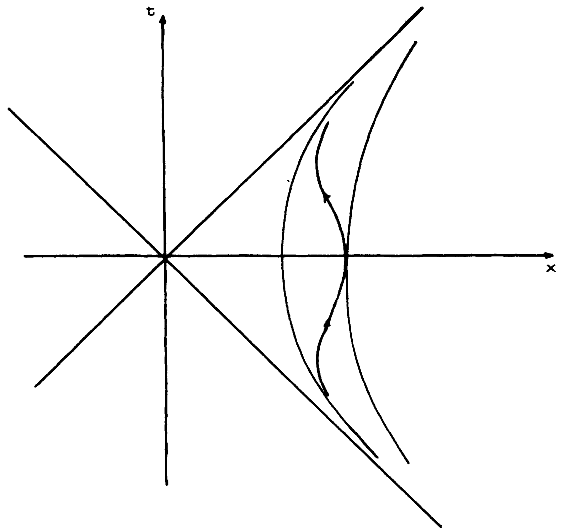

FIG. 5. A representative trajectory for $\lambda>0$ for motion on the separatrix in the exterior well. Note that the curve approaches ρ_1 only asymptotically.

For $\rho^2=\xi$, $d/d\rho=2\sqrt{\xi}d/d\xi$ and $d^2/d\rho^2=2d/d\xi+4\xi d^2/d\xi^2$, so that (at $\xi_1=\rho_1^2$)

$$\frac{d}{d\xi}\left[\frac{b}{4}\xi^2-\frac{a}{2}\xi-\frac{\lambda^2}{2m\xi}\right]\Bigg|_{\xi=\xi_1}=0 ,$$

$$\left[\frac{d}{d\xi}+d\xi\frac{d^2}{d\xi^2}\right]\left[\frac{b}{4}\xi^2-\frac{a}{2}\xi-\frac{\lambda^2}{2m\xi}\right]\Bigg|_{\xi=\xi_1}=0 . \tag{3.16}$$

These are

$$b\xi_1^3-a\xi_1^2+\frac{\lambda^2}{m}=0 \tag{3.17}$$

and

$$3b\xi_1^3-a\xi_1^2-3\frac{\lambda^2}{m}<0 . \tag{3.18}$$

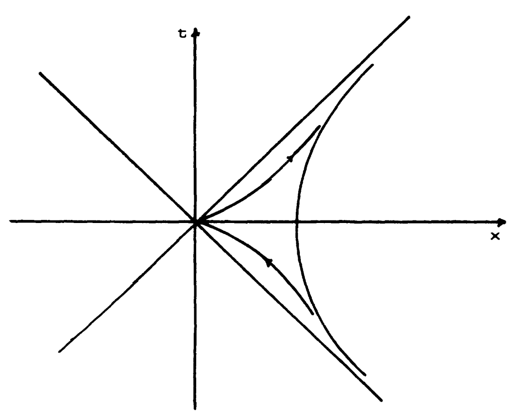

FIG. 6. A representative trajectory for $\lambda>0$ for motion on the separatrix in the interior well. The curve approaches ρ_1 only asymptotically.

The turning point condition (3.13) is

$$K_s-\frac{b}{4}\xi_1^2+\frac{a}{2}\xi_1+\frac{\lambda^2}{2m\xi_1}=0 ; \tag{3.19}$$

with Eq. (3.17), this is

$$K_s=-\tfrac{1}{4}\left[a\xi_1+\frac{3\lambda^2}{m\xi_1}\right]<-\tfrac{3}{4}b\xi_1^2 , \tag{3.20}$$

where the inequality follows from Eq. (3.18).

We now consider the denominator of Eq. (3.11) at $K_{\rm rel}=K_s$. We now define

$$f(\xi)=\xi K_s-\frac{b}{4}\xi^3+\frac{a}{2}\xi^2+\frac{\lambda^2}{2m} . \tag{3.21}$$

Using Eq. (3.19), this becomes

$$f(\xi)=\tfrac{1}{2}(\xi-\xi_1)^2\left[-\frac{b\xi}{2}+\frac{\lambda^2}{m\xi_1^2}\right] . \tag{3.22}$$

The turning point at $\xi_2=\rho_2^2$ is the other zero of $f(\xi)$

$$\xi_2=\frac{2\lambda^2}{mb\xi_1^2} . \tag{3.23}$$

Therefore,

$$f(\xi)=\frac{b}{4}(\xi-\xi_1)^2(\xi_2-\xi) . \tag{3.24}$$

For the external well, $\xi_1\le\xi\le\xi_2$, $f(\xi)\ge0$. We define τ to be the historical time necessary for the system to move from ξ_2 to an arbitrary ξ in the interval (ξ_1,ξ_2). From Eq. (3.11),

$$\tau=-\frac{1}{2}\sqrt{m/2}\int_{\xi_2}^{\xi}\frac{d\xi'}{(\xi'-\xi_1)\sqrt{b/4}(\xi_2-\xi')^{1/2}} . \tag{3.25}$$

This integral can be evaluated by elementary means [19]. We obtain, after inverting the function $\tau(\xi)$, the separatrix solution (recall $\xi_1=\rho_1^2$)

$$\rho^2=\xi_1+\frac{\Delta}{\cosh^2(\tau\sqrt{b\Delta/2m})} , \tag{3.26}$$

where $\Delta=\xi_2-\xi_1$.

For the inner well, we calculate the historical time for the system to move from a point on the light cone to some position in $0\le\rho^2\le\xi_1$. According to Eq. (3.11), this is, on the separatrix

$$\tau=+\tfrac{1}{2}\sqrt{m/2}\int_0^{\xi}\frac{d\xi'}{\left[\xi'K_s-\frac{b}{4}\xi'^3+\frac{a}{2}\xi^{2\prime}+\frac{\lambda^2}{2m}\right]^{1/2}} . \tag{3.27}$$

Again, using Eq. (3.24) for the denominator function, but noting that $\xi\le\xi_1$, we obtain

$$\tau=\tfrac{1}{2}\sqrt{m/2}\int_0^{\xi}\frac{d\xi'}{(\xi_1-\xi')\sqrt{b/4}(\xi_2-\xi')^{1/2}} . \tag{3.28}$$

Again, using the formulas of Ref. [19], we find

$$\rho^2=\xi_1-\frac{\Delta}{\sinh^2(\tau\sqrt{b\Delta/2m}+\delta/2)}\ , \tag{3.29}$$

where

$$\sinh^2\frac{\delta}{2}=\frac{\Delta}{\xi_1}\ . \tag{3.30}$$

We have computed this result for $\tau\geq 0$. For $\tau<0$, the equation corresponding to (3.28) can be obtained by integrating $d\tau$ from $-|\tau|$ to 0 and from ξ to 0 over the corresponding differential expression in ρ. One obtains Eq. (3.29) with τ replaced by $+|\tau|$, and hence the emotion $\rho(\tau)$ is even in τ even in the presence of the term $\delta\neq 0$.

IV. DRIVING AND DISSIPATIVE FORCES

Forces, as for nonrelativistic dynamics, are defined as associated with acceleration with respect to the invariant world time. This identification results in the notion of conservative (or nonconservative) systems consistent with the structure of the Hamilton equations, e.g., for one particle with an external potential $V(x^\mu)$,

$$\frac{dx^\mu}{d\tau}=\frac{p^\mu}{M}\frac{dp^\mu}{d\tau}=-\frac{\partial V}{\partial x_\mu}\ ,$$

so that

$$F^\mu\equiv M\ddot{x}^\mu=-\frac{\partial V}{\partial x_\mu}$$

defines a "force." The integral (in this case)

$$\int_A^B F^\mu dx_\mu=-[V(x_B^\mu)-V(x_A^\mu)]$$

is clearly independent of the path. We therefore add an external driving force to the equations of motion for the two particles we are studying here in the form

$$M_1\ddot{x}_1^\mu=-\frac{\partial V}{\partial x_{1_\mu}}+f_1^\mu\sin\omega\tau\ ,$$
$$M_2\ddot{\xi}_2^\mu=-\frac{\partial V}{\partial x_{2_\mu}}+f_2^\mu\sin\omega\tau\ . \tag{4.1}$$

Dividing the first equation by M_1 and the second by M_2, we obtain

$$\ddot{x}^\mu=-\left[\frac{1}{M_1}\frac{\partial V}{\partial x_{1_\mu}}-\frac{1}{M_2}\frac{\partial V}{\partial x_{2_\mu}}\right]+\left[\frac{1}{M_1}f_1^\mu-\frac{1}{M_2}f_2^\mu\right]\sin\omega\tau\ . \tag{4.2}$$

Since $\partial V/\partial x_{2_\mu}=-\partial V/\partial x_{1_\mu}=-\partial V/\partial x_\mu$, Eq. (4.2) becomes

$$\ddot{x}^\mu=-\frac{1}{m}\frac{\partial V}{\partial x_\mu}+f^\mu\sin\omega\tau\ , \tag{4.3}$$

where we have defined $f^\mu=1/M_1f_1^\mu-1/M_2f_2^\mu$. Using Eq. (2.20), one obtains, in $\mathrm{I}_\pm$,

$$(\rho\dot{\beta}^2+\ddot{\rho})\cosh\beta+(\rho\ddot{\beta}+2\dot{\rho}\dot{\beta})\sinh\beta=-\frac{1}{m}\frac{\partial V}{\partial\rho}\cosh\beta+f^x\sin\omega\tau\ ,$$
$$(\rho\dot{\beta}^2+\ddot{\rho})\sinh\beta+(\rho\ddot{\beta}+2\dot{\rho}\dot{\beta})\cosh\beta=-\frac{1}{m}\frac{\partial V}{\partial\rho}\sinh\beta+f^t\sin\omega\tau\ . \tag{4.4}$$

Hence

$$\rho\dot{\beta}^2+\ddot{\rho}+\frac{1}{m}\frac{\partial V}{\partial\rho}=(\pm f^x\cosh\beta-f^t\sinh\beta)\sin\omega\tau\ ,$$
$$\rho\ddot{\beta}+2\dot{\rho}\dot{\beta}=(f^t\cosh\beta\mp f^x\sinh\beta)\sin\omega\tau\ . \tag{4.5}$$

The last of (4.5) is

$$\frac{d}{d\tau}(\rho\dot{\beta}^2)=(f^t\cosh\beta\mp f^x\sinh\beta)\sin\omega\tau\ . \tag{4.6}$$

Since the form of the driving force is at our disposal, we may choose (for simplicity in the later calculations) $f_1^\mu=M_1x_1^\mu f$, $f_2^\mu=M_2x_2^\mu f$; it then follows that

$$f^x=\pm f\rho\cosh\beta\ ,\quad f^t=f\rho\sinh\beta$$

of the form of a Lorentz transformation, for each β, from the vector $f^x=fx$, $f^t=0$ in the original frame. Then $d/d\tau(\rho\dot{\beta}^2)=0$, i.e., the angular momentum is conserved by the driving forces, and

$$\ddot{\rho}+\rho\dot{\beta}^2+\frac{1}{m}\frac{\partial V}{\partial\rho}=f\rho\sin\omega\tau\ . \tag{4.7}$$

For dissipation, we assume a friction force that damps only the relative motion of the system, e.g., as for a damping due to dipole radiation of two charged particles. We take this damping force to be proportional to the relative velocity $\dot{x}^\mu$ in the system, and hence the equation of motion is modified to

$$\ddot{x}^\mu=-\frac{1}{m}\frac{\partial V}{\partial x_\mu}-k\dot{x}^\mu\ . \tag{4.8}$$

As above, we then obtain in $\mathrm{I}_\pm$, in hyperbolic coordinates [see Eqs. (2.23)], the equations

$$(\rho\dot{\beta}^2+\ddot{\rho})\cosh\beta+(\rho\ddot{\beta}+2\dot{\rho}\dot{\beta})\sinh\beta=-\frac{1}{m}\frac{\partial V}{\partial\rho}\cosh\beta-k(\rho\dot{\beta}\sinh\beta+\dot{\rho}\cosh\beta)\ ,$$
$$(\rho\dot{\beta}^2+\ddot{\rho})\sinh\beta+(\rho\ddot{\beta}+2\dot{\rho}\dot{\beta})\cosh\beta=-\frac{1}{m}\frac{\partial V}{\partial\rho}\sinh\beta-k(\rho\dot{\beta}\cosh\beta+\dot{\rho}\sinh\beta)\ , \tag{4.9}$$

so that

$$\ddot{\rho}+\rho\dot{\beta}^2=-\frac{1}{m}\frac{\partial V}{\partial\rho}-k\dot{\rho}\ , \tag{4.10}$$

$$\rho\ddot{\beta}+2\dot{\rho}\dot{\beta}+k\rho\dot{\beta}=0\ . \tag{4.11}$$

The first is a dissipative "radial equation." The second corresponds to nonconservation of $\rho^2\dot{\beta}$

$$\frac{d}{d\tau}(\rho^2\dot{\beta})=-k\rho^2\dot{\beta}\ ,$$

and thus

$$\rho^2\dot{\beta}=\pm\frac{\lambda}{m}e^{-k\tau}, \tag{4.12}$$

where we have chosen the constant according to the initial condition $\lambda=\pm m\rho^2\dot{\beta}$ (at $\tau=0$). The result (4.15) implies that for $k\neq 0$, $M^{01}\to 0$ exponentially for $\tau\to\infty$. In this limit the system must, therefore, approach a motion for which $t=\alpha x$ for some constant $\alpha^2<1$ (if the emotion is restricted to $I_\pm$), as pointed out in Sec. III.

V. THE MELNIKOV FUNCTIONS

As discussed in Refs. [3], [5], [6], and [7], the Melnikov function, measuring the signed area in phase space between the stable and unstable manifolds as a function of the phase of the driving forces, is a powerful tool for detecting homoclinic crossings and the existence of a Cantor set produced by mappings in the Smale horseshoe configuration [2,5] (chaotic behavior). We shall calculate (we use the notation of Ref. [6])

$$\Delta(\tau_0)=\Delta_\omega(\tau_0)+\Delta_k, \tag{5.1}$$

for

$$\Delta_\omega(t_0)=-\int_{-\infty}^{\infty}F_\omega(\tau)^\mu\dot{x}_\mu d\tau, \tag{5.2}$$

$$\Delta_k=-\int F_k(\tau)^\mu\dot{x}_\mu d\tau, \tag{5.3}$$

where

$$F_\omega(\tau)^\mu=f^\mu\sin\omega(\tau-\tau_0) \tag{5.4}$$

and

$$F_k(\tau)^\mu=-k\dot{x}^\mu. \tag{5.5}$$

The Melnikov criterion applies to the phase space of the reduced motion. With the choice leading to Eq. (4.7), $f^\mu\dot{x}_\mu=f\rho\dot{\rho}=\frac{1}{2}f\,d\rho^2/d\tau$, and $\dot{x}^\mu\dot{x}_\mu=\dot{\rho}^2-\rho^2\dot{\beta}^2$; furthermore, for both interior and exterior wells, $\rho(\tau)$ [from Eqs. (3.26) and (3.29)] is even. Hence,

$$\Delta_\omega(\tau_0)=f\omega\cos\omega\tau_0\int_0^\infty\rho^2\cos\omega\tau\,d\tau \tag{5.6}$$

and

$$\Delta_k=2k\int_0^\infty(\dot{\rho}^2-\rho^2\dot{\beta}^2)d\tau, \tag{5.7}$$

where we have added the dissipation of the τ forward and reversed contributions to Δ_k, utilizing the symmetry of the solutions found in Sec. III.

We remark that both $\dot{\rho}^2$ and $\rho^2\dot{\beta}^2$ are singular for $\rho\to 0$, but $\dot{\rho}^2-\rho^2\dot{\beta}^2$ is finite. For the oscillator example treated above, for the unperturbed motion,

$$\dot{\rho}^2=\frac{\omega^2A^2\sin^2 2\omega\tau}{\cos 2\omega\tau},$$

and

$$\rho^2\dot{\beta}^2=\frac{A^2\omega^2}{\cos 2\omega\tau}.$$

Hence, $\dot{\rho}^2$ and $\rho^2\dot{\beta}^2$ are separately singular on the light cone, where $\omega\tau\to\pm\pi/4$, but the difference,

$$\dot{\rho}^2-\rho^2\dot{\beta}^2=-A^2\omega^2\cos 2\omega\tau,$$

is finite (zero in this case). This result is quite general; from the quadrature formula (for potentials with finite limit for $\rho\to 0$),

$$\dot{\rho}^2\sim\frac{2}{m}\frac{\lambda^2}{2m\rho^2}=\frac{\lambda^2}{m^2\rho^2}.$$

But, $\dot{\beta}=\lambda/m\rho^2$, so that $\rho^2\dot{\beta}^2$ exactly cancels this singularity. Note, that for $\lambda=0$, $\dot{\rho}^2$ is regular for $\rho\to 0$ (as pointed out above, in this case the trajectory follows a path $t=\alpha x$ through the light cone, and for $\alpha^2<1$, the problem is equivalent to that of the nonrelativistic Duffing oscillator).

First, for the exterior well, we obtain from Eq. (3.26)

$$\Delta_\omega(\tau_0)=f\omega\Delta\cos\omega\tau_0\int_0^\infty\frac{\cos\omega\tau\,d\tau}{\cosh^2\gamma\tau}, \tag{5.8}$$

where

$$\gamma=\sqrt{b\Delta/2m}, \tag{5.9}$$

and we have discarded the (averaged) contribution of the constant ξ_1.

Using the formula [20]

$$\int_0^\infty d\tau\frac{\cos\omega\tau}{\cosh^2\gamma\tau}=\frac{1}{\gamma}\Gamma\left[1+\frac{i\omega}{2\gamma}\right]\Gamma\left[1-\frac{i\omega}{2\gamma}\right] \tag{5.10}$$

and

$$\Gamma(1+z)\Gamma(1-z)=\frac{\pi z}{\sin\pi z}, \tag{5.11}$$

we obtain

$$\Delta_\omega(\tau_0)=\frac{f\omega}{\gamma}\Delta\cos\omega\tau_0\frac{\omega\pi/2\gamma}{\sinh\omega\pi/2\gamma}. \tag{5.12}$$

For the interior well, again discarding the constant term, we have from Eq. (3.29)

$$\Delta'_\omega(\tau_0)=-f\omega\Delta\cos\omega\tau_0\int_0^\infty\frac{\cos\omega\tau\,d\tau}{\sinh^2(\gamma\tau+\delta/2)}. \tag{5.13}$$

To evaluate this integral, we use the series expansion

$$\frac{1}{\sinh^2(\gamma\tau+\delta/2)}=4\sum_{h=0}^{\infty}(k+1)e^{-(2\gamma\tau+\delta)(k+1)} \tag{5.14}$$

and obtain directly the convergent expansion

$$\Delta'_\omega(\tau_0)=-8\gamma f\omega\Delta\cos\omega\tau_0\sum_{k=0}^{\infty}\frac{(k+1)^2}{4\gamma^2(k+1)^2+\omega^2}\times e^{-2\gamma(k+1)}. \tag{5.15}$$

We now turn to the dissipative contributions. In the presence of dissipation, as pointed out in Eq. (4.15), $\dot{\beta}\to 0$ as $\tau\to\infty$. Neglect of this effect would result in a divergence of the contribution of the $\rho^2\dot{\beta}^2$ term in Eq. (5.7), since $\rho^2\dot{\beta}$ is constant in the unperturbed problem. We are, however, testing for homoclinic points in the $\rho,\dot{\rho}$ sector of phase space, and hence we may treat the angular momentum nonperturbatively in coupling it to the motion in ρ, i.e., we shall use the precise form of Eq.

(4.12). It then follows, for the exterior well, that

$$\Delta_k = 2k\int_0^\infty d\tau \frac{1}{\Delta+\xi_1\cosh^2\gamma\tau}\left[\frac{\gamma^2\Delta^2\sinh^2\gamma\tau}{\cosh^4\gamma\tau} - \frac{\lambda^2}{m^2}\cosh^2\gamma\tau e^{-2k\tau}\right], \tag{5.16}$$

where we have used Eq. (3.26) to obtain

$$\dot\rho = -\frac{\gamma\Delta\sinh\gamma\tau}{\cosh^2\gamma\tau\sqrt{\Delta+\xi_1\cosh^2\gamma\tau}} . \tag{5.17}$$

For simplicity in obtaining an estimate of Δ_k, we take $k=\gamma/2$, a small quantity. Letting $\zeta=\cosh\gamma\tau$, Eq. (5.16) becomes

$$\Delta_k = \frac{1}{\xi_1}\left[\gamma^2\Delta^2 I_1(\zeta) - \frac{\lambda^2}{m^2}I_2(\zeta)\right]\Bigg|_1^\infty , \tag{5.18}$$

where we have defined the indefinite integrals

$$I_1(\zeta) = \int\frac{d\zeta}{\zeta^2+d^2}\frac{\sqrt{\zeta^2-1}}{\zeta^4} , \tag{5.19}$$

$$I_2(\zeta) = \int\frac{d\zeta}{\sqrt{\zeta^2-1}}\frac{1}{\zeta^2+d^2}\zeta^2(\zeta-\sqrt{\zeta^2-1}) , \tag{5.20}$$

and $d\equiv\sqrt{\Delta/\xi_1}$. For the interior well using, from Eq. (3.29),

$$\dot\rho = \frac{\gamma\Delta\cosh\phi}{\sinh^2\phi\sqrt{\xi_1\sinh^2\phi-\Delta}} , \tag{5.21}$$

where $\phi=\gamma\tau+\delta/2$, we obtain

$$\Delta_k' = 2k\int_0^\infty d\tau\frac{1}{\xi_1\sinh^2\phi-\delta}\times\left[\gamma^2\Delta^2\frac{\cosh^2\phi}{\sinh^4\phi} - \frac{\lambda^2}{m^2}\sinh^2\phi e^{-2k\tau}\right] . \tag{5.22}$$

We have started the integration at light cone at $\tau=0$. At this point the divergences in $\dot\rho^2$ and $\rho^2\dot\beta^2$ cancel, even in the presence of dissipation, since then the factor $e^{-k\tau}\rightarrow 1$. We see that, in Eq. (5.20), the denominator vanishes, but the numerator becomes

$$\gamma^2\xi_1^2\left[1+\frac{\Delta}{\xi_1}\right] - \frac{\lambda^2}{m^2}\frac{\Delta}{\xi_1} , \tag{5.23}$$

which vanishes for

$$bm\xi_1^2\xi_2 = 2\lambda^2 ;$$

this is, in fact, the condition (3.23) determining the turning point ξ_2.

Letting $\zeta=\sinh\varphi$, and taking $k=\gamma/2$ as above, we obtain

$$\Delta_k' = \frac{1}{\xi_1}\left[\gamma^2\Delta^2 I_1'(\zeta) - \frac{\lambda^2}{m^2}e^{\delta/2}I_2'(\zeta)\right]\Bigg|_d^\infty , \tag{5.24}$$

where we define the indefinite integrals

$$I_1'(\zeta) = \int\frac{d\zeta}{\zeta^2-d^2}\frac{\sqrt{\zeta^2+1}}{\zeta^4} , \tag{5.25}$$

$$I_2'(\zeta) = \int\frac{d\zeta}{\sqrt{\zeta^2+1}}\frac{\zeta^2(\sqrt{\zeta^2+1}-\zeta)}{\zeta^2-d^2} . \tag{5.26}$$

The integrals (5.19), (5.20) and (5.25), (5.26) are elementary (after some transformation the formulas can be found in Ref. [20]). The solutions are

$$I_1(\zeta) = \frac{1}{d^4}\left[\frac{\sqrt{\zeta^2-1}}{\zeta} + \frac{d^2(\zeta^2-1)^{3/2}}{3\zeta^3} + \frac{\sqrt{1+d^2}}{d}\,\mathrm{Im}\sin^{-1}\left[\frac{d^2+1}{\zeta-id}-id\right]\right] , \tag{5.27}$$

$$I_2(\zeta) = \sqrt{\zeta^2-1}-\zeta-\frac{d^2}{2i\sqrt{1+d^2}}\ln\frac{\sqrt{\zeta^2-1}-i\sqrt{1+d^2}}{\sqrt{\zeta^2-1}+i\sqrt{1+d^2}} + \frac{d}{2i}\ln\frac{\zeta-id}{\zeta+id} \tag{5.28}$$

and (obtained essentially by taking $\zeta\rightarrow i\zeta$)

$$I_1'(\zeta) = \frac{1}{d^4}\left\{\left[\frac{\zeta^2+1}{\zeta}\right]^{1/2} + d^2\frac{(\zeta^2+1)^{3/2}}{3\zeta^3} - \left[\frac{d^2+1}{2d}\right]^{1/2}\left[\sinh^{-1}\left[\frac{d^2+1}{\zeta-2}+d\right] - \sinh^{-1}\left[\frac{d^2+1}{\zeta+d}-d\right]\right]\right\} , \tag{5.29}$$

$$I_2'(\zeta) = \zeta-\sqrt{\zeta^2+1}+\frac{d}{2}\ln\frac{\zeta-d}{\zeta+d} - \frac{d^2}{2\sqrt{1+d^2}}\ln\frac{\sqrt{\zeta^2+1}-\sqrt{1+d^2}}{\sqrt{\zeta^2+1}+\sqrt{1+d^2}} . \tag{5.30}$$

At the limits for the exterior well

$$I_1(1) = 0 ,$$

$$I_1(\infty) = \frac{1}{d^4}\left[1+\frac{d^2}{3}-\frac{\sqrt{1+d^2}}{d}\sinh^{-1}d\right] . \tag{5.31}$$

To evaluate I_2, define

$$\theta = \tan^{-1}\frac{\sqrt{1-d^2}}{\sqrt{\zeta^2-1}} , \quad \phi = \tan^{-1}\frac{d}{\zeta} .$$

Then

$$\ln\frac{\sqrt{\zeta^2-1}-i\sqrt{1+d^2}}{\sqrt{\zeta^2-1}+i\sqrt{1-d^2}} = \begin{cases}-i\pi , & \zeta=1\\ 0 , & \zeta=\infty ,\end{cases}$$

$$\ln\frac{\zeta-id}{\zeta+id}=\begin{cases}-2i\tan^{-1}d\ , & \zeta=1\\ 0\ , & \zeta=\infty\ .\end{cases}$$

Hence,

$$I_2(1)=-1+\frac{d^2\pi}{2\sqrt{1+d^2}}-d\tan^{-1}d\ ,$$
$$I_2(\infty)=0\ . \tag{5.32}$$

In evaluation Δ_k and Δ_k' we use the fact that the expression (5.23) vanishes, i.e., that

$$\frac{\lambda^2}{m^2}=\gamma^2\Delta^2\frac{(1+d^2)}{d^6}\ . \tag{5.33}$$

Then,

$$\Delta_k=\frac{4k^2\Delta}{d^2}\left[1+\frac{d^2}{3}-\frac{\sqrt{1+d^2}}{d}\sinh^{-1}d-\left[\frac{1+d^2}{d^2}\right]\left[1-\frac{d^2\pi}{2\sqrt{1+d^2}}+d\tan^{-1}d\right]\right]. \tag{5.34}$$

For $d\ll 1$ (with $k=\gamma/2$),

$$\Delta_k\simeq\frac{-4k^2\Delta}{d^4}=\frac{-\xi_1^2 b}{2m}\ . \tag{5.35}$$

For the interior well, the lower limit is somewhat delicate; the singularities in I_1' and I_2' cancel due to the relation (5.33). Using the relations

$$\sinh^{-1}\frac{d^2+1}{\zeta-d}\sim\ln 2(d^2+1)-\ln(\zeta-d)\ ,$$
$$\ln(\sqrt{\zeta^2+1}-\sqrt{1+d^2})\sim\ln(\zeta-d)+\ln\frac{d}{\sqrt{1+d^2}}$$

for $\zeta\sim d$, and

$$I_1'(\infty)=\frac{1}{d^4}\left[1+\frac{d^2}{3}-\frac{\sqrt{d^2+1}}{d}\sinh^{-1}d\right],$$
$$I_2'(\infty)=0\ ,$$

we obtain

$$\Delta_k'=\frac{4h^2\Delta}{d^2}\left\{-\frac{1}{d^2}+\frac{d^2}{3}-\frac{\sqrt{d^2+1}}{d}\left[\frac{4}{3}+\frac{d^2}{3}+\sin^{-1}d-\tfrac{1}{2}\sinh^{-1}\frac{1-d^2}{2d}-\ln\sqrt{2(1+d^2)}\right.\right.$$
$$\left.\left.-\sqrt{1+d^2}\ln[2d(d+\sqrt{1+d^2})]^{1/2}-(d+\sqrt{1+d^2})\ln\left[\frac{d}{2(1+d^2)}\right]^{1/2}\right]\right\}. \tag{5.36}$$

For small d, the estimate for Δ_k' is equal to that of Δ_k.

The coefficient of $\cos\omega\tau_0$ in the driving force contribution [Eq. (5.12)] to the Melnikov integral has a maximum at $\omega\simeq 2\gamma/\pi$ (1.914), for which

$$\Delta_\omega(\tau_0)\simeq(1.104)\frac{2f}{\pi}\Delta\cos\omega\tau_0\ . \tag{5.37}$$

Hence, for the exterior well separatrix motion, the Melnikov integral has an infinite number of zeros, at this maximizing frequency, for

$$(1.104)\frac{2f}{\pi}\gtrsim\frac{4k^2}{d^4}\ . \tag{5.38}$$

At sufficiently large ω, for $k\neq 0$, $\Delta_\omega(\tau_0)\sim 0(\omega^2 e^{-\omega\pi/2\gamma})$, so the Melnikov integral has no zero, and hence there is no chaotic behavior.

At very small ω ($\omega\pi/2\gamma\ll 1$) the condition for the existence of zeros is

$$\frac{f}{\gamma}\omega\gtrsim\frac{4k^2}{d^4}\ , \tag{5.39}$$

and for any given f there is sufficiently small ω such that chaotic behavior does not occur.

We now turn to the bounds for the interior well, where $\Delta_\omega'(\tau_0)$ is given by Eq. (5.15). We investigate the bounds for small and large ω.

For $\omega\ll 2\gamma$, the sum is approximated by

$$\frac{1}{4\gamma^2}\sum_{k=0}^{\infty}e^{-2\delta(k+1)}=\frac{1}{4\gamma^2}\frac{1}{e^{2\delta}-1}\simeq\frac{1}{16\gamma^2 d}\ , \tag{5.40}$$

where we have used $\delta\simeq 2d$ for small d in the last relation.

The coefficient of $\cos\omega\tau_0$ in Eq. (5.15) is therefore greater than the estimate (5.35) for Δ_k' if

$$\frac{f\omega}{\gamma}\gtrsim\frac{8k^2}{d^3}\ . \tag{5.41}$$

For sufficiently small ω, this condition cannot be satisfied.

For ω large, we note that the contribution from the series from terms $k+1\gtrsim\omega/2\gamma$ are of the order

$$\frac{1}{4\gamma^2}\sum_{\omega/2\gamma}^{\infty}e^{-2\delta(k+1)}=\frac{1}{4\gamma^2}\frac{e^{-\delta\omega/\gamma}}{e^{2\delta}-1}\ .$$

We thus estimate a lower bound for the series as

$$\frac{1}{\omega^2}\sum_{k=0}^{\omega/2\gamma}(k+1)^2e^{-2\delta(k+1)}\cong\frac{1}{\omega^2}\frac{1}{32d^3}\ ,$$

using again that $\delta\cong 2d$ for small d. Hence, for large ω, the coefficient of $\cos\omega\tau_0$ in Eq. (5.15) is greater than the estimate (5.35) for Δ_k' if

$$\frac{f\gamma}{\omega}\gtrsim\frac{16k^2}{d}\ . \tag{5.42}$$

For sufficiently large ω this condition cannot be satisfied.

We remark that for both large and small ω, the conditions for chaotic behavior are somewhat easier to satisfy for the interior well.

For intermediate, finite values of ω, the criterion for the existence of a horseshoe can always be satisfied with sufficiently large (but still perturbative) driving force. We have therefore analytically demonstrated the existence of relativistic chaotic motion for the system we have studied.

VI. SUMMARY AND DISCUSSION

We have studied the nonlinear classical dynamics of a relativistic two-body system in $1+1$ dimensions with interaction described by a relativistic generalization of the well-known Duffing potential. The framework we have used is based on the canonical relativistic dynamics of Stueckelberg [14], in which the time of occurrence of an event, as well as its position are considered to be dynamical variables. The time of occurrence, or detection of an event, is subject to variation due to the action of forces (as for the gravitational redshift) as well as the usual effect of the relative motion of the detector through the Lorentz transformation [15].

The dynamical equations for the relative coordinates $x(\tau)$, $t(\tau)$ are separable in hyperbolic coordinates. We have solved these equations in quadrature; the Hamiltonian of the system provides a first integral. Closed expressions were found for the "radial" variable $\rho^2(\tau)=x(\tau)^2-t(\tau)^2$ for the unperturbed problem. The hyperbolic angle $\beta=\tanh^{-1}t/x$ is then determined by conservation of the angular momentum M^{01}, corresponding to the generator of Lorentz boosts. Guided by the structure of the expression for the accelerations $\ddot{x}=d^2x/d\tau^2$, $\ddot{t}=d^2t/d\tau^2$, we introduced driving and damping forces as quantities which affect the accelerations linearly. The periodic driving forces were chosen so that M^{01} remains conserved, but dissipative forces result in an exponential decrease of this quantity. In the limit in which $M^{01}\rightarrow 0$, $x(\tau)$ and $t(\tau)$ become proportional, and the two coupled equations of motion reduce to a single one, identical to the nonrelativistic Duffing oscillator (stable if the proportionately constant is such that the relative motion is spacelike). This equation is known to have chaotic behavior near the separatrix, under weak driving and damping, and a strange attractor for stronger perturbation. This would imply that similar behavior should occur in the relativistic generalization of the Duffing problem and was indeed observed in the computer solutions for the motion reported in Ref. [10]. Using the exact solutions for the separatrix orbit, we have shown where that the Melnikov criterion for the existence of chaotic behavior under small driving and damping perturbation demonstrates analytically the existence of chaotic behavior (in this work, we confined the motion to the spacelike region by using an infinite potential in the timelike sector).

The existence of chaos in spacetime has important consequences for our perception of the structure of matter and its evolution. The simpler picture of a smoothly developing world line must, in this case, be replaced by a more complicated, stochastic structure, i.e., initially nearby orbits may follow very different paths, so that the manifold generated by the motion is not smooth.

As an example of the application of this result, the test of reversible motion is in choosing a set of initial conditions that have all velocities reversed and to check whether the ensuring motion retraces the path followed in reaching those conditions at this initial time. In Galilean mechanics, there is no question of principle in achieving this with sufficiently accurate measurements of position, since the time at which the position is observed is that of the laboratory clock, which also serves as the parameter of motion in the equations.

In the relativistic case, the time at which observations are made is, in the same way, the time of the laboratory, but is not identical (as it would be in the Galilean case) to the invariant time parameter governing the motion. Hence, the determination of the τ associated with each t requires a sufficient number of observations in the laboratory to fit the functions of $x^{\mu}(\tau)$ known from the solution of the equations of motion. For sufficiently smooth motion and accurate observation of x, t, this determination can be made in principle. Under conditions of chaotic motion, however, it becomes difficult, if not impossible, to relate observations in the laboratory to the functions $x^{\mu}(\tau)$ with sufficient accuracy to determine the *a posteriori* correspondence of the set of events observed with the parameter of the motion τ. There is then no way of choosing a set of initial conditions which would result in a motion that retraces the path of arrival at these conditions.

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APPENDIX: CONTINUITY CONDITIONS ACROSS THE LIGHT CONE

We rewrite Eqs. (2.31) and (2.33) in regions $I_{\pm}$, $II_{\pm}$ as

$$\dot{\rho}=\sigma_s^{\pm}\sqrt{2/m}\left[K_{\rm rel}-V_{I_{\pm}}(\rho^2)+\frac{\lambda^2}{2m\rho^2}\right]^{1/2} \tag{A1}$$

$$\dot{\rho}'=\sigma_t^{\pm}\sqrt{2/m}\left[-K_{\rm rel}+V_{{\rm II}_\pm}(\dot{\rho}^2)+\frac{\lambda^2}{2m\rho'^2}\right]^{1/2}, \quad \text{(A2)}$$

where we have taken into account the possibility of invariantly choosing different potential functions in each region. In the neighborhood of the light cone, $\rho(\rho')\to 0, \beta(\beta')\to\pm\infty$, we cannot *a priori* neglect the small terms $e^{-|\beta|}(e^{-|\beta'|})$, since they may occur multiplied by $1/\rho(1/\rho')$ [e.g., in the formula (2.17), $e^{-\beta}\dot{\rho}$ cannot be neglected for $\beta\to\infty$, since $\dot{\rho}$ diverges with $1/\rho$]. Expanding the square roots for small $\rho(\rho')$ and using the relations (2.17), (2.18), one obtains in $\mathrm{I}_\pm$,

$$\dot{x}\sim\frac{\lambda}{m\rho}\sinh\beta\pm\sigma_s^{\pm}\left[\frac{|\lambda|}{\rho m}+\frac{\rho}{|\lambda|}(K_{\rm rel}-V_{{\rm I}_\pm})\right]\cosh\beta\ , \quad \text{(A3)}$$

$$\dot{t}\sim\pm\frac{\lambda}{m\rho}\cosh\beta+\sigma_s^{\pm}\left[\frac{|\lambda|}{\rho m}+\frac{\rho}{|\lambda|}(K_{\rm rel}-V_{{\rm I}_\pm})\right]\sinh\beta \quad \text{(A4)}$$

and in $\mathrm{II}_\pm$

$$\dot{x}'=\mp\frac{\lambda}{m\rho'}\cosh\beta' +\sigma_t^{\pm}\left[\frac{|\lambda|}{m\rho'}+\frac{\rho'}{|\lambda|}(-K_{\rm rel}+V_{{\rm II}_\pm})\right]\sinh\beta'\ , \quad \text{(A5)}$$

$$\dot{t}'=-\frac{\lambda}{m\rho'}\sinh\beta' \pm\sigma_t^{\pm}\left[\frac{|\lambda|}{m\rho'}+\frac{\rho'}{|\lambda|}+(-K_{\rm rel}+V_{{\rm II}_\pm})\right]\cosh\beta\ . \quad \text{(A6)}$$

Since $d\tau\geq 0$, $\sigma_s^{\pm}d\rho\geq 0$, $\sigma_t^{\pm}d\rho\geq 0$ ($d\rho$ can change sign only at turning points). For $\rho(\rho')\to 0$, $\beta(\beta')\to\infty$, $\dot{x},\dot{t}$ and $\dot{x}',\dot{t}'$ must be bounded. Hence from (A3)–(A6), we have the sign correlations

$$\sigma_s^{\pm}=\mp\,{\rm sgn}\beta\,{\rm sgn}\lambda\ ,\quad \sigma_t^{\pm}=\pm\,{\rm sgn}\beta'{\rm sgn}\lambda\ . \quad \text{(A7)}$$

It then follows in $\mathrm{I}_\pm$ that for $\beta\to+\infty$

$$\dot{x}\sim\frac{-\lambda}{m\rho}e^{-\beta}-\frac{\rho}{2\lambda}(K_{\rm rel}-V_{{\rm I}_\pm})e^{\beta}\ ,$$
$$\dot{t}\sim\pm\left[\frac{\lambda}{m\rho}e^{-\beta}-\frac{\rho}{2\lambda}(K_{\rm rel}-V_{{\rm I}_\pm})e^{\beta}\right] \quad \text{(A8)}$$

and for $\beta\to-\infty$,

$$\dot{x}\sim\frac{\lambda}{m\rho}e^{-|\beta|}+\frac{\rho}{2\lambda}(K_{\rm rel}-V_{{\rm I}_\pm})e^{|\beta|}\ ,$$
$$\dot{t}\sim\pm\left[\frac{\lambda}{m\rho}e^{-|\beta|}-\frac{\rho}{2\lambda}(K_{\rm rel}-V_{{\rm I}_\pm})e^{|\beta|}\right]. \quad \text{(A9)}$$

In $\mathrm{II}_\pm$, for $\beta'\to+\infty$,

$$\dot{x}'\sim\pm\left[-\frac{\lambda}{m\rho'}e^{-\beta'}+\frac{\rho'}{2\lambda}(K_{\rm rel}+V_{{\rm II}_\pm})e^{\beta'}\right],$$
$$\dot{t}'\sim\left[\frac{\lambda}{m\rho'}e^{-\beta'}+\frac{\rho'}{2\lambda}(-K_{\rm rel}+V_{{\rm II}_\pm})e^{\beta'}\right] \quad \text{(A10)}$$

and for $\beta'\to-\infty$,

$$\dot{x}'\sim\pm\left[-\frac{\lambda}{m\rho'}e^{-|\beta'|}+\frac{\rho'}{2\lambda}(-K_{\rm rel}+V_{{\rm II}_\pm})e^{|\beta'|}\right],$$
$$\dot{t}'\sim-\left[\frac{\lambda}{m\rho'}e^{-|\beta'|}+\frac{\rho'}{2\lambda}(-K_{\rm rel}+V_{{\rm II}_\pm})e^{|\beta'|}\right]. \quad \text{(A11)}$$

Let us designate the four branches of the $1+1$ light cones as (i) on the boundary I_+, II_+, (ii) on II_+, I_-, (iii) on I_-, II_-, and (iv) on II_-, I_+.

The spacetime coordinates must be continuous in passing these boundaries, so that at

$$\text{(i)}\quad x\sim\frac{\rho}{2}e^{\beta}=\frac{\rho'}{2}e^{\beta'}\ ,\quad t\sim\frac{\rho}{2}e^{\beta}=\frac{\rho'}{2}e^{\beta'}\ ,$$
$$\text{(ii)}\quad x\sim-\frac{\rho}{2}e^{\beta}=-\frac{\rho'}{2}e^{-\beta'}\ ,\quad t\sim\frac{\rho}{2}e^{\beta}=\frac{\rho'}{2}e^{-\beta'}\ ,$$
$$\text{(iii)}\quad x\sim-\frac{\rho}{2}e^{-\beta}=-\frac{\rho'}{2}e^{-\beta'}\ ,\quad t\sim-\frac{\rho}{2}e^{-\beta}=-\frac{\rho'}{2}e^{-\beta'}\ ,$$
$$\text{(iv)}\quad x\sim\frac{\rho}{2}e^{-\beta}=\frac{\rho'}{2}e^{\beta'}\ ,\quad t\sim-\frac{\rho}{2}e^{-\beta}=-\frac{\rho'}{2}e^{\beta'}\ . \quad \text{(A12)}$$

It then follows from (A8)–(A11) that the velocities $\dot{x}$, $\dot{t}$ must change on these boundaries according to [we assume $V_{{\rm I}_\pm}$, $V_{{\rm II}_\pm}$ have finite limits as $\rho(\rho')\to 0$]

$$\text{(i)}\quad \dot{x}_{{\rm II}_\pm}-\dot{x}_{{\rm I}_\pm}=\frac{x}{\lambda}(V_{{\rm II}_+}-V_{{\rm I}_+})\ ,$$
$$\dot{t}_{{\rm II}_\pm}-\dot{t}_{{\rm I}_\pm}=\frac{x}{\lambda}(V_{{\rm II}_+}-V_{{\rm I}_+})\ ; \quad \text{(A13)}$$

$$\text{(ii)}\quad \dot{x}_{{\rm II}_\pm}-\dot{x}_{{\rm I}_-}=-\frac{x}{\lambda}(V_{{\rm II}_+}-V_{{\rm I}_-})\ ,$$
$$\dot{t}_{{\rm II}_+}-\dot{t}_{{\rm I}_-}=\frac{x}{\lambda}(V_{{\rm II}_+}-V_{{\rm I}_-})\ ; \quad \text{(A14)}$$

$$\text{(iii)}\quad \dot{x}_{{\rm II}_-}-\dot{x}_{{\rm I}_-}=\frac{x}{\lambda}(V_{{\rm II}_-}-V_{{\rm I}_-})\ ;$$
$$\dot{t}_{{\rm II}_-}-\dot{t}_{{\rm I}_-}=\frac{x}{\lambda}(V_{{\rm II}_-}-V_{{\rm I}_-})\ ; \quad \text{(A15)}$$

$$\text{(iv)}\quad \dot{x}_{{\rm II}_-}-\dot{x}_{{\rm I}_+}=-\frac{x}{\lambda}(V_{{\rm II}_-}-V_{{\rm I}_+})\ ,$$
$$\dot{t}_{{\rm II}_-}-\dot{t}_{{\rm I}_+}=\frac{x}{\lambda}(V_{{\rm II}_-}-V_{{\rm I}_+})\ . \quad \text{(A16)}$$

With these conditions, the motion can, in principle, be integrated through the light cone from the quadrature solution with general potential function in all sectors.

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