Conservation laws in superfluorescence

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Conservation laws for Maxwell-Bloch equations applied to describe superfluorescence are studied. Modifications due to the initial boundary conditions are discussed. For initial polarization in the form of a constant tipping angle, conservation laws are integrated to give exact dependence of all field moments on the propagation length. Three regions of propagation are distinguished: the linear region, where the field energy is negligibly small, the region of rapid exponential growth, and the saturation region. The characteristic distance, where maximum radiation occurs, is explicitly evaluated; it defines the threshold length for superfluoresence. A small chirping, proportional to the square of the tipping angle, is found. The change of the pulse width with propagation length is estimated.

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I. INTRODUCTION

Superfluorescence (SF) is one of the collective phenomena of quantum optics. It can be described rather simply. A cylindrical sample filled with a large number of two-level atoms, all initially excited, radiates spontaneously one or several pulses of intense electromagnetic (e.m.) radiation. The radiation is characterized by several parameters, such as, e.g., the delay of its first maximum, width of the pulse, and energy. Since its prediction by Dicke [1] in 1954 SF has been the subject of several experiments [2] and of a large number of theoretical papers, see e.g., [3].

According to the present theory, SF proceeds in two stages [4,5]. At the early stage, spontaneous emission takes place and a small polarization is built from quantum fluctuations. In this process quantum character of the field is essential, but equations may be linearized. The effect of quantum fluctuations leads to random initial polarization with Gaussian probability distribution.

During the second stage the e.m. field evolves and is amplified to form a strong pulse of radiation. This process is semiclassical and causal. It is described by the system of Maxwell-Bloch (MB) equations. Nonlinearity and inhomogeneous broadening of the atomic line are essential at this stage. The present paper deals only with the second part of the problem. Statistical properties of the field will not be considered.

In the majority of papers propagation has been treated in an approximate way. Equations were linearized and (or) sharp-line approximation was used. In the sharp-line limit exact self-similar solutions were studied [6]. Some information on SF pulses such as, e.g., the statistics of the delay time have been found on the basis of the linear theory [5]. A satisfactory description of pulse formation and its shape can, however, be given only by nonlinear equations.

Maxwell-Bloch equations belong to the class of equations integrable by the inverse-scattering method (ISM) [7,8]. This method describes perfectly propagation of e.m. pulses in absorbing media [9–11]. Its application to SF [12,13] is not straightforward because the initial boundary conditions here are different. In the case of SF the initial values for scattering data are trivial. Their evolution with distance along the sample is essential. Gabitov, Zakharov, and Mikhailov [14] generalized the evolution equations for the MB system to the case of SF. They considered pulses on the whole time axis. Steudel [15] modified these equations for the scattering on the half-time axis. These equations are basic for the SF process. When the scattering data are known it remains to solve the inverse problem. This is difficult because of the presence of a continuous spectrum together with the discrete spectrum. Steudel [15] solved the problem asymptotically for large time and distance of propagation. This solution is based on the linearized scattering problem and basically applies to the sharp-line case. Inhomogeneous broadening appears as a small correction. These approximations are the reason for the self-similar character of the solution which has the form of a secant hyperbolic with delay time related to random initial polarization. It resembles the solution found from the sine-Gordon equation by Gabitov, Zakharov, and Mikhailov [6].

Self-similar solutions predict that the shape of the pulse is independent of the sample length. The ISM can give information on the field evolution inside the sample and, consequently, more precise description of SF pulses. As a first step of such a study I consider, in this paper, the conservation laws for the MB equations applied to SF. Under the assumption of a constant tipping angle as initial condition, the exact form of the dependence of field energy and higher moments on the distance of propagation is found. It is shown that there are three stages of amplification: (1) very slow linear growth, proportional to initial polarization (2) rapid exponential growth becoming independent of initial conditions, and (3) linear growth due to saturation. The change from first to second stage is abrupt and the corresponding value of x may be interpreted as the threshold value for the sample length.

In Sec. II the problem is formulated and evolution equations are derived. This section is based on the work of Gabitov, Zakharov, and Mikhailov [14] and Steudel [15]. In Sec. III A conservation laws for SF are derived

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and modifications introduced by initial conditions are discussed. In Sec. III B they are integrated to give the exact expression for pulse energy and first spectral moment. Finally, in Sec. III C the contribution of discrete, soliton-like eigenvalues to the radiation is discussed.

In conclusion, the validity of certain approximations used in SF theory is discussed on the basis of the results for energy propagation.

II. MAXWELL-BLOCH EQUATIONS AND THE INVERSE-SCATTERING PROBLEM

Consider a cylindrical sample of length L and diameter d filled with two-level atoms of density n. The wavelength of the atomic transition is $\lambda=c/\omega_0$. In the one-dimensional model of SF it is assumed that the Fresnel number $F=d^2/L\lambda$ is close to 1. Therefore L is related to the total number of atoms $\mathcal{N}=n\lambda L^2$. In the following, arbitrary L (or $\mathcal{N}$) will be considered and a fixed value of n. Interaction of the e.m. field with two-level atoms in the slowly varying envelope approximation is described by the MB equations. These equations depend on one characteristic frequency $\nu_c=[2\pi n\mathcal{P}^2\omega_0/\hbar]^{1/2}$, where $\mathcal{P}$ denotes the transition matrix element. In dimensionless notation, MB equations take the form

$$E_x=\langle\lambda\rangle\ , \tag{2.1}$$

$$\lambda_t=-2i\omega\lambda+NE\ , \tag{2.2}$$

$$N_t=-\tfrac{1}{2}(E\lambda^*+E^*\lambda)\ , \tag{2.3}$$

where t is the retarded time, $E(t,x)$, $\lambda(t,x,\omega)$, and $N(t,x,\omega)$ denote, respectively, the field envelope, complex polarization, and population inversion in the medium. All quantities are dimensionless, scaled by $\tau_c=\nu_c^{-1}$ or $c\tau_c$. Angle brackets denote average over the inhomogeneously broadened atomic line

$$\langle\lambda\rangle=\int_{-\infty}^{\infty}\lambda(\omega)g(\omega)d\omega \tag{2.4}$$

with the Gaussian distribution

$$g(\omega)=\frac{T}{\sqrt{\pi}}e^{-\omega^2T^2}\ , \tag{2.5}$$

where 2ω is the difference between the atomic frequency and ω_0. The initial boundary conditions for SF read

$$E(t,0)=E(0,x)=0\ , \tag{2.6}$$

$$\lambda(0,x,\omega)=\lambda^0\ ,\quad N(0,x,\omega)=N^0\ . \tag{2.7}$$

In general λ^0 depends on x and ω and it is a random function.

In the ISM a system of equations equivalent to Eqs. (2.1)–(2.3) is used. Let Φ denote a 2×2 matrix function of t, x, and ζ satisfying the equations

$$\mathbf{\Psi}_t+\mathbf{U\Psi}=0\ , \tag{2.8}$$

$$\mathbf{\Psi}_x+\mathbf{V\Psi}=0\ , \tag{2.9}$$

where

$$\mathbf{U}=i\zeta\sigma_3+\tfrac{1}{2}(E\sigma_+-E^*\sigma_-)\ , \tag{2.10}$$

$$\mathbf{V}=\frac{i}{4}\int_C\frac{1}{\alpha-\zeta}\rho(\alpha)g(\alpha)d\alpha\ ,\quad \rho=N\sigma_3+\lambda\sigma_++\lambda^*\sigma_-\ , \tag{2.11}$$

C denotes the contour of integration from $\alpha=-\infty$ to $\alpha=+\infty$ indenting under the pole $\alpha=\zeta$, $\sigma_i (i=1,2,3)$ are Pauli matrices, and $\sigma_\pm=\frac{1}{2}(\sigma_1\pm i\sigma_2)$. The compatibility condition of Eqs. (2.8) and (2.9) requires that

$$\mathbf{U}_x-\mathbf{V}_t+[\mathbf{U},\mathbf{V}]=0\ , \tag{2.12}$$

which is equivalent to Eqs. (2.1)–(2.3). Matrices **U** and **V** are the Lax pair for MB equations.

The scattering problem on the time axis is given by Eq. (2.8) where E is the "potential." Following Steudel [15] let us define the Jost functions on the half t axis as solutions of (2.8), satisfying

$$\chi(t=0)=\mathbf{I}\ ,\quad \chi^+\to\exp(-i\zeta\sigma_3t)\ \text{ as } t\to\infty\ . \tag{2.13}$$

The scattering matrix $\mathbf{S}(x,\zeta)$ is defined by the relation

$$\chi^+\mathbf{S}=\chi^0\ , \tag{2.14}$$

S may be written in the form

$$\mathbf{S}=\begin{bmatrix} a & -\bar{b} \\ b & \bar{a}\end{bmatrix}\ , \tag{2.15}$$

where $a\bar{a}+b\bar{b}=1$. At $x=0$, $E=0$, hence

$$a(x=0,\zeta)=1\ ,\quad b(x=0,\zeta)=0\ . \tag{2.16}$$

Function Ψ which satisfies (2.8) and (2.9) can be written as a superposition of the Jost functions

$$\mathbf{\Psi}=\chi^0\phi^0=\chi^+\phi^+\ , \tag{2.17}$$

where ϕ^0 and ϕ^+ are matrix functions of x and ζ. The **S** matrix can now be written as

$$\mathbf{S}=\phi^+(\phi^0)^{-1}\ . \tag{2.18}$$

To find the x dependence of **S** one uses Eq. (2.9) at $t=0$ and $t\to\infty$. As $t\to\infty$, $\lambda\to\lambda^+\exp(-2i\zeta t)$, while

$$\rho\to\exp(-i\zeta\sigma_3t)\rho^+\exp(i\zeta\sigma_3t)\ , \tag{2.19}$$

where ρ^+ is independent of time. Multiplying Eq. (2.9) by $\exp(i\sigma_3\zeta t)$ and taking the limit $t\to\infty$ one gets

$$\phi_x^+=\mathbf{R}^+\phi^+\ , \tag{2.20}$$

where

$$\mathbf{R}^+=\frac{i}{4}\lim_{t\to\infty}\left\{\int_C\frac{1}{\zeta-\alpha}\exp[-i\sigma_3(\alpha-\zeta)t]\times\rho^+\exp[i\sigma_3(\alpha-\zeta)t]g(\alpha)d\alpha\right\}\ . \tag{2.21}$$

From Eq. (2.21) one finds

$$R_{12}^+=0\ ,\quad R_{21}^+=\frac{\pi}{2}\lambda^+g(\zeta)\ ,\quad R_{11}^+=-R_{22}^+=\tilde{N}^+\ .$$

At $t=0$,

$$\phi_x^0 = \mathbf{R}^0 \phi^0 , \tag{2.22}$$

where

$$\mathbf{R}^0 = \frac{i}{4} \int_C \frac{1}{\zeta - \alpha} \rho^0(\alpha) g(\alpha) d\alpha . \tag{2.23}$$

From Eqs. (2.18), (2.20), and (2.22) it follows that

$$\mathbf{S}_x = \mathbf{R}^+ \mathbf{S} - \mathbf{S} \mathbf{R}^0 . \tag{2.24}$$

In particular,

$$a_x = a(\tilde{N}^+ - \tilde{N}^0) - \bar{b} \tilde{\lambda}^{0*} , \tag{2.25}$$

$$\bar{b}_x = a \tilde{\lambda}^0 + \bar{b}(\tilde{N}^0 + \tilde{N}^+) , \tag{2.26}$$

where operation $\sim$ is defined by

$$\tilde{f} = \frac{i}{4} \int_C \frac{1}{\zeta - \alpha} f(\alpha) g(\alpha) d\alpha . \tag{2.27}$$

Equations (2.25) and (2.26) form a closed system of equations for $a(x,\zeta)$ and $\bar{b}(x,\zeta)$ with analytical coefficients in the upper half plane (UHP) of ζ. The scattering amplitude $c=\bar{b}/a$ satisfies the following equation [15]:

$$c_x = \tilde{\lambda}^0 + 2c\tilde{N}^0 - c^2 \tilde{\lambda}^{0*} , \tag{2.28}$$

where the coefficients depend only on the initial conditions. Note that in the sharp-line limit this equation becomes singular.

The system (2.25) and (2.26) is highly nonlinear. This can be seen when N^+ and λ^+ are expressed in terms of a and $\bar{b}$. Let us observe that Eqs. (2.2) and (2.3) may be written as

$$\rho_t = [\mathbf{U}, \rho] , \tag{2.29}$$

where

$$\rho = \Psi \rho^0 \Psi^{-1} . \tag{2.30}$$

This relation shows that the scattering problem (2.8) constitutes a part of MB dynamics. Making use of (2.20) and (2.30) one obtains

$$\rho^+ = \mathbf{S} \rho^0 \mathbf{S}^{-1} , \tag{2.31}$$

provided $[\rho^0, \phi^0]=0$. In particular,

$$N^+ = N^0(a\bar{a} - b\bar{b}) - \bar{a}\bar{b}\lambda^{0*} - ab\lambda^0 , \tag{2.32}$$

$$\lambda^+ = 2a\bar{b}N^0 + a^2\lambda^0 - \bar{b}^2 \lambda^{0*} . \tag{2.33}$$

In the next section N^+ will be expressed in terms of c and explicitly evaluated.

III. CONSERVATION LAWS

An infinite set of conservation laws for MB equations has been derived by Lamb [7] and applied to self-induced transparency [16–19]. It has been shown that solitons are responsible for lossless propagation. In the conservation laws constant discrete values related to solitons contribute to the field spectral moments. The continuous spectrum plays a minor role decaying with growing propagation length. In the case of SF analogous conservation laws can be derived. Modifications appear which are due to the initial condition at $t=0$ instead of at $t=-\infty$, and the fact that discrete, x-dependent eigenvalues do not correspond to stable soliton solutions.

A. General form of conservation laws

Consider the first pair of Eq. (2.8),

$$\Psi_{11}, t = -i\zeta\Psi_{11} - \tfrac{1}{2} E \Psi_{21} , \tag{3.1}$$

$$\Psi_{21}, t = i\zeta\Psi_{21} + \tfrac{1}{2} E^* \Psi_{11} , \tag{3.2}$$

and define

$$\mu = \ln(e^{i\zeta t}\Psi_{11}/\phi_{11}^0) . \tag{3.3}$$

Using Eqs. (2.17) and (2.18) one finds the limiting values of μ,

$$\mu(t=0)=0 ,$$

$$\mu \to \ln[a(1-cf)] \quad \text{as } t\to\infty , \tag{3.4}$$

where

$$f(x) = \phi_{21}^0/\phi_{11}^0 \tag{3.5}$$

is uniquely defined by Eq. (2.22) (see Appendixes A and B).

By elimination of Ψ_{21} from Eqs. (3.1) and (3.2) a Ricatti equation for μ_t is found:

$$2i\zeta\mu_t = \mu_t^2 - \tfrac{1}{4}|E|^2 + \tfrac{1}{2} E \left[\frac{\mu_t}{E} \right]_t . \tag{3.6}$$

Following the Zakharov and Shabat method [20] it can be shown that asymptotically, for large ζ,

$$\mu_t \sim -\frac{i}{2} \sum_{n=0}^{\infty} \frac{C_n}{(2\zeta)^{n+1}} , \tag{3.7}$$

where C_m depend only on the field and are determined by a recurrence formula resulting from Eq. (3.6). For $n=0$ and 1 one gets

$$C_0 = \tfrac{1}{2}|E|^2 , \quad C_1 = \frac{i}{2} E E_t^* . \tag{3.8}$$

On the other hand, from the first pair of Eq. (2.9),

$$\Psi_{11,x} = \tilde{N}\Psi_{11} + \tilde{\lambda}\Psi_{21} , \tag{3.9}$$

$$\Psi_{21,x} = \tilde{\lambda}^* \Psi_{11} - \tilde{N}\Psi_{21} , \tag{3.10}$$

by elimination of Ψ_{21}, one obtains

$$\partial_x(\mu_t) = \partial_t(\tilde{N} - 2\tilde{\lambda}\mu_t/E) . \tag{3.11}$$

Integrating this equation over t, one finds, asymptotically

$$\partial_x F_n = (M_n^0 - M_n^+) - \Lambda_n , \tag{3.12}$$

where

$$F_n = \int_0^\infty C_n dt \tag{3.13}$$

denote the field "moments." M_n^0 and M_n^+ are moments of the population inversion N,

$$M_n(x,t)=\int_{-\infty}^{\infty}\alpha^n(t,x,\alpha)g(\alpha)d\alpha\ , \tag{3.14}$$

taken, respectively, at $t=0$ and $+\infty$. Λ_n denote the contribution from the second term of Eq. (3.11),

$$\tfrac{1}{2}\lim_{t\to 0}\tilde{\lambda}\mu_t/E=\Lambda\ . \tag{3.15}$$

From the definition (3.4) of μ one gets

$$\Lambda=\tfrac{1}{2}\tilde{\lambda}^0 f\ . \tag{3.16}$$

Asymptotically,

$$\Lambda\sim\sum_{n=0}^{\infty}\frac{\Lambda_n}{(2\zeta)^{n+1}}\ . \tag{3.17}$$

It is worth noticing that the set of moments (3.19) is well defined only for finite inhomogeneous broadening time T and Gaussian shape of the atomic line.

B. Exact solution for the field energy

Ricatti equation (2.28) for the scattering amplitude c can be easily solved when λ^0 is independent of x. Assuming constant initial polarization $\lambda^0=\sin\Theta$, $N^0=\cos\Theta$ one finds

$$\tilde{\lambda}^0=\frac{i}{4}\sin\Theta\int_C\frac{1}{\zeta-\alpha}g(\alpha)d\alpha=\tfrac{1}{2}W\sin\Theta\ , \tag{3.18}$$

$$\tilde{N}^0=\frac{i}{4}\cos\Theta\int_C\frac{1}{\zeta-\alpha}g(\alpha)d\alpha=\tfrac{1}{2}W\cos\Theta\ , \tag{3.19}$$

where $W=\frac{1}{2}\sqrt{\pi}Tw(\zeta T)$, $\mathrm{Im}\zeta>0$, and w denotes the probability error function [21]

$$w(z)=\frac{i}{\pi}\int_C\frac{e^{-t^2}}{z-t}dt\ . \tag{3.20}$$

Equation (2.28) can now be written in the form

$$c_x=\tfrac{1}{2}W(\sin\Theta+2\cos\Theta c-\sin\Theta c^2)\ . \tag{3.21}$$

Its solution, vanishing at $x=0$, is

$$c=\sqrt{\gamma}\frac{e^{Wx}-1}{1+\gamma e^{Wx}}\ , \tag{3.22}$$

where

$$\gamma=\frac{1-\cos\Theta}{1+\cos\Theta}\ .$$

The inversion N^+ given by Eq. (2.32) can now be written in the form

$$N^+=\cos\Theta(|a|^2-|b|^2)-\sin\Theta(ab+a^*b^*)\ . \tag{3.23}$$

On the real axis, $\mathrm{Re}\zeta=k$, $\bar{a}=a^*$, $\bar{b}=b^*$, and

$$|a|^2-|b|^2=1-2|c|^2(1+|c|^2)^{-1}\ , \tag{3.24}$$

$$ab+a^*b^*=(c+c^*)(1+|c|^2)^{-1}\ . \tag{3.25}$$

Finally, Eq. (3.23) takes the form

$$N^0-N^+=[2\cos\Theta|c|^2+\sin\Theta(c+c^*)](1+|c|^2)^{-1}\ . \tag{3.26}$$

By introducing Eq. (3.22) into (3.26) one obtains, after some manipulation,

$$N^0-N^+=\frac{2\gamma}{1+\gamma}\frac{e^{\Omega x}-1}{1+\gamma e^{\Omega x}}\ , \tag{3.27}$$

where

$$\Omega=2\,\mathrm{Re}W=\pi g(kT)\ . \tag{3.28}$$

N^0-N^+ has the meaning of energy transmitted to the field per unit length. Note that its dependence on Ω is exactly the same as that of c on the complex function W. The point x_m at which $N^+=0$ is characterized by maximum radiation, $\lambda^+(x_m)=1$. From (3.26) one finds

$$x_m=\Omega^{-1}\ln(\gamma^{-1})\ . \tag{3.29}$$

It corresponds to the half-deexcited state in the Dicke model of SF [1], and may be interpreted as the threshold length for SF. In the limit of the sharp atomic line x_m tends to zero. Equation (3.27) for $k=0$ is depicted in Fig. 1. Figure 2 shows the peak value of N^0-N^+ for different values of γ.

To evaluate Λ_n one has to solve Eq. (2.22). This is easily done for the constant initial polarization (see Appendix A). The result is

$$f(x)=\frac{\sinh Wx}{\cosh(Wx+\delta)}\ , \tag{3.30}$$

where $\delta=\frac{1}{2}\ln(\gamma^{-1})$. Introducing (3.30) into (3.16) and making use of the expansion of the probability error function for large argument

$$w(z)=\frac{i}{\sqrt{\pi}}\frac{1}{z}\left[1+\frac{1}{2z^2}+\cdots\right] \tag{3.31}$$

one finds the moments of Λ:

$$\Lambda_0=0\ , \tag{3.32}$$

$$\Lambda_1=-\tfrac{1}{8}(\sin\Theta)^2 x\ . \tag{3.33}$$

Integrating Eq. (3.12) for $n=0$ over x one finds from (3.27) the field energy

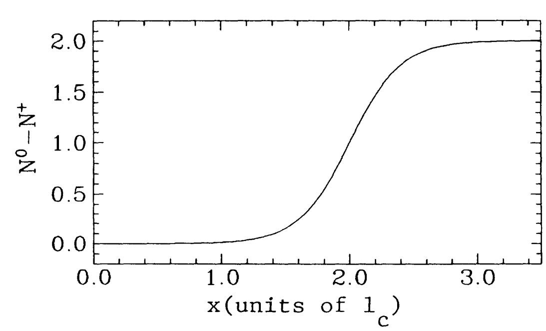

FIG. 1. Plot of Eq. (3.27) for $\ln(\gamma^{-1})=20$ showing atomic energy transmitted to the field as a function of distance measured in $l_c=5c\tau_c^2/\sqrt{\pi}T$.

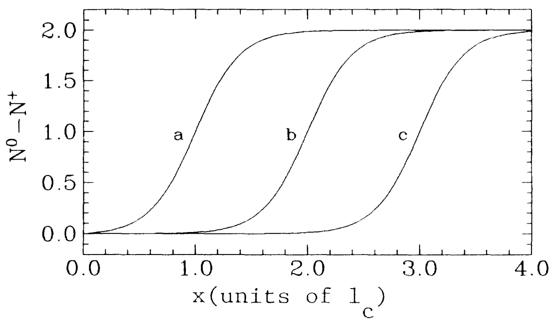

FIG. 2. Same as Fig. 1 for different values of γ. Curves a, b, and c correspond to $\ln(\gamma^{-1})=10$, 20, and 30, respectively. Note that in the region of maximum growth the shape of N^0-N^+ does not depend on γ.

$$F_0=\frac{\pi}{2}\int_{-\infty}^{\infty}\ln\left[\frac{1+\gamma e^{\Omega x}}{1+\gamma}-\frac{\gamma}{1+\gamma}\Omega x\right]dk\ . \qquad (3.34)$$

The function

$$\mu(x,k)=\ln\left[\frac{1+\gamma e^{\Omega x}}{1+\gamma}-\frac{\gamma}{1+\gamma}\Omega x\right] \qquad (3.35)$$

may be interpreted as the spectral density of the field. Figure 3 shows the peak value of μ as a function of x measured in units $l_c=5c\tau_c^2/\sqrt{\pi}T$. In the linear region, $x<l_c$ radiation is negligible. Amplification starts at $x_0\approx 2l_c$, for $2l_c<x<3l_c$ μ grows rapidly, and for $x>3l_c$ the growth is linear. This region corresponds to saturation of N^0-N^+. For the parameters of the cesium experiment [2] $l_c\approx 10^{-1}$ cm and all stages should occur inside the sample of length $L=2$ cm.

The shape of F_0 as a function of x is similar to that for μ. This is due to the scaling property

$$\mu(x,k)=\mu(xe^{-k^2T^2})\ . \qquad (3.36)$$

This property may be used to investigate the spectral width of μ. Taking the inverse function $y(\mu)$, $y=xe^{-k^2T^2}$, one may define the spectral width, at different x, as

$$k_R=T^{-1}\{\ln[y(\mu)/y(\tfrac{1}{2}\mu)]\}^{1/2}\ ,\quad 0<y<x\ . \qquad (3.37)$$

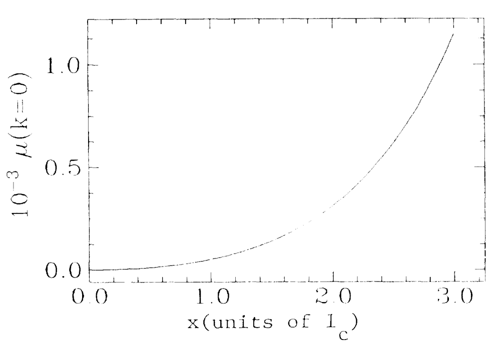

FIG. 3. Plot of Eq. (3.35) for $k=0$ showing the peak value of the energy density μ as a function of distance measured in $l_c=5c\tau_c^2/\sqrt{\pi}T$; $\ln(\gamma^{-1})=20$.

The study of k_R as a function of x shows that k_R decreases in the region $0<x<x_m$, gets a minimum value for $x\approx x_m$, and grows in the saturation region to reach the asymptotic value $k_R=(1/\sqrt{2})T^{-1}$. The growth of k_R near saturation corresponds to the pulse narrowing in time. This effect is due to slower saturation of modes with large k values.

Equation (3.12) for the first moment reads

$$\partial_x F_1=M_1^0-M_1^+-\Lambda_1\ . \qquad (3.38)$$

For symmetrical $g(k)$ $M_1=0$. The first moment is due to Λ_1,

$$F_1=\tfrac{1}{16}(\sin\Theta)^2x^2\ . \qquad (3.39)$$

Writing F_1 in the form

$$F_1=\int|E|^2\varphi_t dt\ , \qquad (3.40)$$

where φ denotes the phase of the e.m. field one sees that the field is chirped.

C. Field energy in terms of the scattering data

In the theory of pulse propagation in absorbing media the field moments can be related to $\ln a$ and written as a sum of continuous spectrum contribution and discrete eigenvalues. In the case of SF, according to Eq. (3.14) $\ln a$ is replaced by $\ln[a(1-cf)]$. The function $a(1-cf)$ has no zeros in the UHP. Corrections due to $(1-cf)$ are, however, of the order of ζ^{-2} for large ζ and they do not contribute to the zero-order moment. The field energy can be written in the usual form

$$F_0=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\ln[1+|c|^2]dk+4i\sum_j(\zeta_j^*-\zeta_j)\ , \qquad (3.41)$$

where ζ_j are solutions to

$$a(\zeta_j)=0\ ,\quad \mathrm{Im}\zeta_j>0\ ,\quad j=1,2,\ldots\ .$$

As shown in Sec. II the amplitudes a and $\bar{b}$ are analytical functions in the UHP. The poles ζ_j of c are, therefore, the zeros of a. Writing Eq. (3.22) in the form

$$c(x,\zeta)=\frac{\sinh Wx}{\cosh(Wx-\delta)}\ , \qquad (3.42)$$

one can, readily, find the condition for ζ_j,

$$W(\zeta_j)=[\delta\pm i(2j+1)\pi]\frac{1}{x}\ ,\quad j=0,1,2,\ldots$$

or, in terms of the probability error function

$$\bar{x}\ \mathrm{Re}w(\zeta_jT)=\delta\ , \qquad (3.43)$$

$$\bar{x}\ \mathrm{Im}w(\zeta_jT)=\pm(2j+1)\pi\ , \qquad (3.44)$$

where $\bar{x}=\frac{1}{2}\sqrt{\pi}Tx$.

There are no solutions for ζ_j when $\bar{x}$ is smaller than the threshold value

$$\bar{x}_{\rm th}=(\delta^2+\pi^2)^{1/2}\ . \qquad (3.45)$$

This value is close to x_m given by (3.29). For $x>x_{\rm th}$ a breatherlike pair $(\zeta_0, -\zeta_0^*)$ appears and grows with x. New eigenvalues appear when

$$x_{j\,\mathrm{th}} > [\delta^2+(2j+1)^2\pi^2]^{1/2}\ , \quad j=1,2\ldots .$$

They can be evaluated from (3.43) and (3.44) using tables of the w function [21].

The contribution from discrete eigenvalues is an oscillating function of x. This can be seen from Eq. (3.41),

$$4i\sum_j (\zeta_j^*-\zeta_j)=F_0-\frac{1}{\pi}\int \ln(1+|c|^2)dk\ , \tag{3.46}$$

where, according to (3.22),

$$(1+|c|^2)=\frac{(1+\gamma)(1+\gamma e^{\Omega x})}{(1+\gamma e^{Wx})(1+\gamma e^{W^* x})}\ . \tag{3.47}$$

While F_0 is a monotonous function of x, the "soliton" contribution and the continuous spectrum oscillate. For large x, $\ln(1+|c|^2)$ saturates to a constant value and "soliton" eigenvalues grow linearly in x.

IV. DISCUSSION AND CONCLUSIONS

The results of this paper show that conservation laws in SF can be derived essentially along the same lines as in pulse propagation. Significant differences appear, however. The continuous spectrum plays a dominant role, the relation of field moments to the scattering data is different and, in general, field moments are not additive functions of the continuous and discrete spectra.

Explicit x dependence of the field energy has been found. It determines the characteristic threshold distance for the SF process. Contrary to the linear theory or self-similar solutions, where x and t are interrelated, the characteristic value of propagation length is found without any assumptions about the time evolution of the SF pulse.

The field energy associated with Steudel's solution [15] grows linearly with x. This indicates that it is a good solution in the saturation region, $x > x_s$. On the other hand, this solution has been evaluated using the linearized scattering amplitude c, i.e., for $x \ll x_s$.

The present approach also shows that a consistent description of the amplification process by the ISM requires a finite value of the inhomogeneous broadening time T. Infinitely large T leads to singularities and degeneracy of the transmitted energy moments M_n (3.14). Finite T seems also to be responsible for pulse narrowing during amplification.

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APPENDIX A: EVALUATION OF $\mathbf{\Phi}^0(x)$

Matrix function Φ^0 satisfies Eq. (2.22),

$$\mathbf{\Phi}_x^0=\mathbf{R}^0\mathbf{\Phi}^0\ . \tag{A1}$$

For constant initial polarization the matrix $\mathbf{R}^0$ (2.23) takes the simple form

$$\mathbf{R}^0=W\begin{bmatrix}\cos\Theta & \sin\Theta\\ \sin\Theta & -\cos\Theta\end{bmatrix}\ . \tag{A2}$$

All components of Φ satisfy the equation

$$\Phi_{xx}^0=W^2\Phi\ . \tag{A3}$$

From the compatibility condition (2.31) one has

$$\Phi_{21}^0=\Phi_{12}^0\ ,\quad \Phi_{22}^0=\Phi_{11}^0-2\cot\Theta\Phi_{12}^0\ . \tag{A4}$$

Assuming $\Phi_{11}^0(0)=1$ and $\Phi_{21}(0)=0$ one finds

$$\Phi_{11}=\tfrac{1}{2}(1+\cos\Theta)e^{Wx}+\tfrac{1}{2}(1-\cos\Theta)e^{-Wx}\ , \tag{A5}$$

$$\Phi_{21}=\tfrac{1}{2}\sin\Theta(e^{Wx}-e^{-Wx})\ , \tag{A6}$$

and

$$f(x)=\frac{\Phi_{21}^0}{\Phi_{11}^0}=\frac{\sinh Wx}{\cosh(Wx+\delta)}\ , \tag{A7}$$

where $\delta=\frac{1}{2}\ln(\gamma^{-1})$, $\gamma=(1-\cos\Theta)/(1+\cos\Theta)$.

APPENDIX B: UNIQUENESS OF $f(x)$

The values of $\mathbf{\Phi}^0$ at $x=0$ are not determined by boundary conditions (2.6) and (2.7). One may ask if the conservation laws depend on the choice of $f(0)$.

The MB equations are invariant under the general transformation

$$\mathbf{\Psi}'=\mathbf{h}\mathbf{\Psi}\ , \tag{B1}$$

$$\mathbf{U}'=\mathbf{h}\mathbf{U}\mathbf{h}^{-1}+\mathbf{h}_t\mathbf{h}^{-1}\ , \tag{B2}$$

$$\mathbf{V}'=\mathbf{h}\mathbf{V}\mathbf{h}^{-1}+\mathbf{h}_x\mathbf{h}^{-1}\ , \tag{B3}$$

where $\mathbf{h}=\mathbf{h}(x,t)$. Choosing constant, diagonal $\mathbf{h}_1$,

$$\mathbf{h}=\tfrac{1}{2}(h_1+h_2)\mathbf{I}+\tfrac{1}{2}(h_1-h_2)\sigma_3\ , \tag{B4}$$

one has

$$\Phi_{11}'=h_1\Phi_{11}\ ,\quad \Phi_{21}'=h_2\Phi_{21}\ . \tag{B5}$$

Under this transformation

$$\mathbf{U}'=-i\zeta\boldsymbol{\sigma}_3+\frac{1}{2}\frac{h_1}{h_2}E\boldsymbol{\sigma}_+-\frac{1}{2}\frac{h_2}{h_1}E^*\boldsymbol{\sigma}_-\ , \tag{B6}$$

$$\mathbf{V}'=\boldsymbol{\sigma}_3\tilde{N}+\frac{h_1}{h_2}\tilde{\lambda}\boldsymbol{\sigma}_++\frac{h_2}{h_1}\tilde{\lambda}^*\boldsymbol{\sigma}_-\ ,$$

which is equivalent to

$$E'=\frac{h_1}{h_2}E\ ,\quad E^{*\prime}=\frac{h_2}{h_1}E^*\ ,$$

and

$$\lambda'=\frac{h_1}{h_2}\lambda\ .$$

It follows that $\mu_x = V_{12}\Psi_{21}/\Psi_{11}$ is unchanged and, as a consequence, the form of conservation laws does not depend on $f(x=0)$.

APPENDIX C: COMPATIBILITY OF EQ. (2.25) WITH EQ. (3.11)

From the evolution of the S-matrix elements (2.25) one finds

$$(\ln a)_x = \tilde{N}^+ - \tilde{N}^0 + W \sin\Theta c \ . \tag{C1}$$

On the other hand, integrating Eq. (3.12), one gets

$$[\ln a(1-cf)]_x = \tilde{N}^+ - \tilde{N}^0 + W \sin\Theta f \ . \tag{C2}$$

From (C1) and (C2) the compatibility condition follows:

$$[\ln(1-cf)]_x = W \sin\Theta(f+c) \ . \tag{C3}$$

This condition is exactly satisfied by the solutions (3.22) and (3.30) for c and f, respectively.

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