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Analysis and synthesis of synchronous periodic and chaotic systems

Rong He and P. G. Vaidya*
Department of Mechanical and Materials Engineering, Washington State University, Pullman, Washington 99164-2920
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Chaotic systems are known for their sensitivity to initial conditions. However, Pecora and Carroll [Phys. Rev. Lett. **64**, 821 (1990); Phys. Rev. A **44**, 2374 (1991); IEEE Trans. Circuits Syst. **38**, 453 (1991)] have recently shown that a system, consisting of two Lorenz oscillators exhibiting chaos, could achieve synchronization if a portion of the second system is driven by the first. In this paper, a necessary and sufficient condition for synchronization is presented. This condition has been used to create a high-dimensional chaotic system with a nonlinear subsystem. This system shows synchronization both when it exhibits periodic limit cycles and when it turns chaotic.

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I. INTRODUCTION

Perhaps the most well known of the properties that characterize chaotic systems is their sensitivity to initial conditions. Consider two identical chaotic systems that are started from virtually identical initial conditions. In a short time they would be observed to diverge from one another. This divergence is often observed, even for identical initial conditions, if the two systems are integrated, using the two different routines of integration.

However, in their fascinating papers, Pecora and Carroll [1–3] have recently described a system of two oscillators. We could metaphorically describe them as a master and slave system. The master system, undergoing chaos, drives a part of the slave system. The slave system had different initial conditions than the master system. And yet, incredibly enough, the two systems soon achieve a perfect synchronization, and maintain it as time marches on. The recent work of Kowalski, Albert, and Gross [4] and Matthews and Strogatz [5] also show similar synchronization in chaotic systems.

Synchronization is a fascinating phenomenon. It has been observed in many diverse systems. The phenomenon of synchronization in chaotic systems may bring many interesting possibilities in practical applications. For example, neural signals in the brain are known to be chaotic; their potential synchronization is certainly worth studying [6–8].

One new area of application would be cryptography. For example, the nondriven parts of the chaotic signals can be used as a pseudorandom sequence to construct *keys,* which can each be used, on a one-time basis, for coding short sequences of a "plaintext" to be transmitted.

In this paper, we are using the metaphor of a master and slave to describe Pecora and Carroll's synchronization, so as to distinguish it from another type of synchronization that we have recently discovered. We have called it the "fracternal synchronization" [9].

There is yet another type of synchronization. It occurs when two systems are not chaotic, but follow periodic limit cycles. While this might not be as fascinating, at first, as chaotic synchronization, it still holds promise of several important practical applications.

One of the central themes of this paper is that of asymptotic stability. We have used this idea to develop the necessary and sufficient condition for synchronization of linear or nonlinear systems. One highly useful technique of exploring asymptotic stability is that of using Lyapunov functions.

In what follows, we have developed an appropriate Lyapunov function and used it to prove the synchronization of the Pecora and Carroll system. From an epistemological perspective, this method is related to the one based on the Lyapunov exponents that Pecora and Carroll have used in their paper.

The Lyapunov function approach has been further used to create a high-dimensional chaotic system, with a nonlinear subsystem. This system shows synchronization, both when it exhibits periodic limit cycles, and also when it turns chaotic.

II. PECORA AND CARROLL'S RESULTS

Pecora and Carroll used the Lorenz equations to create the master system:

$$\dot{x}=\sigma(y-x),\quad \dot{y}=\rho x-y-xz,\quad \dot{z}=xy-\beta z\ . \tag{1}$$

The slave system had an identical set of equations for y and z; however, the initial conditions on y and z were not the same. The slave system did not have an independent equation in x. Instead, the signal $x(t)$ from the master system was used to drive the slave system. Thus the slave system was

$$x'=x,\quad \dot{y}'=\rho x-y'-xz',\quad \dot{z}'=xy'-\beta z' . \tag{2}$$

Note: primes, in this paper, do not indicate differentiation.

It is well known that the above systems undergo chaos for a certain range of parameters. In particular, chaos is observed for these values: $\sigma=10$, $\rho=60$, and $\beta=\frac{8}{3}$. For these values, it was found that in spite of differences in the initial conditions of y and z, the unprimed and primed systems synchronized, so that when $t\rightarrow\infty$, the differences $y-y'$ and $z-z'\rightarrow 0$.

Similar synchronization was observed with a "y" feedback from the master system. It was noticed, however, that the "z" feedback failed to create synchronization.

III. GENERALIZATION

Consider a master system governed by the following set of differential equations:

$$\dot{\mathbf{x}}=\mathbf{f}(t,\mathbf{x}) , \tag{3}$$

where $\mathbf{x}$, $\mathbf{f}$ are the vectors

$$\mathbf{x}=\{x_1(t),\ldots,x_n(t)\}^T,\quad \mathbf{f}=\{f_1(t,\mathbf{x}),\ldots,f_n(t,\mathbf{x})\}^T .$$

In what follows, we will restrict ourselves to those cases in which there exists a unique solution

$$\mathbf{x}=\mathbf{x}(t)=\mathbf{x}(t;t_0,\mathbf{x}_0) , \tag{4}$$

which satisfies Eq. (3) and the initial conditions

$$\mathbf{x}(t_0;t_0,\mathbf{x}_0)=\mathbf{x}_0 . \tag{5}$$

The theorems of existence and uniqueness, which hold for Eq. (3), can be found in Ref. [10].

The generalized setup for synchronization is shown in Fig. 1. The master system governed by Eq. (3) is divided into two interdependent subsystems $\mathbf{u}$ and $\mathbf{v}$. These components are used to create a slave system that also consists of two subsystems $\mathbf{u}'$ and $\mathbf{v}'$, whose functional form is identical to the corresponding master system.

The master system controls the slave system by totally overriding the $\mathbf{u}'$ component. The other component is allowed to have (at least initially) different conditions.

Let the division of the master system be given by

$$\dot{\mathbf{u}}=\mathbf{h}(t,\mathbf{u},\mathbf{v}) \tag{6}$$

and

$$\dot{\mathbf{v}}=\mathbf{g}(t,\mathbf{u},\mathbf{v}) . \tag{7}$$

Further, let the slave system be governed by these two subsystems:

$$\mathbf{u}'=\mathbf{u} \tag{8}$$

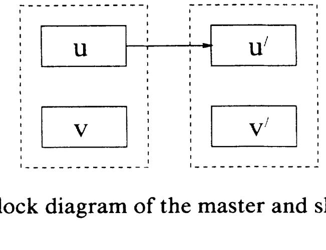

FIG. 1. Block diagram of the master and slave systems.

and

$$\dot{\mathbf{v}}'=\mathbf{g}(t,\mathbf{u},\mathbf{v}') . \tag{9}$$

Equation (9) is quite similar to Eq. (7) for $\mathbf{v}$. However, $\mathbf{v}$ and $\mathbf{v}'$ have different initial conditions. Thus the master system consists of Eqs. (6) and (7); the slave system consists of Eqs. (8) and (9). In the specific case involving Lorenz system, mentioned above,

$$\mathbf{u}'=\mathbf{u}=x=x' , \tag{10}$$

$$\mathbf{v}=\begin{bmatrix} y \\ z \end{bmatrix} , \tag{11}$$

$$\mathbf{v}'=\begin{bmatrix} y' \\ z' \end{bmatrix} , \tag{12}$$

and

$$\mathbf{h}=\sigma(y-x) ,\qquad \mathbf{g}=\begin{bmatrix} \rho x-y-xz \\ xy-\beta z \end{bmatrix} . \tag{13}$$

IV. ASYMPTOTIC STABILITY

Precise definitions of asymptotic stability and synchronization are given in the Appendix. However, for now we would use the term synchronization to denote an eventual coincidence of two different systems, which start with different initial conditions. And asymptotic stability would refer to a condition for a given system, to reach the same eventual state at a fixed, (but sufficiently far enough) time, no matter what the initial conditions were.

Asymptotic stability is most commonly seen in linear damped forced systems. When the transients have died away and only the forced part of the solution remains, we often say that the systems have *forgotten* the initial conditions. This state of forgetting or being totally insensitive to the initial conditions leads to asymptotic stability.

Now, in the case of chaos, asymptotic stability for the total chaotic system would be impossible almost by definition. Chaotic systems very much *remember* the initial conditions. It is this sensitivity, coexisting with their boundedness, that characterizes the chaotic systems.

However, it is quite conceivable that a subsystem of a chaotic system can demonstrate asymptotic stability. If this is the case, we will show that subsystem could play the role of $\mathbf{v}$ and the remaining part can be used as $\mathbf{u}$ so that together they can generate a master-slave system that is synchronous.

It can thus be seen that the two ideas, synchronization and asymptotic stability, are very much related to one another. In fact, in the Appendix we have stated a theorem that shows that this connection is almost inseparable. We have shown that the necessary and sufficient condition for synchronization is that the subsystem that remains undriven ($\mathbf{v}'$ in our notation) must be asymptotically stable for synchronization to take place.

V. LYAPUNOV FUNCTION

One of the practical ways to establish the asymptotic stability of the subsystem is to find a Lyapunov function [11]. Its use can be shown, at first, by considering this function in connection with the subsystem given by Eq. (11).

Subsystem (11) can be written as

$$\dot{\mathbf{v}}=\begin{bmatrix}\dot{y}\\ \dot{z}\end{bmatrix}=\begin{bmatrix}-1 & -x\\ x & -\beta\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}+\begin{bmatrix}\rho x\\ 0\end{bmatrix} . \tag{14}$$

The subsystem for the slave system $\mathbf{v}'$ is also given by the same equation, except that all variables are replaced by their primed counterparts. Now, let us denote the difference between the unprimed and primed quantities by "starring" them. Thus we have

$$\dot{\mathbf{v}}^*=\begin{bmatrix}\dot{y}^*\\ \dot{z}^*\end{bmatrix}=\begin{bmatrix}-1 & -x\\ x & -\beta\end{bmatrix}\begin{bmatrix}y^*\\ z^*\end{bmatrix} . \tag{15}$$

Consider the Lyapunov function given by

$$E=\tfrac{1}{2}(y^{*2}+z^{*2}) . \tag{16}$$

We have

$$\begin{aligned}\dot{E}&=y^*\dot{y}^*+z^*\dot{z}^*\\ &=y^*(-y^*-xz^*)+z^*(xy^*-\beta z^*)\\ &=-y^{*2}-\beta z^{*2}\leq 0 .\end{aligned}$$

The equality sign applies only at the origin, therefore the subsystem (11) is globally asymptotically stable [11]. It follows, therefore, from the theorem (see Appendix) that the master and slave systems synchronize.

VI. SYNTHESIS OF SYNCHRONOUS SYSTEMS

Let us now consider the possibility of synthesizing new synchronous systems using the Lyapunov function approach. We would illustrate this by describing the steps taken to obtain a specific system, given in the next section.

In the case of the Lorenz equation, it can be readily recognized that the subsystem $\mathbf{v}$ is linear. However, the theorem and its applications are not limited to linear systems. In fact, the new system possesses a nonlinear subsystem.

We began by assuming an outline of the system. Thus we had assumed that the number of equations was going to be five. We now wanted to adjust the specific form of $\mathbf{f}(t,\mathbf{x})$ so that we would generate a master system and a slave system that would be synchronous.

We speeded up the trial and error process that followed by taking these steps: First, we made sure that the divergence of the system was negative. This restricted us to dissipative systems. The form of function $\mathbf{f}$ was further guided by the necessity to obtain a nonpositive definite Lyapunov function. The next step was to calculate the rest points of the system, and the corresponding Jacobian and eigenvalues. It was essential to ensure that none of the rest points were stable. It would also have been desirable to ensure that the system under development was bounded. In some cases, this could also be done using a Lyapunov function as shown, for example, by Sparrow [12]. However, in this case, numerical verification was found to be the most expedient course. Once these criteria were satisfied, it was clear that we would obtain either synchronized limit cycles, or chaos.

VII. EXAMPLE OF A SYSTEM WITH A NONLINEAR v

The system we obtained that is related to the Lorenz system was given by

$$\begin{aligned}\dot{x}_1&=\sigma(x_2-x_1)+x_5 ,\\ \dot{x}_2&=\rho x_1-x_2-x_1x_3 ,\\ \dot{x}_3&=x_1x_2-\beta x_3 ,\\ \dot{x}_4&=-x_4^3+x_5 ,\\ \dot{x}_5&=-x_1-x_4-8x_5 .\end{aligned} \tag{17}$$

It can be readily seen that the system has a negative divergence. Further, let the rest points of Eq. (17) be $(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4,\bar{x}_5)$, then the Jacobian matrix of this system is

$$\begin{bmatrix}-\sigma & \sigma & 0 & 0 & 1\\ \rho-\bar{x}_3 & -1 & -\bar{x}_1 & 0 & 0\\ \bar{x}_2 & \bar{x}_1 & -\beta & 0 & 0\\ 0 & 0 & 0 & -3\bar{x}_4^2 & 1\\ -1 & 0 & 0 & -1 & -8\end{bmatrix} .$$

When $\sigma=10$, $\rho=20$, and $\beta=\frac{1}{3}$, the solution of this system is periodic. We get a limit cycle in the phase plane, as shown in Fig. 2. When $\sigma=10$, $\rho=60$, and $\beta=\frac{8}{3}$, the system (17) is a chaotic system, as Fig. 3 shows. The rest points and the corresponding Jacobian eigenvalues are shown in Table I. It can be seen that all rest points are unstable.

The subsystem with x_1 drive is

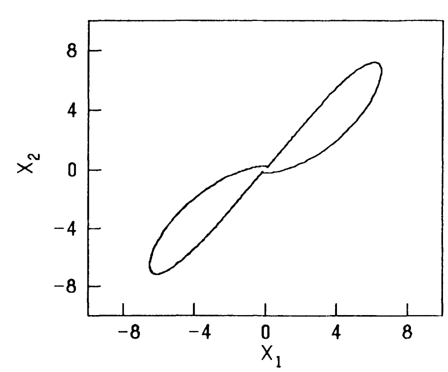

FIG. 2. Limit cycle of system (17) in the phase plane of x_1 via x_2, where $\sigma=10$, $\rho=20$, and $\beta=\frac{1}{3}$.

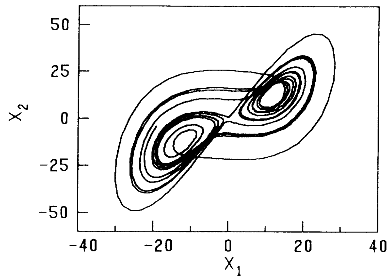

FIG. 3. Chaotic solution of system (17) in the phase plane of x_1 via x_2, where $\sigma=10$, $\rho=60$, and $\beta=\frac{8}{3}$.

$$\begin{aligned} \dot{x}_2' &= \rho x_1 - x_2' - x_1 x_3' , \\ \dot{x}_3' &= x_1 x_2' - \beta x_3' , \\ \dot{x}_4' &= -x_4'^3 + x_5' , \\ \dot{x}_5' &= -x_1 - x_4' - 8x_5' . \end{aligned} \tag{18}$$

In spite of the nonlinearity of the subsystem (18), the synchronization could be observed in both the periodic and chaotic systems, as shown in Figs. 4 and 5. Figures 4 and 5 only show the synchronization of x_2 and x_2'. Meanwhile, the synchronization also exists between the components of x_3 and x_3', x_4 and x_4', and x_5 and x_5'.

Again, let us denote the difference between the primed and unprimed quantities by starring them. Thus we have

$$\begin{aligned} \dot{x}_2^* &= -x_2^* - x_1 x_3^* , \\ \dot{x}_3^* &= x_1 x_2^* - \beta x_3^* , \\ \dot{x}_4^* &= -(x_4'^3 - x_4^3) + x_5^* \\ &= -(x_4^2 + x_4 x_4' + x_4'^2) x_4^* + x_5^* , \\ \dot{x}_5^* &= -x_4^* - 8x_5^* . \end{aligned} \tag{19}$$

We know that

$$\begin{aligned} x_4^2 + x_4 x_4' + x_4'^2 &= (x_4 + x_4')^2 - x_4 x_4' \\ &= (x_4 - x_4')^2 + 3x_4 x_4' . \end{aligned}$$

Therefore, for negative, positive, and null values of the product $x_4 x_4'$,

$$x_4^2 + x_4 x_4' + x_4'^2 \geq 0 .$$

Let $a = x_4^2 + x_4 x_4' + x_4'^2$, then rewrite Eq. (19) as

$$\begin{aligned} \dot{x}_2^* &= -x_2^* - x_1 x_3^* , \\ \dot{x}_3^* &= x_1 x_2^* - \beta x_3^* , \\ \dot{x}_4^* &= -a x_4^* + x_5^* , \\ \dot{x}_5^* &= -x_4^* - 8x_5^* . \end{aligned} \tag{20}$$

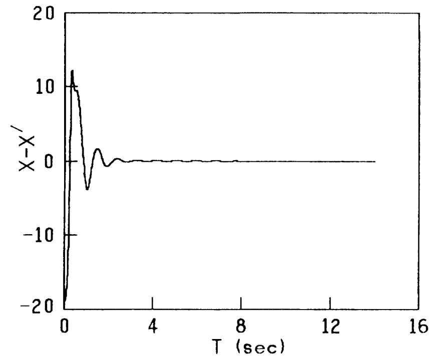

FIG. 4. Synchronization of the periodic system (17) and (18), where $\sigma=10$, $\rho=20$, and $\beta=\frac{1}{3}$. When $t \to +\infty$, $x_2 - x_2' \to 0$.

Consider the Lyapunov function given by

$$E = \tfrac{1}{2}(x_2^{*2} + x_3^{*2} + x_4^{*2} + x_5^{*2}) . \tag{21}$$

We have

$$\begin{aligned} \dot{E} &= x_2^* \dot{x}_2^* + x_3^* \dot{x}_3^* + x_4^* \dot{x}_4^* + x_5^* \dot{x}_5^* \\ &= -x_2^{*2} - \beta x_3^{*2} - a x_4^{*2} - 8x_5^{*2} \leq 0 . \end{aligned}$$

Only at the origin $\dot{E}=0$. Therefore the subsystem (18) is globally asymptotically stable. From the theorem in the Appendix the master system (17) and slave system (18) synchronize. This theorem can be applied for both linear and nonlinear systems and for both autonomous and nonautonomous systems.

VIII. CONCLUSIONS

A criterion based on the asymptotic stability has been developed as a necessary and sufficient condition for the synchronization of periodic and chaotic systems. This criterion is shown to be especially useful if an appropriate Lyapunov function could be found. In the case for the system developed by Pecora and Carroll, such a function has been found and used to prove its synchronization.

This criterion has been further used to create a high-dimensional chaotic system with a nonlinear system. This system shows synchronization, both when it exhibits periodic limit cycles, and also when it turns chaotic. This ability to create synchronous systems could be expected to be of practical use in modeling complex systems and in the case of cryptographic application, mentioned in the introductory part of this paper.

ACKNOWLEDGMENTS

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TABLE I. Rest points and Jacobian eigenvalues of Eq. (17).

Rest points $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)$	Eigenvalues
0,0,0,0,0	-30.38, -7.88, -2.67, -0.13, 19.39
-12.471, -12.613, 58.989, 1.123, 1.418	-15.02, -7.85, -4.03, $0.73 \pm 14.37i$
12.471, 12.613, 58.989, -1.123, -1.418	-15.02, -7.85, -4.03, $0.73 \pm 14.37i$

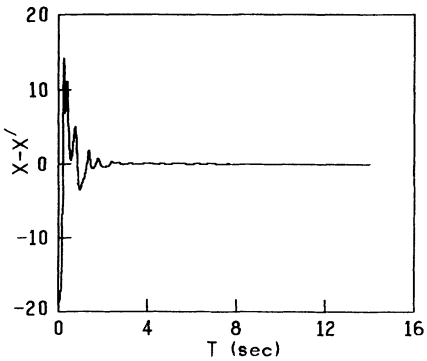

FIG. 5. Synchronization of the chaotic system (17) and (18), where $\sigma=10$, $\rho=60$, and $\beta=\frac{8}{3}$. When $t\to+\infty$, $x_2-x_2'\to 0$.

S. Manoranjan, Department of Mathematics, W.S.U., for helpful discussions.

APPENDIX: NECESSARY AND SUFFICIENT CONDITION OF SYNCHRONIZATION

Before analyzing the necessary and sufficient condition of synchronization, the general background of stability definition is briefly reviewed now. Consider a system of n first-order ordinary differential equations

$$\dot{\mathbf{x}}=\mathbf{f}(t,\mathbf{x}) , \tag{A1}$$

where $\dot{\mathbf{x}}=d\mathbf{x}/dt$ and $\mathbf{x}$, $\mathbf{f}$ are the vectors:

$$\mathbf{x}=(x_1(t),\ldots,x_n(t)),\quad \mathbf{f}=(f_1(t,\mathbf{x}),\ldots,f_n(t,\mathbf{x})) .$$

Let us denote by $\|\mathbf{u}\|$ the *norm* of any vector $\mathbf{u}$. If t_0, $\mathbf{x}_0=(x_1(t_0),\ldots,x_n(t_0))$, $b>0$ are given, we define sets S (tubes) by $S=[t_0\le t\le+\infty, \|\mathbf{x}-\mathbf{x}\|\le b]$.

If $\mathbf{f}(t,\mathbf{x})$ satisfies the Lipschitz condition $|\mathbf{f}(t,\mathbf{x}_1)-\mathbf{f}(t,\mathbf{x}_2)|\le K|\mathbf{x}_1-\mathbf{x}_2|$ for all $(t,\mathbf{x}_1),(t,\mathbf{x}_2)\in S$, and some constant $K>0$, then the solution of Eq. (A1), represented by $\mathbf{x}(t;t_0,\mathbf{x}_0)$, is uniquely determined by the initial condition $\mathbf{x}(t_0)=\mathbf{x}_0$ [10]. Furthermore, $\mathbf{x}(t;t_0,\mathbf{x}_0)$ is said to be *stable* if (i) there exists a $b_1>0$, such that every solution $\mathbf{x}(t;t_0,\mathbf{x}_1)$ exists in $t_0\le t\le+\infty$ and $[t,\mathbf{x}(t)]\in S$ for all $t\ge t_0$ whenever the initial vector $\mathbf{x}_1$ satisfies $\|\mathbf{x}_1-\mathbf{x}_0\|\le b_1$; and (ii) given $\epsilon>0$, there is a $\delta=\delta(\epsilon;\mathbf{f},\mathbf{x}_0)$, $0<\delta\le b_1$, such that $\|\mathbf{x}_1-\mathbf{x}_0\|\le\delta$ implies $\|\mathbf{x}(t;t_0,\mathbf{x}_1)-\mathbf{x}(t;t_0,\mathbf{x}_0)\|\le\epsilon$ for all $t_0\le t<+\infty$. The same solution $\mathbf{x}(t;t_0,\mathbf{x}_0)$ is said to be *asymptotically stable* [10] if (i) and (ii) hold, and (iii) there exists a $\delta=\delta(\mathbf{f},\mathbf{x}_0)$, $0<\delta\le b_1$, such that $\|\mathbf{x}_1-\mathbf{x}_0\|\le\delta$ implies

$$\|\mathbf{x}(t;t_0,\mathbf{x}_1)-\mathbf{x}(t;t_0,\mathbf{x}_0)\|\to 0 \quad\text{as } t\to+\infty .$$

In this paper, if there exists a subset of R^n called $D(t_0)$ such that for initial conditions $\mathbf{x}_0\in D(t_0)$, the solutions of Eq. (A1) are asymptotically stable, then we call $D(t_0)$ as the region of asymptotic stability.

We say that the solution $\mathbf{x}(t;t_0,\mathbf{x}_0)$ is *globally asymptotically stable* if the region is the entire R^n. In order to analyze synchronization, we should define synchronization mathematically. In this paper, we give the following definition of synchronization.

Definition. Consider two dynamic systems $\dot{\mathbf{x}}=\mathbf{F}(t,\mathbf{x})$ and $\dot{\mathbf{x}}'=\mathbf{F}'(t,\mathbf{x}')$, where t is time and $\mathbf{x},\mathbf{x}'\in\mathbb{R}^n$. Let the solutions of the systems be given by $\mathbf{x}(t;t_0,\mathbf{x}_0)$ and $\mathbf{x}'(t;t_0,\mathbf{x}_0')$, respectively. We say that $\mathbf{F}(t,\mathbf{x})$ *synchronizes* with $\mathbf{F}'(t,\mathbf{x}')$ if there exists a subset of R^n, denoted by $D(t_0)$, such that $\mathbf{x}_0,\mathbf{x}_0'\in D(t_0)$ implies

$$\|\mathbf{x}(t;t_0,\mathbf{x}_0)-\mathbf{x}'(t;t_0,\mathbf{x}_0')\|\to 0 \quad\text{as } t\to+\infty .$$

The synchronization is defined as *global* if $D(t_0)$ spans the whole space, i.e., $D(t_0)=R^n$. It is defined as *local* if $D(t_0)$ is a proper subset of R^n. Also we can call $D(t_0)$ as the *region* of synchronization.

As shown in Sec. III, the system (A1) is subdivided into two subsystems: $\mathbf{u}$ and $\mathbf{v}$. Using these subsystems, a master and a slave system are created. The master system simply consists of these two interacting subsystems. The slave system is quite similar, except that a part of it is totally driven by the master system. We write them again.

The master system is given by

$$\dot{\mathbf{u}}=\mathbf{h}(t,\mathbf{u},\mathbf{v}),\quad \dot{\mathbf{v}}=\mathbf{g}(t,\mathbf{u},\mathbf{v}) , \tag{A2}$$

and the slave system is given by

$$\mathbf{u}'=\mathbf{u},\quad \dot{\mathbf{v}}'=\mathbf{g}(t,\mathbf{u},\mathbf{v}') . \tag{A3}$$

Now we have the following theorem.

Theorem. If the solutions of (A2) and (A3) are in sets (tubes) S, and $\mathbf{h}$ and $\mathbf{g}$ satisfy the Lipschitz condition in S, the slave system (A3) synchronizes with the master system (A2) if and only if there exists a subset $D(t_0)\subset\mathbb{R}^n$ such that when the initial conditions of the nondriven part of the slave system $\dot{\mathbf{v}}'=\mathbf{g}(t,\mathbf{u},\mathbf{v}')$ fall in $D(t_0)$, the solutions of $\mathbf{v}'$ are asymptotically stable.

Proof. If $\mathbf{h}$ and $\mathbf{g}$ satisfy the Lipschitz condition in sets S, the solutions of Eq. (A2), then Eq. (A3), are uniquely determined by the initial conditions. They can be written as

$$\mathbf{u}(t)=\mathbf{u}(t,\mathbf{v};t_0,\mathbf{u}_0,\mathbf{v}_0) , \tag{A4}$$

$$\mathbf{v}(t)=\mathbf{v}(t;\mathbf{u};t_0,\mathbf{u}_0,\mathbf{v}_0) , \tag{A5}$$

$$\mathbf{v}'(t)=\mathbf{v}'(t,\mathbf{u};t_0,\mathbf{u}_0,\mathbf{v}_0') . \tag{A6}$$

Let the initial conditions be

$$\mathbf{u}(t_0)=\mathbf{u}_0 , \tag{A7}$$

$$\mathbf{v}(t_0)=\mathbf{v}_0 , \tag{A8}$$

$$\mathbf{v}'(t_0)=\mathbf{v}_0' . \tag{A9}$$

Since the structures of $\dot{\mathbf{v}}'=\mathbf{g}(t,\mathbf{u},\mathbf{v}')$ and $\dot{\mathbf{v}}=\mathbf{g}(t,\mathbf{u},\mathbf{v})$ are exactly the same, for the rest of this section the crucial parameters are these initial conditions. As a first step, it follows from the uniqueness that were we to use for the slave subsystem $\mathbf{v}'$ the same initial conditions as for the master subsystem $\mathbf{v}$, i.e., $\mathbf{v}_0'=\mathbf{v}_0$, we would get the same solutions:

$$\mathbf{v}'(t,\mathbf{u};t_0,\mathbf{u}_0,\mathbf{v}_0')=\mathbf{v}'(t,\mathbf{u};t_0,\mathbf{u}_0,\mathbf{v}_0)=\mathbf{v}(t,\mathbf{u};t_0,\mathbf{u}_0,\mathbf{v}_0) . \tag{A10}$$

In general, $\mathbf{v}_0' \neq \mathbf{v}_0$, so the two solutions could differ in the beginning.

Sufficient condition. If there exists a subset $D(t_0) \subset \mathbb{R}^n$ such that when the initial conditions of the nondriven part of the slave system $\dot{\mathbf{v}}' = \mathbf{g}(t, \mathbf{u}, \mathbf{v}')$ fall in $D(t_0)$, the solutions of $\mathbf{v}'$ are asymptotically stable, then, by the definition of the asymptotic stability previously mentioned, we obtain that when $t \to +\infty$,

$$\|\mathbf{v}'(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0') - \mathbf{v}'(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0)\| \to 0 . \quad \text{(A11)}$$

It follows from Eqs. (A10) and (A11) together that

$$\|\mathbf{v}'(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0') - \mathbf{v}(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0)\| \to 0 \quad \text{as } t \to +\infty .$$

Therefore, by the definition of synchronization given in this paper, the master and slave systems are synchronized.

Necessary condition. Suppose there exists a subset $D(t_0) \subset \mathbb{R}^n$ such that when the initial conditions of the nondriven part of the slave system $\dot{\mathbf{v}}' = \mathbf{g}(t, \mathbf{u}, \mathbf{v}')$ fall in $D(t_0)$, the master and slave systems are synchronized. By the definition of synchronization outlined in this paper, we have

$$\|\mathbf{v}'(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0') - \mathbf{v}(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0)\| \to 0 \quad \text{as } t \to +\infty . \quad \text{(A12)}$$

Suppose that when the initial conditions of the nondriven part of the slave system $\dot{\mathbf{v}}' = \mathbf{g}(t, \mathbf{u}, \mathbf{v}')$ fall in $D(t_0)$ the solutions of $\mathbf{v}'$ are not asymptotically stable. Then, by the definition of asymptotic stability,

$$\|\mathbf{v}'(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0') - \mathbf{v}'(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0)\| \not\to 0 \quad \text{as } t \to +\infty . \quad \text{(A13)}$$

Because of the uniqueness of the solutions,

$$\mathbf{v}'(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0) = \mathbf{v}(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0) . \quad \text{(A14)}$$

Therefore,

$$\|\mathbf{v}'(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0') - \mathbf{v}(t, \mathbf{u}; t_0, \mathbf{u}_0, \mathbf{v}_0)\| \not\to 0 \quad \text{as } t \to +\infty .$$

This result is in contradiction with Eq. (A12). So we can conclude that when the initial conditions of the nondriven part of the slave system $\dot{\mathbf{v}}' = \mathbf{g}(t, \mathbf{u}, \mathbf{v}')$, fall in $D(t_0)$, the solutions of $\mathbf{v}'$ must be asymptotically stable.

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*Electronic address: vaidya@wsuvm1.csc.wsu.edu Fax: (509)-335-4662.

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