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Unitarity and irreversibility in chaotic systems

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We analyze the spectral properties of the Perron-Frobenius operator U, associated with some simple highly chaotic maps. We obtain a spectral decomposition of U in terms of generalized eigenfunctions of U and its adjoint. The corresponding eigenvalues are related to the decay rates of correlation functions and have magnitude less than one, so that physically measurable quantities manifestly approach equilibrium. To obtain decaying eigenstates of unitary and isometric operators it is necessary to extend the Hilbert-space formulation of dynamical systems. We describe and illustrate a method to obtain the decomposition explicitly.

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I. INTRODUCTION

One of the oldest and most important problems in the field of nonequilibrium statistical mechanics is the derivation of the thermodynamic approach to equilibrium from the basic laws of dynamics. This problem has been studied for some years by Prigogine and his co-workers, who argue that irreversibility arises from dynamics, rejecting explanations based on extradynamical considerations [1]. His group has recently developed a "complex spectral theory" of unstable physical systems in which irreversibility is seen as a spectral property of the generator of time evolution [2,3]. The characteristic quantities of irreversible behavior, such as lifetimes, decay rates, and transport coefficients, appear as eigenvalues in a generalized spectral decomposition.

For the so-called "large Poincaré systems"—a large class of systems with continuous spectrum—they construct a spectral representation of the Hamiltonian or Liouvillian operator which includes generalized decaying eigenstates. The corresponding eigenvalues are related to lifetimes and cross sections. Their method, which is based on a time-ordering rule for the analytic continuation of a perturbation series [4,5], has been applied to quantum systems involving unstable particles and scattering, and is applicable to more general problems involving interacting fields [6–8].

In this paper, we obtain analogous results for highly chaotic maps. The analysis is not perturbative, so our method appears different from the method used to analyze large Poincaré systems. In particular, we do not encounter small denominators and therefore do not need an analytic continuation based on a time-ordering rule. Nevertheless, the two methods produce very similar mathematical structures. We believe our method is valid for highly chaotic systems such as Anosov diffeomorphisms and strong perturbations of integrable systems where the Kol'mogorov-Arnol'd-Moser (KAM) tori have been destroyed by resonances.

Chaotic maps can display at least three types of "thermodynamic" behavior usually associated with large or many-body systems. First, if a map is mixing, probability densities "approach equilibrium" in the sense that they eventually appear uniform according to any finite-precision measurement. Second, correlation functions can decay exponentially, sometimes with long-time tails. Third, trajectories can diffuse in phase space.

The simplest systems which display thermodynamic behavior in the sense described above are piecewise linear chaotic maps, which are mixing and for which correlation functions decay exponentially. Through an analysis of two such systems—the Bernoulli map and the baker transformation—we will show that irreversibility is associated with generalized decaying eigenstates of the time-evolution operator.

It is generally accepted that the Liouville theorem is a strong obstacle to a derivation of irreversibility from fundamental principles. The theorem means that distribution functions always retain complete memory of their initial state, or, mathematically, that time evolution is given by a group of unitary operators acting on a Hilbert space. For area-preserving maps, the unitary Perron-Frobenius operator U governs the time evolution of a probability distribution describing an ensemble of trajectories. The appropriate Hilbert space is the space of Lebesgue square-integrable functions on phase space. Spectral values of unitary operators lie on the unit circle $|z|=1$, implying that the time evolution is just a mixing of phases. These eigenvalues contain no time scales related to the approach to equilibrium, such as Lyapunov exponents and decay rates. For an approach to equilibrium, there apparently must be eigenvalues with magnitude $|z|<1$.

We will show that, for systems which display an approach to equilibrium, one can construct a natural extension of the Hilbert space which contains generalized decaying eigenstates whose eigenvalues lie inside the unit circle. The eigenvalues are related to decay rates of correlation functions and therefore give information about the approach to equilibrium. We refer to the spectral decomposition of U in terms of these eigenstates as

the "physical" spectral representation. This representation introduces time symmetry breaking and thermodynamic behavior into the description of the system at a fundamental level.

The generalized decaying eigenstates are Schwartz distributions (continuous linear functionals) defined on a set of test functions $\mathcal{T}$, which is dense in the Hilbert space. The subspace $\mathcal{T}$ contains only smooth (infinitely differentiable) functions and can be given a topology in which it is complete. To calculate correlation functions and expectation values using the physical spectral representation, distribution functions ("states") and observables must be restricted to $\mathcal{T}$. The mathematical framework is the rigged Hilbert space which has been used to describe quantum resonance phenomena [9].

By requiring that distribution functions and observables be elements of a space of test functions, one acknowledges that not all states approach equilibrium. For instance, the Dirac δ distribution, which describes a trajectory, is not in the space of test functions. (The δ function is not in the original Hilbert space, either, but the domain of the Perron-Frobenius operator can easily be extended from the Hilbert space to δ functions, while the domain of the physical representation cannot be extended.) The theory determines what kinds of functions can be observed if one accepts the second law of thermodynamics. Alternatively, if one takes the (philosophical) point of view that nature is smooth or that, even if not, lack of infinite-precision measurements make it appear smooth, then the restriction to smooth test functions is a rejection of unphysical or unobservable quantities. We note that $\mathcal{T}$ is dense in the original space and therefore the restriction is not a coarse-graining.

We will present a method, based on the subdynamics approach of Prigogine and co-workers, for constructing the physical spectral representation. The method is based on an analysis of the resolvent $R(z)=1/(z-U)$ of the time-evolution operator U. Poles of the resolvent are located at eigenvalues of U. (By a "pole" of an operator-valued function, we mean that a matrix element, which is an ordinary function of z, has a pole.) The "residues" are related to the corresponding eigenfunctions. In our method, we evaluate matrix elements of the resolvent for $|z|>1$ and analytically continue them inside the unit circle, where we locate the poles and evaluate the residues. For the unitary operator U of Sec. IV, which has an absolutely continuous spectrum on the circle $|z|=1$, matrix elements of $R(z)$ are analytic in the entire complex plane except at $|z|=1$, where they are discontinuous. To restore analyticity, we define these matrix elements on a two-sheeted Riemann surface with a cut on the unit circle. The matrix elements are analytic on the first sheet (the "physical" sheet—corresponding to the original complex plane), but have poles on the second, which we find by analytic continuation through the cut. We will show that the second-sheet poles can be associated with generalized decaying eigenvectors which lie outside the Hilbert space. If a pole is not simple, the eigenvalue is degenerate and we construct a generalized eigenspace. There is a close relation between the second-sheet poles discussed here and second-sheet resonance poles of the S matrix of scattering theory.

We will illustrate the method with an analysis of two paradigm systems—the Bernoulli map and the baker transformation. The baker transformation is a well-known example of a highly unstable K system. Its associated time-evolution operator is unitary and has continuous spectrum on the unit circle. The Bernouli map is a one-dimensional projection of the baker. Its time-evolution operator is not unitary, but the adjoint operator is isometric, so that the system shares many mathematical features with the baker case. We start with the Bernoulli map because it is simpler, so that a complete and thorough analysis is possible. We will find a spectral representation of the Perron-Frobenius operator in terms of a set of exponentially decaying left and right eigenstates and will show how to decompose the time evolution of expectation values and correlation functions in terms of these eigenstates. We will then analyze the baker transformation, showing that because its symmetries cause the eigenvalues to be degenerate, an eigenspace, rather than an eigenstate, is associated with each resonance. This fact leads to power-law corrections to exponential decay. In the two examples here, our method gives exact results. It also gives an exact solution for the so-called "multi-Bernoulli" map [10] and the Arnold cat map [11]. It has been applied perturbatively to the standard map to discuss diffusion [12,13]. Work on nonlinear perturbations of the maps discussed in this paper is in progress.

The work we present here is closely related to results obtained via the "thermodynamic formalism" (involving *equilibrium* statistical mechanics) of Ruelle [14]. The decaying eigenvalues we discuss are the same as the resonances introduced by Ruelle [15] and others to describe the power spectra of correlation functions, and are related to the poles of dynamical ζ functions (see Appendix E). The essential difference is that their results are existence proofs while our method gives an explicit construction of the generalized eigenstates. Our point of view is quite different since we see the results as being of fundamental importance to *nonequilibrium* statistical mechanics. In the Ruelle approach, this is obscured by the emphasis on formal equivalence to mathematical structures in equilibrium statistical mechanics.

In our work, the emphasis is on the construction of generalized eigenstates and on the spectral decomposition of U. Many papers discuss how to calculate the positions of the resonances [16,17]. Mori *et al.* [18] have made some preliminary attempts to find the eigenstates themselves. Dörfle [19] has done a thorough analysis of the spectrum and eigenfunctions of the Perron-Frobenius operator associated with the tent map, but does not consider generalized eigenstates not in L_2. Misra, Prigogine, and Courbage [20] constructed the unitary representation of the Perron-Frobenius operator. Moreover, they have shown that for K systems a nonunitary transformation leads to a Markov process. We show the existence of more than one representation for the original, untransformed distribution function. We have recovered the nonunitary transformations through a different approach. We have given a preliminary description of our results in two previous Letters [21,22].

II. RESOLVENT FORMALISM

In this section we introduce some basic notation and give a broad outline of the method. The details will be clarified in specific examples. We start with a dynamical system defined by a map f: $M \to M$ for some space M (which in our two examples will be the interval [0,1] of the real line and the unit square $[0,1]\times[0,1]$ in $\mathbb{R}^2$). We will study the evolution of ensembles of trajectories described by a probability distribution $\rho(x)$. Since $\rho(x)$ is a probability density, it should be integrable, i.e., it should belong to the function space $L_1(M)$. We will instead follow the usual approach [23] and restrict $\rho(x)$ to the Hilbert space $L_2(M)$ of square-integrable functions, where there is a rich mathematical structure. Most of the results from L_2 can easily be extended to L_1.

We use a bra-ket notation [24], representing the distribution function ρ by the ket $|\rho)$. The expectation value of an observable A is denoted by the inner product $(A|\rho)$. For convenience, we sometimes use the notation $(x|\rho)=(\rho|x)^*\equiv\rho(x)$ for $x\in M$. Using this notation, we can evaluate inner products by writing

$$(A|\rho)=\int_{x\in M} dx\,(A|x)(x|\rho)=\int_M dx\; A^*(x)\rho(x)\ .$$

In this paper we will usually consider real-valued functions, but the formalism is valid for complex-valued functions, where $(A|B)=(B|A)^*$.

The time evolution of ρ is governed by the Perron-Frobenius operator U, which is defined by

$$\begin{aligned}\rho_{n+1}(x)&=U\rho_n(x)\\&=\int_M dy\;\delta(x-f(y))\rho_n(y)\\&=\sum_{y=f^{-1}(x)}\frac{\rho_n(y)}{|f'(y)|}\ .\end{aligned}\tag{2.1}$$

The sum is over the inverse branches of the possibly many-to-one map f.

To analyze (2.1), it is usually sufficient to study the matrix elements $(i|U^n|i')$ of U^n with respect to some orthonormal basis $\{|i)\}$. An orthonormal basis $\{|i)\}$ for $L_2(M)$ satisfies $1=\sum_i|i)(i|$ and $(i|i')=\delta_{i,i'}$. Inserting the identity operator twice into (2.1), the time evolution of ρ can be written in terms of the matrix elements by

$$\begin{aligned}\rho_n(x)&=U^n\rho_0(x)\\&=\sum_{i,i'}(x|i)(i|U^n|i')(i'|\rho_0)\ .\end{aligned}\tag{2.2}$$

If the system is ergodic (with uniform invariant distribution), then the matrix elements $(i|U^n|i')$, which contain an integral over phase space, are equivalent to time correlation functions, containing an integral over time, between "observables" $(x|i)$ and $(x|i')$ measured n time steps apart.

We start by rewriting the matrix elements in terms of the resolvent of U, $R(z)=1/(z-U)$:

$$(i|U^n|i')=\frac{1}{2\pi i}\oint_C dz\,(i|\frac{z^n}{z-U}|i')\ ,\tag{2.3}$$

where the contour C is taken counterclockwise just outside the circle $|z|=1$. [The notation we use in this paper is different from that used in Refs. [21,22]. There we defined a resolvent $1/(e^{-iz}-U)$ so that everything would be consistent with the continuous time case [1]. In this paper, e^{iz} has been replaced by z and the integration is on a circle in the complex plane rather than on the real axis.] Equation (2.3) is valid for $n\geq 0$ if the spectrum of U is inside or on the unit circle. The proof is given in Appendix A.

Now we deform the contour C by shrinking it (see Fig. 1), and assume (demonstrating in specific examples) that the matrix elements of the resolvent contain isolated poles (i.e., there is a neighborhood of each pole that does not contain any other poles) in the region $|z|\leq 1$. Each time the contour passes a pole of the resolvent, we get a counterclockwise contour around that pole. We label the poles by an integer m and define $e^{-\gamma_m}$ ($\mathrm{Re}\gamma_m>0$) to be the position of the mth pole. After passing a finite number of poles, the contour is expressed as a finite number of contours around isolated poles, plus a "background" integral:

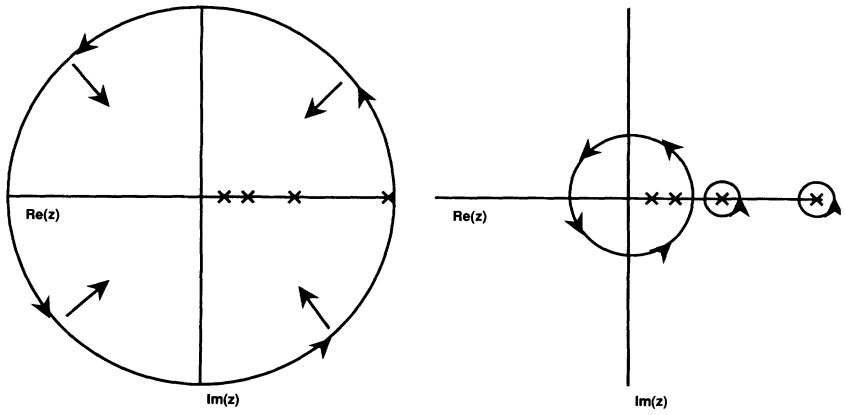

FIG. 1. The left-hand side shows the contour just outside $|z|=1$ in Eq. (2.3). Poles of the matrix element are shown at $z=1,\frac{1}{2},\frac{1}{4},\frac{1}{8}$ (which is appropriate for the Bernoulli map). On the right-hand side, the contour has been deformed by shrinking it past the first two poles, so that it is in the form in Eq. (2.4).

$$(i|U^n|i')=\sum_{\substack{m \\ e^{-\gamma_m}>e^{-\Gamma}}}\frac{1}{2\pi i}\oint_{z=e^{-\gamma_m}}dz(i|\frac{z^n}{z-U}|i') + \frac{1}{2\pi i}\oint_{|z|=e^{-\Gamma}}dz(i|\frac{z^n}{z-U}|i') \ . \tag{2.4}$$

In our two examples each matrix element contains a finite number of poles so that we can enclose all of them and eliminate the background integral. Equation (2.4) then becomes

$$(i|U^n|i')=\sum_m \frac{1}{2\pi i}\oint_{z=e^{-\gamma_m}}dz(i|\frac{z^n}{z-U}|i') \ . \tag{2.5}$$

The procedure described above appears straightforward but there are some subtle issues. We will be interested in situations where U has a continuous spectrum on or inside the unit circle. The resolvent operator itself becomes singular at $|z|=1$ but in our examples its matrix elements with respect to the basis $\{|i)\}$ are well behaved.

Consider the baker map, which we will discuss in detail in Sec. IV, where U is unitary and has continuous spectrum on $|z|=1$. Matrix elements of $1/(z-U)$ are analytic in the entire complex z plane (including infinity) but are discontinuous at $|z|=1$. They remain finite as $|z|$ approaches 1 because there are no eigenfunctions in Hilbert space corresponding to the (generalized) eigenvalues making up the continuous spectrum. We can therefore define the matrix elements to be functions on a two-sheeted Riemann surface with a cut on the unit circle. In deforming the contour as described above we avoid the discontinuity by performing an analytic continuation onto the second sheet. It is well known that a function that is analytic in the entire complex plane (including infinity) is a constant, so there must be singularities on the second Riemann sheet. In the case of the baker map, there are poles, and we will show that these poles correspond to generalized decaying eigendistributions which are not in the Hilbert space.

Equation (2.5) can be understood in terms of a set of "projection" operators $\Pi^{(m)}$, formally (we will usually consider matrix elements, not the operator itself) defined by

$$\Pi^{(m)}\equiv\frac{1}{2\pi i}\oint_{z=e^{-\gamma_m}}dz\frac{1}{z-U} \ . \tag{2.6}$$

Since U commutes with its resolvent, $U\Pi^{(m)}=\Pi^{(m)}U$. In Appendix B we show that these operators are orthogonal and idempotent, i.e.,

$$\Pi^{(\alpha)}\Pi^{(\beta)}=\Pi^{(\alpha)}\delta_{\alpha,\beta} \ , \tag{2.7}$$

and that

$$U^n\Pi^{(m)}=\oint_{z=e^{-\gamma_m}}dz\frac{z^n}{z-U} \ . \tag{2.8}$$

As we shall see, however, they are not Hermitian and in general can map functions outside the Hilbert space.

In terms of these operators, (2.5) can be written

$$(i|U^n|i')=\sum_m (i|U^n\Pi^{(m)}|i') \ . \tag{2.9}$$

From (2.9), one is tempted to write a completeness relation $\sum_m\Pi^{(m)}=1$, but this leads to problems of the definition of the domain of $\Pi^{(m)}$, which can be avoided by always looking at matrix elements.

If the resolvent has a simple pole at $z=e^{-\gamma_m}$, then [from (2.8)] $U\Pi^{(m)}=e^{-\gamma_m}\Pi^{(m)}$. In this case, $\Pi^{(m)}$ can be written

$$\Pi^{(m)}=|\gamma_m)(\tilde{\gamma}_m| \ , \tag{2.10}$$

where $|\gamma_m)$ and $(\tilde{\gamma}_m|$ are generalized left and right eigenstates of U satisfying

$$\begin{aligned} U|\gamma_m)&=e^{-\gamma_m}|\gamma_m) \ , \\ (\tilde{\gamma}_m|U&=e^{-\gamma_m}(\tilde{\gamma}_m| \ . \end{aligned} \tag{2.11}$$

Equation (2.11) will be proven when we discuss the Bernoulli map.

For unitary operators U, such as the Perron-Frobenius operator of the baker map (see Sec. IV), eigenstates whose eigenvalues have magnitudes less than 1 cannot be states in L_2. For example, because U is unitary, $(\gamma_m|U^\dagger U|\gamma_m)=(\gamma_m|\gamma_m)$. But from (2.11), $(\gamma_m|U^\dagger U|\gamma_m)=e^{-2\gamma_m}(\gamma_m|\gamma_m)$. This is a contradiction unless $(\gamma_m|\gamma_m)$ is zero or infinite. In our case, it turns out to be infinite. Therefore $|\gamma_m)$ is not in L_2. The same argument holds for the left eigenstate, $(\tilde{\gamma}_m|$. We will show that when $|\gamma_m)$ or $(\tilde{\gamma}_m|$ is not in L_2, it can be interpreted as continuous linear functional on a dense subset of L_2, i.e., as Schwartz distributions. A related statement holds for eigenstates of isometric operators, such as the adjoint of the Perron-Frobenius operator for the Bernoulli map (see Sec. III C).

III. THE BERNOULLI MAP

In this section we illustrate the formalism with a discussion of the Bernoulli map. We derive a physical spectral representation of the Perron-Frobenius operator in terms of its left and right eigenstates.

A. General discussion

The dyadic Bernoulli map takes the interval [0,1] into itself according to the rule $f(x)=2x \pmod 1$. It is the simplest example of a chaotic map. One can also define an asymmetric Bernoulli map. The analysis is essentially the same and has been outlined in a separate Letter [21].

The Perron-Frobenius operator associated with f [defined by (2.1)] is given by

$$\bar{U}\rho_n(x)=\frac{1}{2}\left[\rho_n(\frac{x}{2})+\rho_n(\frac{x}{2}+\frac{1}{2})\right] \ . \tag{3.1}$$

Because f is not one-to-one, $\bar{U}$ is not unitary. We have placed a bar over it to emphasize this point. One can show (see Appendix C) that the adjoint operator $\bar{U}^\dagger$ acts according to

$$\bar{U}^\dagger\rho(x)=\rho(f(x)) \ . \tag{3.2}$$

Although $\bar{U}$ is not unitary, $\bar{U}^\dagger$ is isometric ($\bar{U}\,\bar{U}^\dagger=1$ but

$\overline{U}^{\dagger}\overline{U}\neq 1$). This fact gives the Bernoulli system much in common with unitary systems.

For the reasons presented in Sec. II, we assume that distribution functions and observables live in the Hilbert space $L_2(0,1)$ of Lebesque square-integrable functions on [0,1]. We introduce a basis $\{|i)\}$ for $L_2(0,1)$ consisting of modified Legendre polynomials. A polynomial basis is in some sense natural because $\overline{U}$, which is essentially a scale transformation, leaves the order of a polynomial unchanged. The modified Legendre polynomials $\tilde{P}_i(x)$ are defined by $(x|i)=\tilde{P}_i(x)=\sqrt{2i+1}P_i(1-2x)$, where $P_i(x)$ is the ith Legendre polynomial. Orthonormality and completeness are expressed in our notation by

$$(i|i')=\delta_{i,i'} \tag{3.3}$$

and

$$1=\sum_{i=0}^{\infty}|i)(i| \ . \tag{3.4}$$

The matrix elements of $\overline{U}$ in this basis are (see Appendix D)

$$\begin{aligned}(i|\overline{U}|i')&=\int_0^1 dx\ \tilde{P}_i(x)\overline{U}\tilde{P}_{i'}(x)\\&=\tfrac{1}{2}[I_{ii'}+(-1)^{i+i'}I_{ii'}] \ ,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}I_{ii'}&\equiv\int_0^1 dx\ \tilde{P}_i(x)\tilde{P}_{i'}(x/2)\\&=\begin{cases}\dfrac{[(2i+1)(2i'+1)]^{1/2}}{2^i}\displaystyle\sum_{l=0}^{i'-i}(-\tfrac{1}{2})^l\frac{(i'+i+l)!}{(i'-i-l)!(2i+l+1)!l!}, & i\leq i'\\ 0, & i>i'\end{cases}\end{aligned} \tag{3.6}$$

The matrix elements (3.5) obey the "nonrecurrence" property:

$$(i|\overline{U}^{k}|i')(i'|\overline{U}^{l}|i)=0, \quad k,l\geq 1, \quad i\neq i' \ . \tag{3.7}$$

We say that $(i|\overline{U}|i')\neq 0$ contains a *transition* from the state $|i')$ to the state $|i)$. Equation (3.7) says that a transition $|i)\rightarrow|i')$ can never be followed by a transition $|i')\rightarrow|i)$; once the system leaves a state, it never returns. In this case, nonrecurrence follows from the fact that $I_{ii'}$, and therefore $(i|\overline{U}|i')$, is upper triangular.

The nonrecurrence property is essential for the rest of the calculation because it allows us to use the techniques of perturbation theory. In ordinary perturbation theory, one decomposes a system into a known (diagonal) part plus a perturbation characterized by a small parameter. The exact solution involves an infinite series in powers of the parameter. If the parameter is small enough, one can make a good approximation by truncating at a finite order. In our case, we do not have a small parameter, but the nonrecurrence condition (3.7) guarantees that the perturbation series will terminate at some finite order, so that arbitrary truncation is not needed. To use our method to examine systems in which one cannot find a nonrecurrent basis, it may be possible to obtain results using ordinary perturbation theory by expressing the system as a small perturbation from one in which (3.7) holds.

Following the procedure of Sec. II, we write matrix elements of $\overline{U}^{n}$ as an integral over the resolvent:

$$(i|\overline{U}^{n}|i')=\frac{1}{2\pi i}\oint_C dz(i|\frac{z^n}{z-\overline{U}}|i') \ . \tag{3.8}$$

To evaluate (3.8), we decompose the Perron-Frobenius operator into an "unperturbed" part given by the diagonal elements in the Legendre basis, and a "perturbation" composed of the off-diagonal elements:

$$\begin{aligned}\overline{U}&=\overline{U}_0+\delta\overline{U} \ ,\\ \overline{U}_0&\equiv\sum_i|i)(i|\overline{U}|i)(i|\\ &=\sum_i\frac{1}{2^i}|i)(i|\\ &\equiv\sum_i e^{-\gamma_i}|i)(i| \ ,\\ \delta\overline{U}&\equiv\sum_{i<i'}|i)(i|\overline{U}|i')(i'| \ .\end{aligned} \tag{3.9}$$

Note that there is no degeneracy—the diagonal elements of $\overline{U}_0$ are distinct.

We will use extensively the operator expansion

$$\frac{1}{z-A-B}=\frac{1}{z-A}\sum_{k=0}^{\infty}\left[B\frac{1}{z-A}\right]^k \ , \tag{3.10}$$

where A and B are operators and z is a complex number. Equation (3.10) converges only for "small" B, i.e., $\|B/(z-A)\|<1$. We will use (3.10) to calculate matrix elements of the resolvent $1/(z-U-\delta U)$ for $|z|>1$. Because of the nonrecurrence condition, the resulting formulas for matrix elements converge for all z (except for isolated poles) and thus effectively define an analytic continuation of the operator to $|z|<1$.

Substituting the decomposition (3.9) into (3.8) and expanding the resolvent operator in powers of $\delta\overline{U}$ using (3.10) we obtain

$$(i|\bar{U}^n|i') = \frac{1}{2\pi i}\oint_C dz\, z^n (i|\frac{1}{z-\bar{U}_0-\delta\bar{U}}|i')$$
$$= \frac{1}{2\pi i}\oint_C dz\, z^n \sum_{j=0}^{\infty} (i|\frac{1}{z-\bar{U}_0}\left[\delta\bar{U}\frac{1}{z-\bar{U}_0}\right]^j|i') . \tag{3.11}$$

Matrix elements in (3.11) always have the form

$$(i|\frac{1}{z-\bar{U}_0}\delta\bar{U}\frac{1}{z-\bar{U}_0}\delta\bar{U}\cdots\delta\bar{U}\frac{1}{z-\bar{U}_0}|i')$$
$$= \sum_{l_1,l_2,\ldots,l_j} \frac{1}{z-e^{-\gamma_i}}(i|\delta\bar{U}|l_1)\frac{1}{z-e^{-\gamma_{l_1}}}$$
$$\times (l_1|\delta\bar{U}|l_2)\cdots(l_j|\delta\bar{U}|i')\frac{1}{z-e^{-\gamma_{i'}}} , \tag{3.12}$$

where the nonrecurrence property requires that $i<l_1<l_2\cdots<l_j<i'$. If $j>i'-i$, this condition cannot be satisfied and (3.12) vanishes. Therefore although the terms in expansion (3.11) are not characterized by a small parameter, the series terminates because of the nonrecurrence property.

Since the expansion becomes a finite series, there is no renormalization of the positions of the poles of the resolvent. The matrix elements $(i|U^n|i')$ therefore have simple poles at $z=e^{-\gamma_m}$ for $i\le m\le i'$ and no other singularities. We show in Appendix E that the positions of the poles of the resolvent are the same as the resonances described by Ruelle [15] which can be calculated by standard methods [16,17].

Now we deform the contour in (3.8) as described in Sec. II. At each step, the matrix element is expressed as the sum of a finite number of contour integrals, each enclosing a separate pole $e^{-\gamma_m}$, plus a background integral containing all the rest. Since each matrix element has a finite number of poles, we can enclose all of them and obtain

$$(i|\bar{U}^n|i') = \sum_{m=i}^{i'} \frac{1}{2\pi i}\oint_{z=e^{-\gamma_m}} dz (i|\frac{z^n}{z-\bar{U}}|i') . \tag{3.13}$$

As we have seen, the fact that matrix elements in the Legendre basis involve a finite number of poles can be traced to the nonrecurrence condition. Matrix elements with respect to an arbitrary basis not satisfying this condition will in general contain an infinite number of poles which accumulate at $z=0$. In this case the background integral cannot be eliminated. The spaces of test functions we will define in Sec. III D are essentially the sets of functions for which the background integral can be eliminated or vanishes in the limit $\Gamma\to\infty$.

To evaluate (3.13) we use a projective decomposition of the resolvent in conjunction with the perturbation expansion (3.10). [In principle, one can evaluate (3.13) directly using (3.10). We introduce the projective decomposition because the elements of the decomposition are useful for defining the generalized eigenstates and because it simplifies the description of more complicated systems such as the baker transformation discussed in the next section.] The decomposition isolates a pole of the resolvent by expressing the resolvent roughly in the form $1/(z-U)\approx S(z)/(z-e^{-\gamma_m})$ where $S(z)$ is an operator which is regular at $e^{-\gamma_m}$. With the resolvent in this form, the Cauchy residue theorem can be used directly to evaluate the integrals (3.13). We now take a small detour to discuss the projective decomposition.

A projection operator $P^{(m)}$ satisfies $(P^{(m)})^2=P^{(m)}$. [We include the superscript (m) because it will be needed later. At this point, it is not relevant to the argument.] We define its complement $Q^{(m)}=1-P^{(m)}$ where 1 is the identity operator. For any such operator, one can prove the following identity [1,25]:

$$\frac{1}{z-U} = [P^{(m)}+\mathcal{C}^{(m)}(z)]\frac{1}{z-\Psi^{(m)}(z)}[P^{(m)}+\mathcal{D}^{(m)}(z)]$$
$$+\mathcal{P}^{(m)}(z) , \tag{3.14}$$

where the operators $\mathcal{C}^{(m)}(z)$, $\mathcal{D}^{(m)}(z)$, $\mathcal{P}^{(m)}(z)$, and $\Psi^{(m)}$ are defined by

$$\Psi^{(m)}(z) \equiv P^{(m)}UP^{(m)} + P^{(m)}UQ^{(m)}\frac{1}{z-Q^{(m)}UQ^{(m)}}Q^{(m)}UP^{(m)} ,$$
$$\mathcal{C}^{(m)}(z) \equiv \frac{1}{z-Q^{(m)}UQ^{(m)}}Q^{(m)}UP^{(m)} ,$$
$$\mathcal{D}^{(m)}(z) \equiv P^{(m)}UQ^{(m)}\frac{1}{z-Q^{(m)}UQ^{(m)}} , \tag{3.15}$$
$$\mathcal{P}^{(m)}(z) \equiv Q^{(m)}\frac{1}{z-Q^{(m)}UQ^{(m)}}Q^{(m)} .$$

For a proof, see, for example, [26].

To evaluate the integrals in (3.13), we isolate the effect of the pole by introducing a projection operator $P^{(m)}\equiv|m)(m|$ and decompose the resolvent operator according to Eq. (3.14). We show below that $\Psi^{(m)}(z)$ is independent of z and is given by

$$\Psi^{(m)}(z) = e^{-\gamma_m}P^{(m)} \tag{3.16}$$

and that the other operators $\mathcal{C}^{(m)}(z)$, $\mathcal{D}^{(m)}(z)$, and $\mathcal{P}^{(m)}(z)$ are not singular at $z=e^{-\gamma_m}$. We are free to decompose the resolvent using a different projection operator at each resonance, and use the operator $P^{(m)}$ to decompose the resolvent near the point $z=e^{-\gamma_m}$. This gives

$$(i|\bar{U}^n|i')=\sum_m \frac{1}{2\pi i}\oint_{z=e^{-\gamma_m}} dz (i|[P^{(m)}+\mathcal{C}^{(m)}(z)]\frac{z^n}{z-e^{-\gamma_m}}[P^{(m)}+\mathcal{D}^{(m)}(z)]+\mathcal{P}^{(m)}(z)|i') \ . \tag{3.17}$$

Using Cauchy's theorem to evaluate the integrals, and explicitly writing out the factor $P^{(m)}=|m)(m|$ in the center, (3.17) becomes

$$(i|\bar{U}^n|i')=(i|\sum_m [P^{(m)}+\mathcal{C}^{(m)}(e^{-\gamma_m})]|m)e^{-\gamma_m n} \times (m|[P^{(m)}+\mathcal{D}^{(m)}(e^{-\gamma_m})]|i') \ . \tag{3.18}$$

Equation (3.18) can be reinterpreted in terms of left and right eigenstates of $\bar{U}$: $|\gamma_m)$ and $(\tilde{\gamma}_m|$. Defining these states by

$$\begin{aligned} |\gamma_m) &\equiv \Pi^{(m)}|m)=[P^{(m)}+\mathcal{C}^{(m)}(e^{-\gamma_m})]|m) \ , \\ (\tilde{\gamma}_m| &\equiv (m|\Pi^{(m)}=(m|[P^{(m)}+\mathcal{D}^{(m)}(e^{-\gamma_m})] \ , \end{aligned} \tag{3.19}$$

Eq. (3.18) can be rewritten

$$(i|\bar{U}^n|i')=\sum_m (i|\gamma_m)e^{-n\gamma_m}(\tilde{\gamma}_m|i') \ . \tag{3.20}$$

Equation (3.20) is essentially a spectral representation of the operator $\bar{U}$. The eigenstates satisfy

$$\begin{aligned} &\bar{U}|\gamma_m)=e^{-\gamma_m}|\gamma_m) \ , \\ &(\tilde{\gamma}_m|\bar{U}=e^{-\gamma_m}(\tilde{\gamma}_m| \ , \\ &(\tilde{\gamma}_m|\gamma_{m'})=\delta_{m,m'} \ . \end{aligned} \tag{3.21}$$

Equations (3.21) follow from the definition (3.19) and (2.8), using the fact that the resolvent has a simple pole at $z=e^{-\gamma_m}$. Before discussing the eigenstates, we discuss how to evaluate the operators defined in (3.15).

The result (3.16) is derived in the following way. Using the definition of $P^{(m)}$, the first term in the definition of $\Psi^{(m)}(z)$ in (3.15) can be written $|m)(m|U|m)(m|$, which is just $e^{-\gamma_m}P^{(m)}$. We then use the expansion (3.10) applied to the resolvent of $Q^{(m)}\bar{U}Q^{(m)}$. This gives, for the second term,

$$\sum_{k=0}^{\infty} P^{(m)}UQ^{(m)}\frac{1}{z-Q^{(m)}\bar{U}_0Q^{(m)}}\left[Q^{(m)}\delta\bar{U}\,Q^{(m)}\frac{1}{z-Q^{(m)}\bar{U}_0Q^{(m)}}\right]^k Q^{(m)}UP^{(m)} \ . \tag{3.22}$$

From the completeness relation (3.4), $Q^{(m)}=\sum_{m'\neq m}|m')(m'|$. Using this form for $Q^{(m)}$ it is clear that (3.22) vanishes because of the nonrecurrence property (3.7). Therefore we obtain (3.16).

Matrix elements of $\mathcal{C}^{(m)}(z)$ can be derived from the definition (3.15). Because of the factor $P^{(m)}$ on the right, the only nonzero matrix elements will be $(i|\mathcal{C}^{(m)}(z)|m)$. Because $\bar{U}$ contains transitions $m\to m'$ for $m\geq m'$, only matrix elements of the form $(m-k|\mathcal{C}^{(m)}(z)|m)$ (k a non-negative integer) will be nonzero. Furthermore, because of the factor of $Q^{(m)}$ on the left-hand side, we cannot have $k=0$.

From the definition,

$$\begin{aligned} (m-k|\mathcal{C}^{(m)}(z)|m) &= (m-k|\frac{1}{z-Q^{(m)}\bar{U}Q^{(m)}}Q^{(m)}\bar{U}|m) \\ &= \frac{1}{z-e^{-\gamma_{m-k}}}(m-k|(z-Q^{(m)}\bar{U}_0Q^{(m)})\frac{1}{z-Q^{(m)}\bar{U}Q^{(m)}}Q^{(m)}\bar{U}|m) \\ &= \frac{1}{z-e^{-\gamma_{m-k}}}(m-k|(z-Q^{(m)}\bar{U}Q^{(m)}+Q^{(m)}\delta\bar{U}\,Q^{(m)})\frac{1}{z-Q^{(m)}\bar{U}Q^{(m)}}Q^{(m)}\bar{U}|m) \\ &= \frac{1}{z-e^{-\gamma_{m-k}}}\left[(m-k|\bar{U}|m)+(m-k|\delta\bar{U}\,Q^{(m)}\frac{1}{z-Q^{(m)}\bar{U}Q^{(m)}}Q^{(m)}\bar{U}|m)\right] , \end{aligned} \tag{3.23}$$

where we assumed $z\neq e^{-\gamma_{m-k}}$ and used $\bar{U}_0=\bar{U}-\delta\bar{U}$ from (3.9). Substituting

$$Q^{(m)}=\sum_{k'\neq 0}|m-k')(m-k'| \ ,$$

using the definition of $\mathcal{C}^{(m)}(z)$, and using the nonrecurrence condition, we obtain a recursion relation for matrix elements of $\mathcal{C}^{(m)}(z)$:

$$(m-k|\mathcal{C}^{(m)}(z)|m)=\frac{1}{z-e^{-\gamma_{m-k}}}\left[(m-k|\bar{U}|m)+\sum_{k'=1}^{k-1}(m-k|\overline{m\bar{U}}|m-k')(m-k'|\mathcal{C}^{(m)}(z)|m)\right] . \tag{3.24}$$

It is clear from (3.24) that the matrix elements of $\mathcal{C}^{(m)}(z)$ are not singular at $z=e^{-\gamma_m}$, although they do have singulari-

ties at $z=e^{-\gamma_{m'}}$ for $m'<m$.

A similar recursion relation exists for the matrix elements of $\mathcal{D}^{(m)}(z)$ and the matrix elements for $\mathcal{P}^{(m)}(z)$. These are summarized in Appendix F. The derivations are very similar.

B. Right eigenstates

The right eigenstates $|\gamma_m)$ defined in (3.19) can be written in terms of Legendre polynomials using the matrix elements of $\mathcal{C}^{(m)}(z)$ given in Eq. (3.24).

$$|\gamma_m)=|m)+\sum_{i=0}^{m-1}|i)(i|\mathcal{C}^{(m)}(e^{-\gamma_m})|m)$$
$$=|m)-|m-2)\sqrt{(2m+1)(2m-3)}\frac{m-2}{3}+|m-4)\sqrt{(2m+1)(2m-7)}\frac{(m-4)(2m-1)(7m-15)}{90}+\cdots . \tag{3.25}$$

The eigenstate $|\gamma_m)$ is therefore a polynomial of degree m. Explicit forms for the first seven eigenstates calculated from (3.24) and (3.25) are

$$\begin{aligned}
&(x|\gamma_0)=1 ,\\
&(x|\gamma_1)=\sqrt{3}(1-2x) ,\\
&(x|\gamma_2)=\sqrt{5}(1-6x+6x^2) ,\\
&(x|\gamma_3)=\sqrt{7}(-10x+30x^2-20x^3) ,\\
&(x|\gamma_4)=3(-\tfrac{7}{3}+70x^2-140x^3+70x^4) ,\\
&(x|\gamma_5)=\sqrt{11}(42x-420x^3+630x^4-252x^5) ,\\
&(x|\gamma_6)=\sqrt{13}(22-462x^2+2310x^4-2772x^5+924x^6) ,\\
&(x|\gamma_7)=\sqrt{15}(-572x+4004x^3-12\,012x^5\\
&\qquad +12\,012x^6-3432x^7) .
\end{aligned} \tag{3.26}$$

Direct computation using the explicit form of the Perron-Frobenius operator given in (3.1) shows that these are indeed eigenstates with eigenvalues $e^{-\gamma_m}$. It has recently been pointed out that the eigenstates (3.25) are related to the well-known Bernoulli polynomials $B_m(x)$ by [27,28]

$$(x|\gamma_m)=(-1)^m\sqrt{2m+1}(2m)!/(m!)^2B_m(x) . \tag{3.27}$$

Properties of the Bernoulli polynomials are summarized in Appendix I. The Bernoulli polynomials satisfy the important property $dB_m(x)dx=mB_{m-1}(x)$.

We define $|a)_N=|a)/\sqrt{(a|a)}$ to be the normalized state $|a)$. The first seven eigenstates $|\gamma_i)_N$ are plotted in Fig. 2.

The right eigenstates have a property we call "exponential degeneracy." Successive eigenstates $|\gamma_m)_N$ and $|\gamma_{m+2})_N$ become close exponentially with m. This is true in both the uniform and the L_2 norms. Thus there are states that are very similar but decay with exponents that are very different. This fact becomes important when approximating certain functions using a large but finite number of the right eigenstates. The coefficients of high-m eigenstates can depend very strongly on the degree of the approximation. In this case it is not very meaningful to treat the high-degree modes separately.

We show in Appendix I that the eigenstates approach $\sin 2\pi x$ and $\cos 2\pi x$ asymptotically, i.e.,

$$\begin{aligned}
&\lim_{m\to\infty}\frac{(-1)^m}{\sqrt{2}}(x|\gamma_{2m+1})_N=\sin 2\pi x ,\\
&\lim_{m\to\infty}\frac{(-1)^{m+1}}{\sqrt{2}}(x|\gamma_{2m})_N=\cos 2\pi x .
\end{aligned} \tag{3.28}$$

Moreover, one can construct sequences which approach $\cos 2n\pi x$ or $\sin 2n\pi x$ for any integer $n>1$. For example,

$$\lim_{m\to\infty}\frac{(-1)^{m+1}}{\sqrt{2}}(x|[|\gamma_{2m+2})_N+|\gamma_{2m})_N]_N=\cos 4\pi x . \tag{3.29}$$

It is natural to ask what space is spanned by the right eigenstates $|\gamma_m)$. We will denote this space by $\mathcal{T}$. It will be defined precisely in Sec. III D. For now we state only that $\mathcal{T}$ contains only analytic functions, that it is dense in L_2, and that it can be given a topology in which it is complete.

C. Left eigenstates

We now discuss the left eigenstates in detail. We have called $(\tilde{\gamma}_m|$ a "left eigenstate" of $\bar{U}$. More precisely it is an eigenstate of the adjoint operator $\bar{U}^\dagger$, defined on L_2 by $(A|\bar{U}|B)=(\bar{U}^\dagger A|B)$ for all $|A),|B)\in L_2$. As mentioned earlier, $\bar{U}^\dagger$ is isometric. We showed after (2.11) that an eigenstate in L_2 of a unitary operator has an eigenvalue with magnitude 1. The same argument holds for an isometric operator. Therefore $(\tilde{\gamma}_m|$ cannot be an

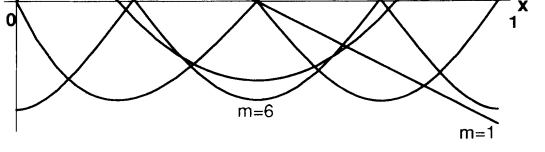

FIG. 2. The first seven right eigenstates $|\gamma_m)_N$ for the Bernoulli map.

ordinary function in Hilbert space. We will show that $(\tilde{\gamma}_m|$ can be interpreted as a Schwartz distribution on the subspace $\mathcal{T}$ of L_2 which is spanned by the right eigenstates, and therefore belongs to the dual space $\mathcal{T}^*$.

A distribution is well defined if we can specify its action on a space of functions. By definition, for any $f(x)$ in $\mathcal{T}$ we can write

$$|f)=\sum_m f_m|\gamma_m) \ . \tag{3.30}$$

From the biorthogonality condition (3.21)

$$(\tilde{\gamma}_m|f)=f_m \tag{3.31}$$

for $f\in\mathcal{T}$. In Sec. III D we will introduce a topology for $\mathcal{T}$ in which $(\tilde{\gamma}_m|$ is continuous. $(\tilde{\gamma}_m|$ therefore defines a distribution on $\mathcal{T}$.

This argument is in some sense formal, since the coefficients f_m are usually not known and in practice are computed from (3.31). We now give two constructive interpretations of the left eigenstates.

Evaluating (3.19) using Eqs. (F1) for $\mathcal{D}^{(m)}(z)$ produces an explicit formula for the left eigenstates ($m\geq 1$):

$$(\tilde{\gamma}_m|=\sum_{l=0}^{\infty}\left[\frac{2m+4l+1}{2m+1}\right]^{1/2} \times\frac{(2m+2l-1)!}{(2m-1)!(2l+1)!}(m+2l| \ . \tag{3.32}$$

The coefficients in this expansion grow as $l^{2m-3/2}$ for large l. Certainly the sum does not converge. This is consistent with our interpretation that $(\tilde{\gamma}_m|$ is a distribution, since singular distributions are characterized by diverging series expansions. For instance, the Dirac δ function is often expanded in a basis of trigonometric functions. The expansion coefficients are constant so that the sum does not converge. Derivatives of the δ function have even worse convergence properties. The expansions make sense under integration, however. We note that a diverging series can be interpreted as a distribution only when it is an expansion in orthogonal functions. For instance, a Taylor series outside its radius of convergence cannot be interpreted as a distribution in a straightforward way.

Equation (3.32) can be given meaning by applying it to a test function. Using the completeness relation (3.4) we have

$$\begin{aligned}(\tilde{\gamma}_m|f)&=\sum_{i=0}^{\infty}(\tilde{\gamma}_m|i)(i|f)\\&=\sum_{l=0}^{\infty}\left[\frac{2m+4l+1}{2m+1}\right]^{1/2}\\&\quad\times\frac{(2m+2l-1)!}{(2m-1)!(2l+1)!}(m+2l|f) \ .\end{aligned} \tag{3.33}$$

Since the coefficients grow as $l^{2m-3/2}$, $(\tilde{\gamma}_m|$ is well defined for functions f such that $(m+2l|f)$ ("Fourier" coefficients of f in a Legendre basis) fall off faster than $l^{-2m+3/2}$. Functions which contain discontinuities do not have this property (for all m). The series (3.30) terminates for all polynomials of finite degree.

There is another way to calculate the left eigenstates which gives us a visual picture and an easier way to compute $(\tilde{\gamma}_m|f)$. We can invert (3.32) to solve (formally) for the Legendre polynomials in terms of the left eigenstates:

$$|m)=|\tilde{\gamma}_m)+\sum_{i=1}^{\infty}a_i^{(m)}|\tilde{\gamma}_{m+i}) \ . \tag{3.34}$$

As usual, (3.34) makes mathematical sense for individual components, i.e., after taking its inner product with $(i|$. Operating with $(\bar{U}^{\dagger}e^{\gamma_m})^n$ on both sides we get

$$\begin{aligned}(\bar{U}^{\dagger}e^{\gamma_m})^n|m)&=(\bar{U}^{\dagger}e^{\gamma_m})^n\left[|\tilde{\gamma}_m)+\sum_{i=1}^{\infty}a_i^{(m)}|\tilde{\gamma}_{m+i})\right]\\&=|\tilde{\gamma}_m)+\sum_{i=1}^{\infty}a_i^{(m)}e^{-n(\gamma_{m+i}-\gamma_m)}|\tilde{\gamma}_{m+i}) \ .\end{aligned} \tag{3.35}$$

As $n\rightarrow\infty$, the second term vanishes, leaving

$$|\tilde{\gamma}_m)=\lim_{n\rightarrow\infty}(\bar{U}^{\dagger}e^{\gamma_m})^n|m) \ . \tag{3.36}$$

For finite n, (3.36) describes a function that contain 2^n copies of the mth Legendre polynomial in the interval [0,1] which are scaled by a height 2^{nm} (see Fig. 3). Obviously it is not a function in the limit $n\rightarrow\infty$, but its inner product with smooth functions can still be defined.

Equation (3.36) can be used to calculate $(\tilde{\gamma}_m|f)$ by computing it for finite n and taking the limit at the end of the calculation. Using Fig. 3 as a guide, we divide the interval [0,1] into 2^n intervals starting at $x=k/2^n$, $k=0,1,\ldots,2^n-1$, and obtain

$$(\tilde{\gamma}_m|f)=\lim_{n\rightarrow\infty}\sum_{k=0}^{2^n-1}\int_0^{1/2^n}dx\ 2^{mn}\tilde{P}(2^nx)f\left[\frac{k}{2^n}+x\right] \ . \tag{3.37}$$

If f is analytic at $k/2^n$, it can be expanded in a Taylor series. Letting $y=2^nx$ gives

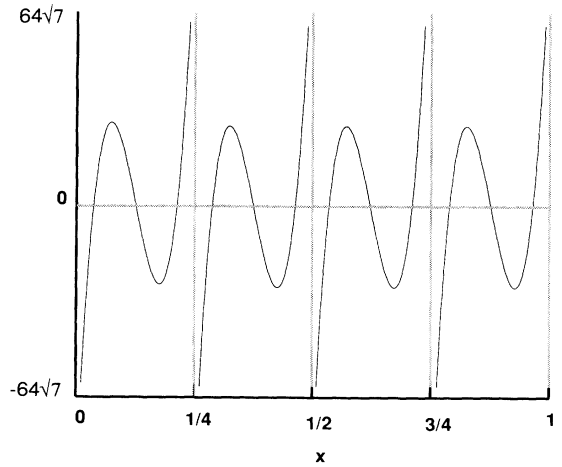

FIG. 3. The function described by Eq. (3.33) for $n=2$, $m=3$, which contains four copies of the third modified Legendre polynomial scaled by a factor $2^{2*3}=64$.

$$(\tilde{\gamma}_m|f)=\lim_{n\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2^n-1}\frac{1}{2^n}f^{(j)}\left[\frac{k}{2^n}\right]\times 2^{n(m-j)}\frac{1}{j!}\int_0^1 y^j\tilde{P}_m(y)\,dy\ . \tag{3.38}$$

Since the Legendre polynomial of degree m is orthogonal to all polynomials of smaller degree, the second integral vanishes for $j<m$. Terms with $j>m$ also vanish since the factor $2^{n(m-j)}$ will go to zero in the $n\to\infty$ limit. Therefore only the term $j=m$ survives. The integral is easy to evaluate because (using, for instance, the Rodriguez formula for the Legendre polynomials)

$$y^m=\frac{(-1)^m(m!)^2}{[(2m)!\sqrt{2m+1}]}\tilde{P}_m(y)+\cdots\ ,$$

where the remaining terms contain lower-order Legendre polynomials which are orthogonal to $\tilde{P}_m(y)$. Finally, we can turn the sum over k into a Riemann integral of $f^{(m)}$ and obtain

$$(\tilde{\gamma}_m|f)=\frac{(-1)^m}{\sqrt{2m+1}}\frac{m!}{(2m)!}[f^{(m-1)}(1)-f^{(m-1)}(0)]\ . \tag{3.39}$$

Equation (3.39) gives another way of calculating $(\tilde{\gamma}_m|f)$. Choosing $f(x)=\tilde{P}_{m+2l}(x)$ reproduces (3.25). Also, $(\tilde{\gamma}_m|f)=0$ for $f(x)=\sin 2\pi nx$ or $\cos 2\pi nx$, which is consistent with Eqs. (3.28).

Evidently [since (3.39) was obtained by integrating $f^{(m)}$], the criterion for $(\tilde{\gamma}_m|f)$ to exist is that f must have at least m derivatives. (For $m=0$, it must be integrable.) If we require that $(\tilde{\gamma}_m|f)$ exist for all m, then f must be infinitely differentiable. This will be discussed in more detail in Sec. III D. Note that analyticity of f is not required for the final result (3.39). The Taylor series is used only in an infinitesimal neighborhood so it is really just an approximation based on the definition of a derivative.

From Eq. (3.39) it is straightforward to show explicitly that $(\tilde{\gamma}_m|$ is an eigenstate of $\bar{U}^\dagger$ with eigenvalue $e^{-\gamma_m}=2^{-m}$. It is sufficient to show that $(\bar{U}^\dagger\tilde{\gamma}_m|f)=e^{-\gamma_m}(\tilde{\gamma}_m|f)$ for any f. Letting $c_m=(-1)^m m!/[\sqrt{2m+1}(2m!)]$ and using the definition of the adjoint operator, we have

$$\begin{aligned}(\bar{U}^\dagger\tilde{\gamma}_m|f)&=(\tilde{\gamma}_m|Uf)\\&=(\tilde{\gamma}_m|\frac{1}{2}[f(\tfrac{x}{2})+f(\tfrac{x+1}{2})])\\&=\frac{1}{2}\frac{c_m}{2^{m-1}}[f^{(m-1)}(\tfrac{1}{2})-f^{(m-1)}(0)+f^{(m-1)}(1)-f^{m-1}(\tfrac{1}{2})]\\&=\frac{1}{2^m}(\tilde{\gamma}_m|f)\ .\end{aligned} \tag{3.40}$$

The distribution $(\tilde{\gamma}_m|$ can be written explicitly in two ways. The first is in terms of derivatives of δ functions

$$(\tilde{\gamma}_m|x)=c_m[\delta^{(m-1)}(x-1)-\delta^{m-1}(x-0)]\ , \tag{3.41}$$

which gives (3.39) immediately if one does not worry about the interpretation of δ functions on the boundary of integration. Equation (3.41) emphasizes the role of the boundary.

We prefer to think of $(\tilde{\gamma}_m|x)$ as a derivative *operator*. Through a slight abuse of notation, we can write

$$(\tilde{\gamma}_m|x)=c_m\frac{d^m}{dx^m}\ . \tag{3.42}$$

This also reproduces (3.39). It is more naturally related to the derivation of (3.39) and can be generalized to an asymmetric version of the Bernoulli map (where it becomes a *fractally weighted* derivative operator).

In the form (3.42), it is clear why $(\tilde{\gamma}_m|$ is an eigenstate of $\bar{U}^\dagger$, since

$$\bar{U}^\dagger d^m/dx^m\approx d^m/d(2x)^m=e^{-\gamma_m}d^m/dx^m\ .$$

The biorthogonality condition in (3.21) also follows directly from (3.39) using properties (I7) and (I5) of the Bernoulli polynomials.

In summary, we have shown that $(\tilde{\gamma}_m|$ is a distribution on $\mathcal{T}$. We have also shown two equivalent ways [(3.32) and (3.39)] of evaluating $(\tilde{\gamma}_m|f)$.

D. Spectral decomposition of $\bar{U}$

We now discuss the space $\mathcal{T}$, which we have defined as the space spanned by the right eigenstates $|\gamma_m)$. Any function $|f)\in\mathcal{T}$ has the decomposition (3.30) with coefficients given by (3.31), so the identity operator on $\mathcal{T}$ may be written

$$1=\sum_m|\gamma_m)(\tilde{\gamma}_m|\ . \tag{3.43}$$

From the discussion so far, it is intuitively clear that (i) $\mathcal{T}$ contains all finite polynomials, (ii) $\mathcal{T}$ does not contain periodic functions of period 1 [for which Eq. (3.43) vanishes], and (iii) $\mathcal{T}$ does not contain functions with compact support in the interior of [0,1] [again, (3.43) vanishes].

From the point of view of the physicist, this is all we need to know. Mathematically, more is required. To be well defined as a space of test functions, $\mathcal{T}$ must be en-

dowed with a topology in which it is complete and nuclear. There is no unique choice for $\mathcal{T}$. While it seems reasonable to choose the largest $\mathcal{T}$ which makes mathematical and physical sense, we do not know how to construct it. We mention here three other possible choices.

Let $\mathcal{P}$ be the set of polynomials defined on [0,1]. Recalling the definition of the normalized eigenstate $|\gamma_m)_N=|\gamma_m)/\sqrt{(\gamma_m|\gamma_m)}$, we define the normalized eigendistribution ${}_N(\tilde{\gamma}_m|$ by the condition ${}_N(\tilde{\gamma}_m|\gamma_{m'})_N=\delta_{m,m'}$. We then introduce an L_2-like norm on $\mathcal{P}$ defined by

$$\|f\|=\sum_{m=0}^{\infty}|{}_N(\tilde{\gamma}_m|f)|^2 \tag{3.44}$$

for $f\in\mathcal{P}$. We then define $\mathcal{T}$ to be the completion of $\mathcal{P}$ under this norm. The test space $\mathcal{T}$ defined in this way appears to satisfy the three conditions above. In particular, sequences of polynomials which converge to $\sin 2n\pi x$ in L_2 [such as those in (3.28), or truncated Taylor expansions] are not Cauchy sequences in $\mathcal{T}$. The test space defined in this way is determined by the dynamics through the left eigenstates.

The choice of the normalized eigendistribution ${}_N(\tilde{\gamma}_m|$ rather than the original $(\tilde{\gamma}_m|$ in (3.44) is crucial. If the norm were based on $(\tilde{\gamma}_m|$, $\mathcal{T}$ would include periodic functions as limits of Cauchy sequences, but they would have zero norm and thus be identified with zero. Test spaces with stronger topologies may not include "reasonable" functions such as $\sin(2-\epsilon)\pi x$ for finite ϵ.

Another choice of topology for $\mathcal{P}$ was suggested by Antoniou and Tasaki [29], who equip $\mathcal{P}$ with a very strong topology (the inductive topology of LF space) for which $\mathcal{T}=\mathcal{P}$. The result is mathematically consistent but to us lacks physical motivation. Also, Gaspard [28] has shown, using a very different approach, that one can define $\mathcal{T}$ to be a Fréchet space of analytic functions of exponential type less than 2π. This space appears to be quite similar to the one we define above.

Equation (3.20) gives the dependence of the matrix elements $(i|U^n|i')$ on n, i.e., the time evolution. It is tempting to generalize this formula and write

$$\bar{U}=\sum_m e^{-\gamma_m}|\gamma_m)(\tilde{\gamma}_m| \tag{3.45}$$

so that correlation functions $(A|U^n|B)$ become

$$\sum_m e^{-n\gamma_m}(A|\gamma_m)(\tilde{\gamma}_m|B)\ . \tag{3.46}$$

From the preceding discussion, $(\tilde{\gamma}_m|$ is only defined on the subspace $\mathcal{T}$ of $L_2(0,1)$. Therefore Eq. (3.46) has meaning only if B belongs to this space.

It is important that $\mathcal{T}$ does not include δ functions. Furthermore, the domain of (3.45) cannot be extended to include δ functions. Therefore trajectories are incompatible with the physical spectral representation.

If we want to calculate a correlation function $(A|U^n|B)$ for an arbitrary (i.e., not necessarily in $\mathcal{T}$) function B, the physical spectral representation can be used to compute an arbitrarily good approximation by approximating B with a finite-degree polynomial. A straightforward way to obtain an approximation is to expand B in terms of Legendre polynomials and truncate at finite order, i.e., by defining a polynomial $\hat{B}$ of order I by

$$|\hat{B})=\sum_{i=0}^{I}|i)(i|B) \tag{3.47}$$

so that $(\tilde{\gamma}_m|B)$ in (3.46) is approximated by

$$\begin{aligned}(\tilde{\gamma}_m|B)&\approx(\tilde{\gamma}_m|\hat{B})\\&=\sum_{i=0}^{I}(\tilde{\gamma}_m|i)(i|B)\ .\end{aligned} \tag{3.48}$$

For nonanalytic functions and periodic functions (see Sec. III E), the approximation in (3.48) may depend sensitively on the order of the approximation (3.47). For B with n continuous derivatives, the first n coefficients $(\tilde{\gamma}_m|\hat{B})$ will be stable with respect to the order of the approximation while the others will not be. For these functions, the decomposition into decaying modes is not physically very meaningful, although it will still give accurate results.

For nonsmooth functions one can give a meaningful spectral decomposition by considering only the first few low-order poles of the resolvent and "resurrecting" the background integral in (2.4). It is then possible to write a spectral decomposition which contains a finite number of terms in (3.45) plus a term related to the background integral. This formulation will be discussed in a forthcoming paper [30].

E. Collapsing functions

In Sec. III B, we showed that sine and cosine functions could be expressed as large-m limits of combinations of right eigenstates. In this section, we take a closer look at these functions and explore their special role.

It is easily verified that $\sin 2\pi x$ and $\cos 2\pi x$ are "collapsing" states, i.e.,

$$\begin{aligned}\bar{U}\sin 2\pi x&=0\ ,\\ \bar{U}\cos 2\pi x&=0\ .\end{aligned} \tag{3.49}$$

Similarly, all states $\sin 2n\pi x$ and $\cos 2n\pi x$ with odd n are collapsing states. These are eigenstates with eigenvalue zero, and therefore cannot be expressed as superpositions of right eigenstates $|\gamma_m)$ which decay at a finite (although, for large m, very large) rate. States with even n are "shifted":

$$\bar{U}\sin 2\pi nx=\sin 2\pi(n/2)x,\quad \text{even } n\ . \tag{3.50}$$

(and similarly for cosine). The norm of such a state stays constant for a finite number of iterations (equal to the number of factors of 2 in the prime factorization of n) and then collapses to zero. It is also not possible to express this behavior exactly in terms of exponentially decaying states.

This verifies our previous statement that sine and cosine functions are not in $\mathcal{T}$ (although, as we have stated before, they can be expressed as limits of functions in $\mathcal{T}$).

Since sine and cosine functions form a basis for $L_2(0,1)$, there is a second, incompatible mathematical structure for describing the dynamics. It is incompatible in the sense that it gives a very different physical picture of the time evolution. Had we started with a Fourier basis instead of a Legendre basis, we would have found an essential singularity of matrix elements of the resolvent at $z=0$ and the subdynamics would have reproduced the shift-collapse picture.

We note finally that Antoniou and Tasaki [29] have shown that it is possible to construct other spectral representations of $\bar{U}$ using the collapsing functions. These representations involve a continuous (and infinitely degenerate) set of eigenvalues on a circle of radius $\lambda \leq 1$. An example of an eigenstate appearing in such a representation is

$$\sum_{n=0}^{\infty} e^{ikn}\lambda^n e^{2\pi i 2^n sx} \tag{3.51}$$

for odd integer s, $\lambda<1$ and $0\leq k\leq 2\pi$, which has eigenvalue λe^{ik}. Although these spectral representations are mathematically correct, the continuous spectrum does not (except coincidentally) contain any physically relevant time scales. The exact connection between the decomposition (3.45) and the "coherent state" representations involving (3.51) will be discussed in [30].

IV. THE BAKER MAP

In this section we discuss the physical spectral representation of the Perron-Frobenius operator for the baker map. We will use extensively the results obtained for the Bernoulli map.

The baker map f is a one-to-one map of the unit square into itself and is given by the rule $f(x,y)=(2x,y/2)$ for $0\leq x<\frac{1}{2}$ and $f(x,y)=(2x-1,(y+1)/2)$ for $\frac{1}{2}\leq x<1$. Since f is bijective and its Jacobian is 1, Eq. (2.1) for the Perron-Frobenius operator reduces to $U\rho(x)=\rho(f^{-1}(x))$. More explicitly this is

$$\begin{aligned} U\rho(x,y)&=\rho(x/2,2y) \quad \text{if } y<\tfrac{1}{2}\ , \\ U\rho(x,y)&=\rho((x+1)/2,2y-1) \quad \text{if } y>\tfrac{1}{2}\ . \end{aligned} \tag{4.1}$$

U is a unitary operator. Phase-space volume is conserved, and in this sense the baker map is very similar to a Hamiltonian system. The Bernoulli map is the projection of this map obtained by integrating over y.

It is well known [31] that for K systems (which include the baker transformation), U has a Lebesgue spectrum: an infinitely degenerate continuous spectrum on the unit circle plus a point eigenvalue at $z=1$. The Lebesgue spectrum does not contain any physically important time scales, such as decay rates of correlation functions, Lyapunov exponents, etc. As we mentioned in Sec. II and will show explicitly in this section, the resolvent operator is regular in the entire z plane, except on $|z|=1$, where it is unbounded, while its matrix elements are analytic everywhere except for a discontinuity at $|z|=1$. We define the matrix elements on a two-sheeted Riemann surface, with values on the second sheet defined by analytic continuation from outside to inside the unit circle (and vice versa). Important time scales are given by the positions of singularities on the second Riemann sheet. In the baker example, matrix elements are meromorphic on the second sheet, and the positions of the poles give the decay rates of correlation functions. The second-sheet poles correspond to generalized decaying eigenspaces which are outside the Hilbert space of square-integrable functions.

The analytic structure of the resolvent operator for discrete time systems is similar to that of more familiar continuous time systems. Motion in Hamiltonian systems is generated by a Hermitian operator H (which may be a Hamiltonian or Liouvillian operator). (In the following we refer to the resolvent operator itself, but it should be understood that the statements are strictly true only for its matrix elements. Analogous statements for the operator itself require a careful treatment of the domain and range.) If H has continuous spectrum, say, on $[0,\infty)$, its resolvent can be defined on a two-sheeted Riemann surface with a branch point at $z=0$ and a cut on the positive real axis (corresponding to the spectrum). Poles on the second sheet correspond to unstable states. If there are bound states, they appear as poles on the negative real axis on the first sheet. In scattering theory, S-matrix elements have the same analytic structure and the poles have the same interpretations. The second sheet is often called "unphysical," although it contains poles which give the observed (physical) decay rates.

The time-evolution operator U_t is related to the generator H by $U_t=e^{-iHt}$. Under this transformation, the cut on the real axis for the resolvent of H becomes a cut on the unit circle for the resolvent of U_t. The situation in the baker map is completely analogous except that the corresponding generator H has continuous spectrum from $-\infty$ to ∞, so that there is a cut but no branch point. It is therefore not possible to get from one-half of a Riemann sheet to the other by going around the branch point, so that each Reimann sheet has two disconnected parts (inside and outside the circle). (There is an additional complication that each infinite strip $[2\pi n,2\pi(n+1)]\otimes(-\infty,\infty)$ is mapped to the entire complex plane under the transformation $z\rightarrow e^{-iz}$, resulting in an infinite-sheeted Riemann surface with another cut on the real axis. The resolvent of the generator of evolution for the baker would be periodic with period 2π so that there are only two Reimann sheets.) This is closely related to the existence of pure exponential decay with no long tail (power-law behavior for long times).

Following the procedure described in Sec. II, we introduce a basis $|i,j)$ for L_2 on the unit square defined by $(x,y|i,j)\equiv\tilde{P}_i(x)\tilde{P}_j(y)$. $\tilde{P}_i(x)$ is the ith modified Legendre polynomial which we used for a basis in the Bernoulli system. In this basis, the matrix elements of U are (see Appendix D)

$$(i,j|U|i'j')=\tfrac{1}{2}[1+(-1)^{i+i'+j+j'}]I_{i,i'}I_{j',j}\ , \tag{4.2}$$

where the functions $I_{\alpha,\beta}$ are defined in (3.6). Because I is upper triangular, the matrix elements satisfy a nonrecurrence property similar to (3.7). There exists a transition from $|i,j)$ to $|i-\delta i,j+\delta j)$ (δi and δj positive in-

tegers) but not from $|i-\delta i,j+\delta j)$ to $|i,j)$.

Before proceeding with the subdynamics analysis, we will illustrate our assertions about the analytic properties of matrix elements of the resolvent. Consider the matrix element $f(z)=(1,1|(z-U)^{-1}|1,1)$ in the baker map, which is defined by

$$f(z)=\int dx\,dy\,\tilde{P}_1(x)\tilde{P}_1(y)\frac{1}{z-U}\tilde{P}_1(x)\tilde{P}_1(y)\ . \tag{4.3}$$

Because the spectrum of U is on the unit circle, $f(z)$ is analytic everywhere except possibly on the unit circle. Using (3.10) to expand the resolvent in the region $|z|>1$, and the nonrecurrence property, we find that

$$f(z)=\sum_{n=0}^{\infty}z^{-(n+1)}[(1,1|U|1,1)]^n$$
$$=\frac{1}{z-\frac{1}{4}},\quad |z|>1\ . \tag{4.4}$$

For $|z|<1$, we can write

$$(z-U)^{-1}=-U^{-1}(1-zU^{-1})^{-1}$$

and use (3.10) to obtain

$$f(z)=\frac{1}{z-4},\quad |z|<1\ . \tag{4.5}$$

As claimed, the matrix element is analytic in the entire complex plane except at $|z|=1$, where it is discontinuous. However, the analytic continuation from the outside to the inside contains a pole and vice versa. [There is also an eigendistribution associated with the pole at $z=4$, but it does not play a role in our analysis because the contour in Eq. (2.3) starts outside the unit circle.] The discontinuity at $|z|=1$ is a property of all matrix elements.

We define the function $f(z)$ on a two-sheeted Riemann surface. On the first sheet, $f(z)$ is the matrix element $(1,1|R(z)|1,1)$. On the second, "unphysical" sheet, it is the analytic continuation from the first sheet through the unit circle. Defining $f(z)$ in this way preserves the analyticity of the function so that we can deform the contour (it cannot be deformed through a nonanalyticity at $|z|=1$).

Returning to an analysis of the time evolution, we write the time evolution of matrix elements of U^n in terms of an integral over the resolvent as in (2.3). Because of the nonrecurrence property, the positions of the poles of the matrix elements of the resolvent are given by the diagonal elements $(i,j|U|i,j)=I_{i,i}I_{j,j}=2^{-(i+j)}=e^{-\gamma_{i+j}}$. The argument is exactly the same as that used for the Bernoulli map. For an integer $m\geq 0$, there are $m+1$ values of i and j which give the same value 2^{-m}. The resonances

$$e^{-\gamma_m}=2^{-m} \tag{4.6}$$

are therefore $(m+1)$-fold degenerate. This degeneracy is a result of the x-y symmetry of the map. The resonances are equivalent to the Ruelle resonances calculated in Appendix D.

We now investigate the matrix elements $(i,j|U^n|i,j)$. For the baker transformation, (2.3) becomes

$$(i,j|U^n|i',j')=\frac{1}{2\pi i}(i,j|\oint_C dz\frac{z^n}{z-U}|i',j')\ . \tag{4.7}$$

As discussed above, the matrix elements will be evaluated at $|z|>1$ and analytically continued inside $|z|=1$ onto the second sheet, where they contain poles. As in the Bernoulli case, the integrand of (4.7) will be analytic except for a finite number of poles at $e^{-\gamma_m}$. Therefore

$$(i,j|U^n|i',j')=\frac{1}{2\pi i}\sum_m\oint_{z=e^{-\gamma_m}}(i,j|\frac{z^n}{z-U}|i',j')\ . \tag{4.8}$$

To evaluate the integrals in (4.8) we again use the projective decomposition (3.14) to isolate the poles. If we choose projection operators $P^{(i,j)}\equiv|i,j)(i,j|$ the decomposition will not isolate the pole $e^{-\gamma_{i+j}}$ because the operators $\mathcal{C}^{(i,j)}(z)$ and $\mathcal{D}^{(i,j)}(z)$ will be singular there. Instead we must choose projection operators which include the degeneracy of the pole. Therefore we define

$$P^{(m)}=\sum_{i+j=m}|i,j)(i,j|\ . \tag{4.9}$$

At each pole we decompose the resolvent according to the projection operator associated with that particular pole. Expression (4.8), using the decomposition (3.14), becomes, for the baker transformation,

$$(i,j|U^n|i',j')=\sum_{m=0}^{\infty}\frac{1}{2\pi i}\oint_{z=e^{-\gamma_m}}dz(i,j|[P(m)+\mathcal{C}^{(m)}(z)]\frac{1}{z-\Psi^{(m)}(z)}[P^{(m)}+\mathcal{D}^{(m)}(z)]|i',j')\ . \tag{4.10}$$

We have dropped the term $\mathcal{P}^{(m)}(z)$ because it has no singularity at $z=e^{-\gamma_m}$ and therefore does not contribute to the integral. Recursion formulas for $\mathcal{C}^{(m)}(z)$ and $\mathcal{D}^{(m)}(z)$ are given in Appendix F.

In the Bernoulli map calculation, the resonances were nondegenerate, so that $\Psi^{(m)}(z)$ was a number and the analog (3.17) of (4.10) could be evaluated directly by Cauchy's theorem. In the baker case, the resonance is degenerate, so that $\Psi^{(m)}(z)$ is a matrix. This introduces several additional steps, which we now discuss.

$\Psi^{(m)}(z)$ is defined in (3.15). The first term in the definition, $P^{(m)}UP^{(m)}$, is the $(m+1)\times(m+1)$ submatrix of U associated with $P^{(m)}$. Explicitly, it can be written

$$P^{(m)}UP^{(m)}=\sum_{i+j=m}\sum_{i'+j'=m}|i,j)(i,j|U|i',j')(i',j'|\ . \tag{4.11}$$

We usually think of it as an $(m+1)\times(m+1)$ matrix and understand implicitly that it is an operator on the full

space with nonzero entries only in the $P^{(m)}$ subspace. $P^{(m)}UP^{(m)}$ is upper triangular (because of the nonrecurrence condition) with $e^{-\gamma_m}$ on the diagonal. The second term can be calculated from formula (3.22), which was used in the Bernoulli map. It is also an upper-triangular square matrix but has zeros on the diagonal. Adding these two parts together, $\Psi^{(m)}$ is an upper-triangular matrix with $e^{-\gamma_m}$ on the diagonal. It is essentially a Jordan block [27].

Denoting by $\Delta^{(m)}$ the part of the matrix above the diagonal, we have

$$\Psi^{(m)}(z)=P^{(m)}e^{-\gamma_m}+\Delta^{(m)} . \tag{4.12}$$

$\Delta^{(m)}$ has the property $(\Delta^{(m)})^{m+2}=0$ [which is true for any strictly upper-triangular (zero-diagonal) matrix of dimension $(m+1)\times(m+1)$]. This allows us to write

$$\frac{P^{(m)}}{z-\Psi^{(m)}(z)}=P^{(m)}\sum_{i=0}^{m+1}\frac{[\Delta^{(m)}(z)]^i}{[z-e^{-\gamma_m}]^{i+1}} . \tag{4.13}$$

Equation (4.13) is an exact identity and there is no problem with convergence because the sum is finite.

Equation (4.10) can now be written

$$(i,j|U^n|i',j')=\frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{k=0}^{m+1}(i,j|\oint_{z=e^{-\gamma_m}}dz[P^{(m)}+\mathcal{C}^{(m)}(z)]\frac{z^n[\Delta^{(m)}(z)]^k}{[z-e^{-\gamma_m}]^{k+1}}[P^{(m)}+\mathcal{D}^{(m)}(z)]|i',j') , \tag{4.14}$$

which can be evaluated by the Cauchy residue theorem. In principle this is straightforward but it leads to an unmanageable expression. We will use the Π operators discussed above to obtain the compact results (4.23).

Using the operators $\Pi^{(m)}$ defined in (2.6), (4.14) can be written as

$$(i,j|U^n|i',j')=\sum_m(i,j|U^n\Pi^{(m)}|i',j') . \tag{4.15}$$

Setting $n=0$ in (4.14) and comparing with (4.13) gives (formally)

$$\Pi^{(m)}=\frac{1}{2\pi i}\sum_{k=0}^{m+1}\oint_{z=e^{-\gamma_m}}dz[P^{(m)}+\mathcal{C}^{(m)}(z)]\times\frac{[\Delta^{(m)}(z)]^k}{[z-e^{-\gamma_m}]^{k+1}}\times[P^{(m)}+\mathcal{D}^{(m)}(z)] . \tag{4.16}$$

Expression (4.16) is dangerous as an operator expression (the domain must be carefully specified) but it is well defined if we consider only its matrix elements.

One can decompose $\Pi^{(m)}$ using the projection operator $P^{(m)}$, writing

$$\Pi^{(m)}=P^{(m)}\Pi^{(m)}P^{(m)}+P^{(m)}\Pi^{(m)}Q^{(m)}+Q^{(m)}\Pi^{(m)}P^{(m)}+Q^{(m)}\Pi^{(m)}Q^{(m)} . \tag{4.17}$$

Equation (4.17) holds for any operator since $P^{(m)}+Q^{(m)}=1$. We now make the following definitions:

$$\begin{aligned}&P^{(m)}\Pi^{(m)}P^{(m)}\equiv A^{(m)} ,\\ &P^{(m)}\Pi^{(m)}Q^{(m)}\equiv A^{(m)}D^{(m)} ,\\ &Q^{(m)}\Pi^{(m)}P^{(m)}\equiv C^{(m)}A^{(m)} .\end{aligned} \tag{4.18}$$

If $A^{(m)}$ is invertible in the $P^{(m)}$ subspace, i.e., if there exists $\overline{A}^{(m)}$ such that $\overline{A}^{(m)}A^{(m)}=A^{(m)}\overline{A}^{(m)}=P^{(m)}$, then there is no loss of generality in writing (4.18). This is shown in Appendix G. We also show there that the component $Q^{(m)}\Pi^{(m)}Q^{(m)}$ is not independent from the other components and can be written

$$Q^{(m)}\Pi^{(m)}Q^{(m)}=C^{(m)}A^{(m)}D^{(m)} .$$

Using this relation, the general decomposition (4.17), and the definitions (4.18), $\Pi^{(m)}$ can be written

$$\Pi^{(m)}=[P^{(m)}+C^{(m)}]A^{(m)}[P^{(m)}+D^{(m)}] . \tag{4.19}$$

The explicit forms of the operators (4.18) for the baker transformation are

$$\begin{aligned}&A^{(m)}=P^{(m)}+\sum_{k=1}^{m}\frac{1}{k!}\frac{d^k}{dz^k}[\Delta^{(m)}(z)]^k\Big|_{z=e^{-\gamma_m}} ,\\ &A^{(m)}D^{(m)}=\sum_{k=0}^{m}\frac{1}{k!}\frac{d^k}{dz^k}\times[\Delta^{(m)}(z)]^k\mathcal{D}^{(m)}(z)\Big|_{z=e^{-\gamma_m}} ,\\ &C^{(m)}A^{(m)}=\sum_{k=0}^{m}\frac{1}{k!}\frac{d^k}{dz^k}\mathcal{C}^{(m)}(z)[\Delta^{(m)}(z)]^k\Big|_{z=e^{-\gamma_m}} .\end{aligned} \tag{4.20}$$

In Appendix B we show that

$$U^n\Pi^{(m)}=[P^{(m)}+C^{(m)}][\Theta^{(m)}]^nA^{(m)}[P^{(m)}+D^{(m)}] , \tag{4.21}$$

where we have defined

$$\Theta^{(m)}=P^{(m)}UP^{(m)}+P^{(m)}UC^{(m)} . \tag{4.22}$$

Using (4.21), (4.16) can be written

$$(i,j|U^n|i',j')=\sum_{m=0}^{\infty}(i,j|[P^{(m)}+C^{(m)}][\Theta^{(m)}]^n\times A^{(m)}[P^{(m)}+D^{(m)}]|i',j') . \tag{4.23}$$

Equation (4.23) is our "final" result. Because of the long derivation of this equation and the layers of definitions, it is easy to lose sight of its simplicity. Equation (4.23) says that the correlation function $(i,j|U^n|i',j')$ can be decomposed into independently evolving modes. The operator $P^{(m)}+D^{(m)}$, associated with mode m, maps the function $|i',j')$ into an $(m+1)$-dimensional subspace associated with $P^{(m)}$. Inside that subspace, time evolution is given by the operator $\Theta^{(m)}$, which is just an $(m+1)\times(m+1)$ matrix. The operator $P^{(m)}+C^{(m)}$ essentially does the in-

verse mapping out of the subspace. The diagonal elements of $\Theta^{(m)}$ are $e^{-\gamma_m}$, so that the time evolution is basically exponential decay. The other nonzero elements of $\Theta^{(m)}$ are above the diagonal, so that it is essentially a Jordan block. The modes within the subspace cannot be decoupled (it is impossible to diagonalize $\Theta^{(m)}$) and their interactions produce power-law corrections to exponential decay.

Although we have derived (4.23) for the baker transformation, its form is more general. For instance, the Bernoulli map is a special case of (4.23) where the dimension of $P^{(m)}$ is 1. In that case, $\Theta^{(m)}$ is simply a number and the eigenstate interpretation (3.20) is possible. The definitions (4.18) reduce to $A^{(m)}=P^{(m)}$, $C^{(m)}=\mathcal{C}^{(m)}(e^{-\gamma_m})$, etc.

In principle, (4.23) is straightforward to evaluate explicitly. Although the layers of definitions make manual calculations quite tedious, it can be evaluated easily by computer programs which do algebraic manipulation. In Appendix H, we illustrate (4.23) with the calculation of a simple correlation function.

A. Test functions and distributions

Until now, we have avoided questions about the domain of $\Pi^{(m)}$, $C^{(m)}$, and $D^{(m)}$ by looking only at matrix elements in the Legendre basis, which are always well defined. Suppose, however, that we want to calculate arbitrary correlation functions

$$(A|U^n|B)\ . \tag{4.24}$$

When is the spectral description (4.23) valid? In the Bernoulli map, the condition was essentially that the sum (3.33) converge. The analog for the baker transformation is that

$$\sum_{\substack{i',j' \\ i'+j'\neq m}} (i,j|D^{(m)}|i',j')(i',j'|B) \tag{4.25}$$

and

$$\sum_{\substack{i',j' \\ i'+j'\neq m}} (A|i',j')(i',j'|C^{(m)}|i,j) \tag{4.26}$$

be finite for all m and $i+j=m$. This provides an operational definition of the test spaces to which A and B belong. From the nonrecurrence property, $(i,j|D^{(m)}|i',j')$ is nonzero only if $j'<j$ and $(i',j'|C^{(m)}|i,j)$ is nonzero only if $i'<i$. This means that $(i,j|D^{(m)}|x,y)$ is a finite polynomial of degree j with respect to y and $(x,y|C^{(m)}|i,j)$ is one of degree i with respect to x. $(i,j|D^{(m)}|x,y)$ and $(x,y|C^{(m)}|i,j)$ are singular distributions with respect to x and y, respectively. For (4.25) and (4.26) to be well defined, it is sufficient that $B(x,y)$ be a finite polynomial with respect to x and $A(x,y)$ be a finite polynomial with respect to y.

To get a better idea of what the test spaces are, we note that $\Theta^{(m)}$, which is essentially a Jordan block, has one eigenstate, so there should be one eigenstate associated with each resonance $e^{-\gamma_m}$. The eigenstates are uniform in the x direction and in the y direction look like the eigenstates $|\tilde{\gamma}_m)$ of $U^{\dagger}_{\text{Bernoulli}}$ for the Bernoulli map, i.e.,

$$|\gamma_m)_{\text{baker}}=1\otimes|\tilde{\gamma}_m)_{\text{Bernoulli}}\ . \tag{4.27}$$

The state (4.27) is an eigenstate because the Perron-Frobenius operator for the baker transformation preserves uniformity in the x direction and looks like $U^{\dagger}_{\text{Bernoulli}}$ in the y direction. From (3.42), one can think of (4.27) as the operator d^m/dy^m.

Using the operators discussed above, the eigenstate can be written

$$|\gamma_m)_{\text{baker}}=(P^{(m)}+C^{(m)})|0,m)\ . \tag{4.28}$$

Since there are no transitions from $|0,m)$ to $|i,j)$ for $i>0$, $|\gamma_m)_{\text{baker}}$ is uniform in x. Since the formula (F2) for $(0,m+k|\mathcal{C}^{(m)}(z)|0,m)$ is essentially the same as formula (F1) for matrix elements of $\mathcal{D}^{(m)}$ in the Bernoulli map and since only the first terms in lines 1 and 3 of (4.20) are nonzero, we obtain (4.27). It is easily verified that this is an eigenstate and eigenvalue $e^{-\gamma_m}=2^{-m}$.

These arguments suggest that the observable A in (4.24) should be restricted to the space $L_2(0,1)\otimes\mathcal{T}$. Similarly, B should be restricted to the space $\mathcal{T}\otimes L_2(0,1)$. These spaces are preserved by U acting to the left and right, respectively. [The domains can be extended by allowing A to be any element of $\mathcal{T}^*\otimes\mathcal{T}$ and B to be any element of $\mathcal{T}\otimes\mathcal{T}^*$, although it is not clear what meaning can be assigned to an observable or distribution in $\mathcal{T}^*$ (but not in L_2).]

B. Time reversal

Equation (4.23) [which is ultimately based on (4.7)] is valid only for positive times. To obtain an expression for negative times one starts with the resolvent of $U^{\dagger}=U^{-1}$, with $n>0$. This leads to a formula identical to (4.23) but with definitions for the operators—the roles of x and y are reversed—which gives the correct dynamics for negative times.

Loschmidt [32] pointed out a "paradox" in any derivation of irreversibility from time-reversible laws. Starting from an initial condition ρ_0, let the system evolve for n time steps, after which the system will be described by a distribution function ρ_n. During this time, we observe thermodynamic behavior—an approach to equilibrium. Now consider an initial distribution equal to ρ_n but with the velocities reversed. Certainly the system will exhibit antithermodynamic behavior. Although the baker map is not a Hamiltonian system, we can effectively "reverse velocities" by interchanging the x and y dependence of $\rho_n(x,y)$.

The paradox is avoided in our generalized spectral decomposition because the velocity-reversed distribution function $\rho_n(y,x)$ will not be in the domain $\mathcal{T}\otimes L_2$. The map preserves smoothness in x but not in y. When velocities are reversed, the distribution function will not be smooth in x. Avoidance of the paradox by a discontinuity is special to the baker map. For a Hamiltonian system we expect a similar situation with a more subtle definition of the test function space. By analogy with results for the quantum Friedrichs model [3], we might expect this

definition to involve analyticity of the spectral density of ρ inside or outside the unit circle.

V. CONCLUSION AND REMARKS

Our goal has been to understand the thermodynamic behavior of dynamical systems—diffusion, decay, transport, and the approach to equilibrium. We have shown that these phenomena can be introduced at a fundamental level into the description of dynamical systems.

In the usual spectral theory in Hilbert space, these "thermodynamic" systems have an absolutely continuous spectrum containing no time scales associated with their instability—Lyapunov exponents, decay rates, or diffusion coefficients. We have presented a construction in which decay rates of correlation functions (Ruelle resonances) are obtained as generalized eigenvalues associated with eigendistributions lying outside the Hilbert space. These eigenfunctions can be used to write a generalized spectral decomposition of the Perron-Frobenius operator if observables and/or distribution functions are restricted to a space of smooth test functions. The distributions themselves are in a sense a mathematical tool, having no physical realization, or even an interpretation in terms of an ensemble of systems. The eigenvalues, however, are real and are easily observed. In the piecewise linear maps discussed in this paper, the Ruelle resonances are the inverses of generalized Lyapunov exponents, although in general this is not true [16].

In this paper we have considered systems with uniform stretching factor. The derivative operator plays a key role in the thermodynamic description of these systems since, as we have shown, the left eigenstates in the position representation are essentially derivative operators, and the derivative operator is a shift operator for the right eigenstates. The derivative operator computes the change in a function in an infinitesimal neighborhood of a point, suggesting a connection between our approach and the usual description in terms of trajectories and Lyapunov exponents. The appearance of a derivative operator is closely related to the importance of the differentiability of observables [30].

The degeneracy of the resonances seen in the baker map always occurs in conservative systems with uniform stretching factor. Since the Lyapunov exponents sum to zero, positive exponents may be expressed as linear combinations of negative exponents. Since the Lyapunov exponents determine the spectrum, it is always degenerate. For nonlinear systems, the degeneracy is removed.

As presented in this paper, our method appears to rely on the upper-triangularity of the Perron-Frobenius operator in some representation (the nonrecurrence condition). Although this simplifies the analysis, it is not essential. Any method of analytically continuing matrix elements inside the unit circle will work. Usually this will be a perturbative method. We have used a perturbative method to explain diffusion in the standard map (for large values of the kick parameter) in terms of exponentially decaying generalized eigenstates [12]. We believe that the method can be used perturbatively to discuss nonlinear perturbations of the maps discussed in this paper. Christiansen, Paladin, and Rugh [16] have calculated the Ruelle resonances of a perturbed Bernoulli map using a monomial basis. For calculation of the resonances and right eigenstates, our method should be equivalent since we use polynomials. To calculate the left eigenstates, the calculations are tedious but there does not seem to be any problem as long as the perturbation is small enough. The situation in the baker map is less clear, because a perturbation will mix the x and y directions. The eigenstates of the baker map are regular with respect to one direction and singular with respect to the other. Eigenstates of a perturbed baker map would be singular with respect to both, so we expect to get a diverging series (representing a distribution) where before we had a finite one. Work on these calculations is in progress [11].

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APPENDIX A: PROOF OF EQ. (2.3)

Equation (2.3) follows from the operator identity

$$\frac{1}{z-U}=\sum_{n=0}^{\infty} z^{-(n+1)}U_n \ , \tag{A1}$$

which is valid for $|z|>\|U\|$. Equation (A1) is a special case of (3.10) for $A=0$.

Because the contour in Eq. (2.3) is outside the spectrum of U, we can expand the right-hand side using (A1):

$$\begin{aligned}\frac{1}{2\pi i}\oint_C \frac{z^n}{z-U} &= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\oint_C dz\, z^n z^{-(k+1)}U^k \\ &= \frac{1}{2\pi i}\int_0^{2\pi} ie^{i\xi}d\xi\, e^{in\xi}e^{-i(k+1)\xi}U^k \\ &= \sum_k U^k\delta_{k,n} \\ &= U^n, \quad n\geq 0 \ . \end{aligned} \tag{A2}$$

In the second line we have used the change of variables $z=(1+\epsilon)e^{i\xi}$, where ϵ is infinitesimal. Q.E.D.

APPENDIX B: PROPERTIES OF $\Pi^{(m)}$

First we show that the $\Pi^{(m)}$ operators are orthogonal projectors:

$$\Pi^{(m)}\Pi^{(m')}=\Pi^{(m)}\delta_{m,m'} \ . \tag{B1}$$

Assume first that $m\neq m'$. From the definition (2.6) we have

$$\Pi^{(m)}\Pi^{(m')} = \frac{1}{(2\pi i)^2}\oint_{z_1=e^{-\gamma_m}} dz_1 \oint_{z_2=e^{-\gamma_{m'}}} dz_2 \frac{1}{z_1-U}\frac{1}{z_2-U} . \tag{B2}$$

Since $e^{-\gamma_m}$ and $e^{-\gamma_{m'}}$ are different, we can always choose the two contours small enough so that they do not overlap. Using a simple identity, (B2) can be rewritten

$$\Pi^{(m)}\Pi^{(m')} = \frac{1}{(2\pi i)^2}\oint_{C_1} dz_1 \oint_{C_2} dz_2 \frac{1}{z_2-z_1} \times \left[\frac{1}{z_1-U} - \frac{1}{z_2-U}\right] . \tag{B3}$$

For the first term, we first do the integral over z_2, considering z_1 to be held fixed. The z_2 dependence of the integrand is $1/(z_2-z_1)$. Since z_1 is always outside the z_2 contour, the integral is zero by the Cauchy residue theorem. By changing the order of the z_1 and z_2 integrations, the second term is also seen to vanish.

For the case $m=m'$, we can choose the contour in the z_1 integral to be entirely inside the contour in the z_2 integral. As before, for the first term we evaluate the z_2 integral first, holding z_1 constant. Now there is a simple pole at $z_2=z_1$ which is inside the contour. After evaluating the integral, what remains is just $\Pi^{(m)}$. For the second term, we evaluate the z_1 integral first, holding z_2 constant. Now, however, the pole at $z_1=z_2$ is outside the z_1 contour, so the integral vanishes. Q.E.D.

To show (2.8) we note that

$$U\oint_{z=e^{-\gamma_m}} dz \frac{z^{n-1}}{z-U} = -\oint dz \frac{z^{n-1}(z-U)}{z-U} + \oint dz \frac{z^n}{z-U} , \tag{B4}$$

where we have added and subtracted the second term. We are allowed to bring U inside the integral because it commutes with the number z. The first term has no singularity, so only the second term remains. The definition (2.6) can be written

$$\Pi^{(m)} = \frac{1}{2\pi i}\oint_{z=e^{-\gamma_m}} dz \frac{z^0}{z-U} . \tag{B5}$$

Applying U^n to (B5) and using (B4) n times, we obtain (2.8). Q.E.D.

Derivation of Eq. (4.21)

Starting with (4.19),

$$\Pi^{(m)}\rho = [P^{(m)}+C^{(m)}]A^{(m)}[P^{(m)}+D^{(m)}]\rho , \tag{B6}$$

and recalling the definitions (4.18) of these operators, we have immediately

$$\begin{aligned} P^{(m)}\Pi^{(m)}\rho &= A^{(m)}[P^{(m)}+D^{(m)}]\rho , \\ Q^{(m)}\Pi^{(m)}\rho &= C^{(m)}A^{(m)}[P^{(m)}+D^{(m)}]\rho \\ &= C^{(m)}P^{(m)}\Pi^{(m)}\rho . \end{aligned} \tag{B7}$$

Using the commutativity with U, the relation $P^{(m)}+Q^{(m)}=1$, and the last line of Eqs. (B7), we have

$$\begin{aligned} P^{(m)}\Pi^{(m)}\rho_{n+1} &= P^{(m)}U\Pi^{(m)}\rho_n \\ &= P^{(m)}UP^{(m)}\Pi^{(m)}\rho_n + P^{(m)}UQ^{(m)}\Pi^{(m)}\rho_n \\ &= P^{(m)}UP^{(m)}P^{(m)}\Pi^{(m)}\rho_n + P^{(m)}UQ^{(m)}C^{(m)}P^{(m)}\Pi^{(m)}\rho_n \\ &= [P^{(m)}UP^{(m)}+P^{(m)}UQ^{(m)}C^{(m)}]P^{(m)}\Pi^{(m)}\rho_n \\ &= \Theta^{(m)}P^{(m)}\Pi^{(m)}\rho_n , \end{aligned} \tag{B8}$$

where we have defined the operator $\Theta^{(m)}$ as in Eq. (4.22). Applying (B8) recursively gives

$$P^{(m)}\Pi^{(m)}\rho_n = [\Theta^{(m)}]^n P^{(m)}\Pi^{(m)}\rho_0 . \tag{B9}$$

From (B6) and (B7),

$$\begin{aligned} Q^{(m)}\Pi^{(m)}\rho_n &= C^{(m)}P^{(m)}\Pi^{(m)}\rho_n \\ &= C^{(m)}[\Theta^{(m)}]^n\Pi^{(m)}\rho_0 . \end{aligned} \tag{B10}$$

Combining (B8) and (B10) we finally obtain

$$\begin{aligned} \Pi^{(m)}\rho_n &= [P^{(m)}+Q^{(m)}]\Pi^{(m)}\rho_n \\ &= [P^{(m)}+C^{(m)}][\Theta^{(m)}]^n P^{(m)}\Pi^{(m)}\rho_0 \\ &= [P^{(m)}+C^{(m)}][\Theta^{(m)}]^n A^{(m)}[P^{(m)}+D^{(m)}]\rho_0 . \end{aligned} \tag{B11}$$

Q.E.D.

APPENDIX C: U, $U^\dagger$, UNITARITY, AND ISOMETRY

In this appendix we review the unitarity properties of the Perron-Frobenius operator and its adjoint. These results are well known, but we repeat them here for completeness. The results are exactly the same for the case of complex-valued functions. By definition, the adjoint $\overline{U}^\dagger$ of $\overline{U}$ is the operator that satisfies

$$(A|\overline{U}|B) = (\overline{U}^\dagger A|B) = (B|\overline{U}^\dagger|A)^* . \tag{C1}$$

From the definition (2.1)

$$(A|\overline{U}|B) = \int_{x\in M} dx\ A^*(x) \sum_{y=f^{-1}(x)} \frac{B(y)}{|f'(y)|} . \tag{C2}$$

Since f might be many to one, we divide M into n disjoint subspaces, $M_1, M_2, \ldots, M_n$ such that f is one-to-one from M_i into M. Then we define define maps f_i: $M_i \to M$

which are one-to-one (but not necessarily onto) by $f_i(x)=f(x)$ for $x\in M_i$. Equation (C2) can then be rewritten

$$\begin{aligned}(A|\overline{U}|B)&=\sum_{i=1}^{n}\int_{x\in f_i(M_i)}dx\ A^*(x)\frac{B(f_i^{-1}(x))}{|f'(f_i^{-1}(x))|}\\&=\sum_{i=1}^{n}\int_{y\in M_i}|f_i'(y)|\,dy\frac{A^*(f_i(y))B(y)}{|f_i'(y)|}\\&=\int_{y\in M}B(y)A^*(f(y))\,dy\\&=\left[\int_{y\in M}B^*(y)A(f(y))\,dy\right]^*.\end{aligned}\tag{C3}$$

Comparing with (C1), we make the identification

$$\overline{U}^{\dagger}\rho(x)=\rho(f(x))\ .\tag{C4}$$

$\overline{U}^{\dagger}$ is called the Koopman operator. The Perron-Frobenius operator and the Koopman operator are the time-evolution operators for states and observables, respectively. Using one or the other corresponds to using the Schödinger or Heisenberg representation in quantum mechanics.

If f is bijective and its determinant is 1, then clearly $\overline{U}^{\dagger}=U^{-1}$ and $\overline{U}$ is unitary. If these conditions do not hold, then $\overline{U}$ has no special properties but $\overline{U}^{\dagger}$ may be isometric. This follows from

$$\begin{aligned}\overline{U}\,\overline{U}^{\dagger}\rho(x)&=\overline{U}[\overline{U}^{\dagger}\rho](x)\\&=\sum_{y=f^{-1}(x)}\frac{(\overline{U}^{\dagger}\rho)(y)}{|f'(y)|}\\&=\sum_{y=f^{-1}(x)}\frac{\rho(f(y))}{|f'(y)|}\\&=\rho(x)\sum_{y=f^{-1}(x)}\frac{1}{|f'(y)|}\ .\end{aligned}\tag{C5}$$

If

$$\sum_{y=f^{-1}(x)}\frac{1}{|f'(y)|}=1$$

for all x, then $\overline{U}\,\overline{U}^{\dagger}=1$, which implies by theorem [33] that $\overline{U}^{\dagger}$ is isometric (preserves norms).

APPENDIX D: MATRIX ELEMENTS OF THE OPERATOR $\overline{U}$ FOR THE BERNOULLI MAP AND U FOR THE BAKER MAP

The modified Legendre polynomials $(x|i)=\tilde{P}_i(x)$ are defined in terms of the usual Legendre polynomials $P_i(x)$ by

$$\tilde{P}_i(x)=\sqrt{2i+1}P_i(1-2x)\ .$$

They are given by the Rodriguez formula:

$$\tilde{P}_i(x)=\frac{\sqrt{2i+1}}{i!}\frac{d^i}{dx^i}x^i(1-x)^i\ .\tag{D1}$$

The polynomials satisfy $(i|j)=\delta_{i,j}$.

For the Bernoulli map, we use expression (3.1) to compute the matrix elements of the Perron-Frobenius operator. They are

$$\begin{aligned}(i|\overline{U}|i')&=\int_0^1 dx\ \tilde{P}_i(x)\overline{U}\tilde{P}_{i'}(x)\\&=\int_0^1 dx\ \tilde{P}_i(x)\frac{1}{2}[\tilde{P}_{i'}(\frac{x}{2})+\tilde{P}_{i'}(\frac{x}{2}+\frac{1}{2})]\ .\end{aligned}\tag{D2}$$

To evaluate the second term in the integral, use the relation [which can be seen from (D1)]

$$\tilde{P}_i(1-x)=(-1)^i\tilde{P}_i(x)\tag{D3}$$

to rewrite the integral as

$$\int_0^1 dx\ \tfrac{1}{2}\tilde{P}_i(x)(-1)^{i'}\tilde{P}_{i'}(\frac{1}{2}-\frac{x}{2})\ .\tag{D4}$$

Then make the change of variable $x\to 1-x$ to obtain

$$\int_0^1 dx\ \tfrac{1}{2}\tilde{P}_i(1-x)(-1)\tilde{P}_{i'}(\frac{x}{2})\ .\tag{D5}$$

Finally again use (D3) to get

$$\int_0^1 dx\ \tfrac{1}{2}\tilde{P}_i(x)(-1)^{i+i'}\tilde{P}_{i'}(\frac{x}{2})\ .$$

Plugging into (D2) we obtain

$$(i|\overline{U}|i')=\tfrac{1}{2}I_{i,i'}+\tfrac{1}{2}(-1)^{i+i'}I_{i,i'}\ ,\tag{D6}$$

where we have defined

$$I_{i,i'}\equiv\int_0^1 dx\ \tilde{P}_i(x)\tilde{P}_{i'}(\frac{x}{2})\ .\tag{D7}$$

The calculation for the baker map is very similar and uses (4.1) as a starting point. The result is given in (4.2).

Now we evaluate $I_{i,i'}$. Using formula (D1), we have

$$\begin{aligned}I_{i,k}&=\frac{\sqrt{(2i+1)(2k+1)}}{i!k!}\int_0^1 dx\left[\frac{d^i}{dx^i}x^i(1-x)^i\right]\left[\frac{d^k}{d(x/2)^k}(x/2)^k[1-(x/2)]^k\right]\\&=\frac{\sqrt{(2i+1)(2k+1)}}{2^k i!k!}\int_0^1 dx\left[\frac{d^i}{dx^i}x^i(1-x)^i\right]\left[\frac{d^k}{dx^k}x^k(2-x)^k\right].\end{aligned}\tag{D8}$$

Integrating by parts i times, we get

$$I_{i,k}=\frac{\sqrt{(2i+1)(2k+1)}}{2^k i!k!}\int_0^1 dx(-1)^i x^i(1-x)^i\left[\frac{d^{k+i}}{dx^{k+1}}x^k(2-x)^k\right] . \tag{D9}$$

If $i>k$, the integral vanishes. Otherwise we use the binomial expansion and the relation

$$\int_0^1 x^j(1-x)^i=\frac{j!i!}{(i+j+1)!}$$

to obtain (3.5).

E: CALCULATION OF THE RUELLE RESONANCES

In this section we show that the resonances described by Ruelle are the same as the eigenvalues we calculate. The Ruelle resonances are the inverses of the zeros of the Fredholm determinant

$$d(z)\equiv\det[1-zU]=\exp\left[-\sum_n (z^n/n)\,\mathrm{Tr}U^n\right]$$
$$=\prod_m (1-ze^{-\gamma_m}) . \tag{E1}$$

For the baker map, the trace of U^n is

$$\mathrm{Tr}U^n=\int dX\,\delta(f^n(X)-X)=\int_0^1 dx_0\int_0^1 dy_0\,\delta(x_n-x_0)\delta(y_n-y_0)$$
$$=\sum_{X=f^n(X)}\left|\frac{\partial x_n}{\partial x_0}-1\right|^{-1}\left|\frac{\partial y^n}{\partial y_0}-1\right|^{-1}=\sum_{X=f^n(X)}(2^n-1)^{-1}\left[1-\frac{1}{2^n}\right]^{-1}=\left[1-\frac{1}{2^n}\right]^{-2}$$
$$=\sum_{m=0}^{\infty}(m+1)\left[\frac{1}{2^m}\right]^n , \tag{E2}$$

where we have written $X\equiv(x,y)$ and used the fact that the baker map has 2^n periodic points of period n. The above calculation of the trace is purely formal.

Therefore the formula for $d(z)$ becomes

$$d(z)=\exp\left[-\sum_n (z^n/n)\sum_m (m+1)2^{-mn}\right]$$
$$=\prod_m \exp\left[-(m+1)\sum_n (z^n 2^{-nm})/n\right]$$
$$=\prod_m \exp[(m+1)\ln(1-z2^{-m})]$$
$$=\prod_m [1-z2^{-m}]^{m+1} . \tag{E3}$$

The inverses of the zeros occur at $e^{-\gamma_m}=2^{-m}$, which are the same as the eigenvalues we have calculated.

For the Bernoulli map, the trace of $\bar{U}$ is

$$\mathrm{Tr}\bar{U}=\sum_{x=f^n(x)}|2^n-1|^{-1}$$
$$=\frac{1}{1-2^{-n}} . \tag{E4}$$

This gives a Fredholm determinant

$$d(z)=\prod_m [1-z2^{-m}] , \tag{E5}$$

which leads to the same resonances $e^{-\gamma_m}=2^{-m}$ but with single multiplicity.

The trace of U can also be calculated using a basis of Legendre polynomials: For the baker map,

$$\mathrm{Tr}U^n=\sum_{i,j}(i,j|U^n|i,j) . \tag{E6}$$

From the nonrecurrence property,

$$(i,j|U^n|i,j)=[(i,j|U|i,j)]^n=e^{-n\gamma_{i+j}} ,$$

which gives (E2) immediately.

APPENDIX F: MATRIX ELEMENTS OF $\mathcal{D}^{(m)}(z)$, $\mathcal{C}^{(m)}(z)$, AND $\Psi^{(m)}(z)$

For the Bernoulli map:

$$\Psi^{(m)}(z)=e^{-\gamma_m}P^{(m)} ,$$
$$(m-k|\mathcal{C}^{(m)}(z)|m)=\frac{1}{z-e^{-\gamma_{m-k}}}[(m-k|\bar{U}|m)+\sum_{k'=1}^{k-1}(m-k|\bar{U}|m-k')(m-k'|\mathcal{C}^{(m)}(z)|m)] , \tag{F1}$$
$$(m|\mathcal{D}^{(m)}(z)|m+k)=\frac{1}{z-e^{-\gamma_{m+k}}}[(m|\bar{U}|m+k)+\sum_{k'=1}^{k-1}(m|\mathcal{D}^{(m)}(z)|m+k')(m+k'|\bar{U}|m+k)] .$$

For the baker map:

$$
\begin{aligned}
(m-j-l,j+l|\Psi^{(m)}(z)|m-j,m)&=(m-j-l,j+l|U|m-j,j)+\sum_{(k',l')=0}^{(l,l)}(m-j-l,j+l|U|m-j-k',j+l')\\
&\quad\times(m-j-k',j+l'|\mathcal{C}^{(m)}(z)|m-j,j)\ ,\\
(m-j-k,j+l|\mathcal{C}^{(m)}(z)|m-j,j)&=\frac{1}{z-e^{-\gamma_{m-k+l}}}[(m-j-k,j+l|U|m-j,j)\\
&\quad+\sum_{(k',l')=0}^{(k,l)}(m-j-k,j+l|U|m-j-k',j+l')\\
&\quad\times(m-j-k',j+l'|\mathcal{C}^{(m)}(z)|m-j,j)]\ ,\\
(m-j,j|\mathcal{D}^{(m)}(z)|m-j+k,j-l)&=\frac{1}{z-e^{-\gamma_{m+k-l}}}[(m-j,j|U|m-j+k,j-l)\\
&\quad+\sum_{(k',l')=0}^{(k,l)}(m-j,j|\mathcal{D}^{(m)}(z)|m-j+k',j-l')\\
&\quad\times(m-j+k',j-l'|U|m-j+k,j-l)]\ .
\end{aligned}
\tag{F2}
$$

The summation in the last two equations of (F2) should be interpreted as

$$
\sum_{(k',l')=0}^{(k,l)}\equiv\sum_{k'=0}^{k}\sum_{l'=0}^{l}(1-\delta_{k',l'})(1-\delta_{k,k'}\delta_{l,l'})\ . \tag{F3}
$$

In the first equation it is the same except that k' runs from 0 to l.

APPENDIX G: DERIVATION OF $Q^{(m)}\Pi^{(m)}Q^{(m)}=C^{(m)}A^{(m)}D^{(m)}$

In this section we prove the decomposition (4.19) of $\Pi^{(m)}$ by showing that $A^{(m)}$ is invertible and that

$$
Q^{(m)}\Pi^{(m)}Q^{(m)}=C^{(m)}A^{(m)}D^{(m)}\ .
$$

From (4.18) we can write

$$
A^{(m)}=P^{(m)}+\delta A^{(m)}\ . \tag{G1}
$$

Because $\delta A^{(m)}$ is upper-triangular with zeros on the diagonal,

$$
(\delta A)^{m+2}=0\ . \tag{G2}
$$

The operator $\overline{A}^{(m)}$, defined by

$$
\overline{A}^{(m)}=P^{(m)}+\sum_{s=1}^{\infty}(-1)^s(\delta A^{(m)})^s\ , \tag{G3}
$$

is the inverse of $A^{(m)}$ on the subspace generate by $P^{(m)}$.

Equation (G3) is just the series expansion for $1/(P^{(m)}+\delta A^{(m)})$. It converges because, by property (G2), it terminates. The fact that (G3) is in fact the inverse of A can be verified by direct calculation.

From definition in (4.18) and the fact that $(\Pi^{(m)})^2=\Pi^{(m)}$ and $P^{(m)}+Q^{(m)}=1$, we have

$$
\begin{aligned}
A^{(m)}&=P^{(m)}\Pi^{(m)}P^{(m)}=P^{(m)}\Pi^{(m)}\Pi^{(m)}P^{(m)}\\
&=P^{(m)}\Pi^{(m)}P^{(m)}\Pi^{(m)}P^{(m)}+P^{(m)}\Pi^{(m)}Q^{(m)}\Pi^{(m)}P^{(m)}\\
&=A^{(m)}A^{(m)}+A^{(m)}D^{(m)}C^{(m)}A^{(m)}\ .
\end{aligned}
\tag{G4}
$$

Multiplication by $\overline{A}^{(m)}$ on the left and right gives

$$
\overline{A}^{(m)}=P^{(m)}+D^{(m)}C^{(m)}\ . \tag{G5}
$$

Note that

$$
\begin{aligned}
C^{(m)}A^{(m)}D^{(m)}&=C^{(m)}A^{(m)}\overline{A}^{(m)}A^{(m)}D^{(m)}\\
&=C^{(m)}A^{(m)}(P^{(m)}+D^{(m)}C^{(m)})A^{(m)}D^{(m)}\\
&=C^{(m)}A^{(m)}A^{(m)}D^{(m)}+(C^{(m)}A^{(m)}D^{(m)})^2
\end{aligned}
\tag{G6}
$$

and that

$$
\begin{aligned}
Q^{(m)}\Pi^{(m)}Q^{(m)}&=Q^{(m)}\Pi^{(m)}Q^{(m)}\Pi^{(m)}Q^{(m)}\\
&\quad+Q^{(m)}\Pi^{(m)}P^{(m)}\Pi^{(m)}Q^{(m)}\\
&=(Q^{(m)}\Pi^{(m)}Q^{(m)})^2\\
&\quad+C^{(m)}A^{(m)}A^{(m)}D^{(m)}\ .
\end{aligned}
\tag{G7}
$$

Comparing (G6) and (G7), we have

$$
Q^{(m)}\Pi^{(m)}Q^{(m)}=C^{(m)}A^{(m)}D^{(m)}\ . \tag{G8}
$$

The uniqueness of (G8) can be verified by a series expan-

sion of (G6) and (G7) obtained by applying the formulas recursively, and using the nonrecurrence relation.

APPENDIX H: EXAMPLE CALCULATION FOR THE BAKER MAP

In this section we illustrate the application of the formalism with an explicit calculation of the correlation function $(0,2|U^n|2,0)$ for the baker map. The calculations involve a great amount of straightforward but tedious algebra and would normally be done with the aid of a computer program which can do symbolic manipulation, such as Mathematica. To make the calculation manageable by hand, we use some shortcuts which unfortunately hide the appearance of the Jordan block $\Theta^{(m)}$ in the final result.

From (4.23), the expression we need to evaluate is

$$(0,2|U^n|2,0)=\sum_{m=0}^{\infty}(0,2|[P^{(m)}+C^{(m)}][\Theta^{(m)}]^n \times A^{(m)}[P^{(m)}+D^{(m)}]|2,0) . \tag{H1}$$

To simplify the calculation, we make use of the fact [see (4.2)] that $(i-k,j+l|U|i,j)$ is nonzero only for $k+l$ even. The possible intermediate states are therefore $|1,1)$ and $|2,2)$. Because of this only the $m=2$ and $m=4$ subspaces contribute, so that (4.2) can be written

$$(0,2|U_n|2,0)=(0,2|[\Theta^{(2)}]^n A^{(2)}|2,0) +(0,2|C^{(4)}[\Theta^{(4)}]^n A^{(4)}D^{(4)}|2,0) . \tag{H2}$$

Inserting all possible intermediate states, we get

$$\begin{aligned}(0,2|U^n|2,0)=&(0,2|[\Theta^{(2)}]^n|0,2)(0,2|A^{(2)}|2,0)\\ &+(0,2|[\Theta^{(2)}]^n|2,0)(2,0|A^{(2)}|2,0)\\ &+(0,2|[\Theta^{(2)}]^n|1,1)(1,1|A^{(2)}|2,0)\\ &+(0,2|C^{(4)}|2,2)(2,2|[\Theta^{(4)}]^n|2,2)(2,2|A^{(4)}|2,2)(2,2|D^{(4)}2,0) .\end{aligned} \tag{H3}$$

We will calculate the first term and most of the second term explicitly, leaving the rest to the reader.

1. Calculation of Eq. (H3), first line

From the definition (4.22) of $\Theta^{(m)}$ (and the nonrecurrence condition), the diagonal element $(0,2|[\Theta^{(2)}]^n|0,2)$ is $(0,2|U^n|0,2)=e^{-\gamma_2 n}$. From the definition (4.20) of $A^{(m)}$, the other matrix element is

$$(0,2|\{P^{(m)}+\frac{d}{dz}[\Delta^{(2)}(z)]+\frac{1}{2}\frac{d^2}{dz^2}[\Delta^{(2)}(z)]^2\}|2,0)\Big|_{z=e^{-\gamma_2}} . \tag{H4}$$

$\Delta^{(2)}$ is a 3×3 matrix with entries only above the diagonal. The first term in (H4) is zero, because $|2,0)$ and $|0,2)$ are orthogonal. From the definition (4.9) of $\Delta^{(m)}$ and the matrix elements of $\Psi^{(m)}$ [first equation of (F2)], then using (F2) for the matrix elements of $C^{(m)}(z)$, we have

$$\begin{aligned}(0,2|\frac{d}{dz}\Delta^{(2)}(z)|2,0)&=\frac{d}{dz}[(0,2|U|2,0)+(0,2|U|2,2)(2,2|C^{(2)}(z)|2,0)]\\ &=(0,2|U|2,2)\frac{d}{dz}\frac{1}{z-e^{-\gamma_4}}(2,2|U|2,0)\\ &=\frac{-1}{(z-e^{-\gamma_4})^2}(0,2|U|2,2)(2,2|U|2,0) .\end{aligned} \tag{H5}$$

The third term vanishes because the element of $[\Delta^{(2)}]^2$ which is nonzero is independent of z. Combining these results,

$$(0,2|U^n|2,0)=e^{-n\gamma_2}\frac{-1}{(e^{-\gamma_2}-e^{-\gamma_4})^2}(0,2|U|2,2)(2,2|U|2,0) . \tag{H6}$$

2. Calculation of Eq. (H3), second line

We evaluate the first term in the second line of Eq. (H3) by decomposing $\Theta^{(2)}=e^{-\gamma_2}P^{(2)}+\Delta\Theta^{(2)}$ into diagonal and off-diagonal parts:

$$\begin{aligned}(0,2|[\Theta^{(2)}]^n|2,0)&=(0,2|[e^{-\gamma_2}P^{(2)}+\Delta\Theta^{(2)}]^n|2,0)\\ &=(0,2|e^{-n\gamma_2}P^{(2)}+ne^{-(n-1)\gamma_2}\Delta\Theta^{(2)}+\frac{n(n-1)}{2}e^{-(n-2)\gamma_2}\Delta\Theta^{(2)}\Delta\Theta^{(2)}|2,0) .\end{aligned} \tag{H7}$$

The first term is zero because $|0,2)$ and $|2,0)$ are orthogonal. In the second term, we insert the definition (4.22) of $\Theta^{(m)}$, and in the third term use the definition and the nonrecurrence property to write down the form immediately:

$$(0,2|[\Theta^{(2)}]^n|2,0)=ne^{-(n-1)\gamma_2}(0,2|U+UC^{(2)}|2,0)+\frac{n(n-1)}{2}e^{-(n-2)\gamma_2}(0,2|U|1,1)(1,1|U|2,0)\ . \tag{H8}$$

The rest of the calculation proceeds in the same manner, inserting definitions and using the formulas for matrix elements given in Appendix F.

The final result is

$$\begin{aligned}(0,2|U^n|2,0)=&e^{-n\gamma_2}\frac{-1}{(e^{-\gamma_2}-e^{-\gamma_4})^2}(0,2|U|2,2)(2,2|U|2,0)\\&+ne^{-\gamma_2(n-1)}\left[(0,2|U|2,0)+\frac{1}{e^{-\gamma_2}-e^{-\gamma_4}}(0,2|U|2,2)(2,2|U|2,0)\right]\\&+\frac{n(n-1)}{2}e^{-\gamma_2(n-2)}(0,2|U|1,1)(1,1|U|2,0)+\frac{e^{-n\gamma_4}}{(e^{-\gamma_4}-e^{-\gamma_2})^2}(0,2|U|2,2)(2,2|U|2,0)\ .\end{aligned} \tag{H9}$$

When we substitute explicit forms for matrix elements of U given by Eq. (4.2) [using (3.6)] it turns out that almost all the terms in (H9) vanish (since $I_{02}=0$), leaving

$$(0,2|U^n|2,0)=\frac{n(n-1)}{2}\frac{45}{64}\left[\frac{1}{4}\right]^{n-2}\ . \tag{H10}$$

Equation (H10) contains exponential decay with a power-law correction.

APPENDIX I: PROPERTIES OF THE BERNOULLI POLYNOMIALS

In this section we summarize the properties of the Bernoulli polynomials, which are related to the right eigenstates of the Bernoulli map by Eq. (3.27).

The Bernoulli polynomials $B_m(x)$ are generated by

$$\frac{te^{xt}}{e^t-1}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}\ . \tag{I1}$$

To first order in t, the left-hand side is

$$t(1+xt+\cdots)\frac{1}{t}\left[1-\frac{t}{2}+\cdots\right] \tag{I2}$$

so that

$$\begin{aligned}&B_0(x)=1\ ,\\&B_1(x)=x-\tfrac{1}{2}\ .\end{aligned} \tag{I3}$$

Also, from (I1) we have

$$\sum_{m=0}^{\infty}\frac{B_m(1)-B_m(0)}{m!}t^m=\frac{t(e^t-1)}{e^t-1} \tag{I4}$$

so that

$$\begin{aligned}&B_1(1)-B_1(0)=1\ ,\\&B_m(1)-B_m(0)=0\ ,\end{aligned} \tag{I5}$$

for $m\neq 1$. Taking $\partial/\partial x$ of (I1), we get

$$\begin{aligned}\frac{t^2e^{xt}}{e^t-1}&=\sum_{m=0}^{\infty}\frac{\frac{d}{dx}B_m(x)}{m!}t^{m-1}\ ,\\\frac{te^{xt}}{e^t-1}&=\sum_{m=0}^{\infty}\frac{\frac{d}{dx}B_{m+1}(x)}{(m+1)!}t^m\ .\end{aligned} \tag{I6}$$

Comparing with (I1) we have

$$\frac{d}{dx}B_{m+1}(x)=(m+1)B_m(x)\ . \tag{I7}$$

To show that the Bernoulli polynomials are eigenstates of the Perron-Frobenius operator we note that [from (3.1)]

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{UB_m(x)}{m!}t^n&=\frac{t}{e^t-1}\frac{1}{2}(e^{xt/2}+e^{(x+1)t/2})\\&=\frac{(t/2)e^{xt/2}}{e^{t/2}-1}\\&=\sum_{m=0}^{\infty}\frac{B_m(x)}{m!}\left[\frac{t}{2}\right]^n\ .\end{aligned} \tag{I8}$$

Therefore

$$UB_m(x)=(\tfrac{1}{2})^mB_m(x)\ . \tag{I9}$$

The Fourier coefficients B_m^l of $B_m(x)$ are defined by

$$\begin{aligned}&B_m(x)=\sum_{l=-\infty}^{\infty}B_m^le^{2\pi ilx}\ ,\\&B_m^l=\int_0^1B_m(x)e^{-2\pi ilx}dx\ .\end{aligned} \tag{I10}$$

For $l=0$,

$$\begin{aligned}B_m^0&=\int_0^1dx\ B_m(x)\\&=\begin{cases}1 \text{ if } m=0\\0 \text{ if } m\neq 0\end{cases},\end{aligned} \tag{I11}$$

where we have used (I5) and (I7). Similarly, one can show by recursive use of (I7) and repeated integration by

parts that

$$B_m^l = \frac{-m!}{(2\pi i l)^m} \tag{I12}$$

for $l \neq 0$.

For $m \neq 0$, we have

$$B_m(x) = -\sum_{l \neq 0} \frac{m!}{(2\pi i l)^m} e^{2\pi i l x} . \tag{I13}$$

As $m \to \infty$, all terms become negligible compared to the $l = \pm 1$ terms, so that

$$\lim_{m \to \infty} B_m(x) \propto \frac{1}{(2\pi i)^m} [e^{2\pi i x} + (-1)^m e^{-2\pi i x}] . \tag{I14}$$

Therefore

$$\lim_{m \to \infty} B_m(x) \propto \begin{cases} \cos 2\pi x & m \text{ even} \\ \sin 2\pi x & m \text{ odd} \end{cases} , \tag{I15}$$

which verifies (3.28). Of course all terms become infinite in the limit. This argument makes sense only if we consider the normalized states $|B_m)_N$.

Equation (3.29) can be verified by examining

$$\frac{(m+2)(m+1)}{(2\pi)^2} B_m(x) + B_{m+2}(x) . \tag{I16}$$

The $l = \pm 1$ terms cancel, so the limit is dominated by $l = \pm 2$, which give terms proportional to $\cos 4\pi x$ and $\sin 4\pi x$, respectively. To get the normalizations in (3.29), we note that $|\gamma_m)_N = |B_m)_N$ and use the large-m normalization $[(B_m|B_m)]^{1/2} \approx \sqrt{2} m!/(2\pi)^m$, which can be obtained from (I13).

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