Configuration-space methods for Coulomb scattering in a laser field

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(Received 6 March 1992; revised manuscript received 6 July 1992)

The Volkov wave function describing the motion of a charged particle in a laser field serves as a modified plane wave in the formulation of external-field collision theory and is widely used in applications. Exact solutions are unavailable for those cases in which the target, as well as the projectile, carries a net charge. It is shown that this difficulty may be overcome through the adoption of a variational formulation of the theory in configuration space. The essential feature of this procedure is the specification of boundary conditions to be satisfied by the trial functions whereby the combined effect of the Coulomb potential and the external field is accounted for with sufficient accuracy. Wave packets constructed from such trial functions satisfy the physical requirement that they follow classical trajectories at asymptotic times. The formalism is applied to the problem of potential scattering in a low-frequency external field and leads to an approximate transition amplitude that serves as a generalization of the Kroll-Watson approximation, reducing to it for potentials having no long-range Coulomb tail. In addition, a relatively simple Coulomb generalization of the cross-section sum rule is obtained. As a second application a low-frequency approximation is derived for the amplitude for laser-assisted electron-impact ionization. It is based on a choice of trial functions that accounts for the effect on the asymptotic motion of these particles of the long-range final-state Coulomb interactions among pairs of charged particles, in the presence of the laser field. The applicability of this approach to multiphoton-ionization processes is discussed briefly.

PACS number(s): 34.80.Qb, 03.65.Nk

I. INTRODUCTION

There appears to be no exact solution to the wave equation for the motion of a charged particle in a Coulomb field in the presence of a time-dependent electric field. Such a solution, were it available, would serve a function, in the theory of laser-assisted collisions and multiphoton ionization, analogous to that of the Volkov solution [1] describing the asymptotic motion of the particle under the influence of the external field alone. In a number of applications [2–4] an approximation has been adopted in which the plane wave appearing in the Volkov solution is replaced by a field-free Coulomb wave function. While such a choice has its merits in certain situations, it lacks a proper basis in the formulation of the theory. There are several indications of the shortcomings of this product form. One first observes that wave packets constructed from such functions fail to follow the appropriate classical asymptotic trajectories. An indication that is perhaps more direct is the fact that when the product function is inserted into the wave equation the result vanishes no more rapidly than the Coulomb potential itself. Finally, one can point to a specific application—the derivation of low-frequency approximations—where asymptotic wave functions of the product form would, if they were adopted, lead to the appearance of spurious near singularities and erroneous conclusions concerning the frequency dependence of the transition amplitude [5].

It is clear that an asymptotic solution is needed that more accurately accounts for the combined effect of both fields on the motion of the charged particle. An improved solution, one that avoids the above-mentioned shortcomings associated with the product form, is proposed here and is discussed in some detail. The improvement is achieved through the introduction of a Kramers-Henneberger transformation [6] acting on a suitably chosen field-free trial function. An approximation procedure for treating atomic interactions with high-frequency laser fields, making use of this space-transition idea, has received a considerable amount of attention in recent years [7]. It is therefore important to emphasize that the present application of the Kramers-Henneberger transformation is not tied to any particular approximation. Rather, it is used to specify the boundary conditions in a configuration-space formulation of the theory of Coulomb scattering in a laser field of arbitrary spectral and polarization properties. In fact, the application that is discussed here is to the development of *low*-frequency approximations, for potential scattering and for laser-assisted electron-impact ionization.

A variational formulation of charged-particle collision theory was presented some time ago [8]. This work is used, in Sec. II, as the basis for a generalization allowing for the presence of a time-dependent external field. The variational approach provides a convenient means for the development of approximations. The essential features of the approximation can often be incorporated in the calculation through an appropriate choice of trial functions, leading to a result of enhanced accuracy. In principle, error estimates can be obtained from an examination of higher-order iterations. These features are illustrated in Sec. III in the derivations of low-frequency approximations. The approximation obtained for potential scattering generalizes a well-known result of Kroll and Watson

[9], reducing to it when the potential is taken to be of short range. The low-frequency approximation provides the basis for the derivation of a cross-section sum rule expressed in a relatively simple form. In the derivation of a low-frequency approximation for electron-impact ionization we base our choice of the final-state trial function on a form that has been used successfully in recent calculations of field-free ionization [10]. This function contains the effects of the dominant postcollision interactions among the pairs of charged particles. In view of this the method adopted here for including the additional particle-field interactions is particularly appropriate since the asymptotic interactions, which are dominant for fields of low frequency, are thereby treated in a manner that is essentially exact.

Section IV contains a summary of results obtained here and includes some brief comments on the applicability of the method to multiphoton-ionization processes.

II. FORMULATION

To simplify the formal presentation we consider the problem of the scattering of an electron by a potential $V(\mathbf{r})$, behaving at great distances as Ze^2/r. (Z may have either sign.) The scattering takes place in the presence of an external field described, in the dipole approximation, by the vector potential $\mathbf{A}(t)$. The S matrix is taken to be

$$S(\mathbf{p}',\mathbf{p})=\lim_{t\to\infty}\int d^3r\, X_{\mathbf{p}'}^{(-)*}(\mathbf{r},t)\Psi_{\mathbf{p}}^{(+)}(\mathbf{r},t)\ . \tag{2.1}$$

Here $\Psi_{\mathbf{p}}^{(+)}$ is the outgoing-wave solution of the wave equation (in units with $\hbar=1$)

$$\left[H-i\frac{\partial}{\partial t}\right]\Psi_{\mathbf{p}}^{(+)}(\mathbf{r},t)=0\ , \tag{2.2}$$

corresponding to an incident wave of momentum $\mathbf{p}$; the Hamiltonian is

$$H=\frac{1}{2m}(-i\nabla-e\,\mathbf{A}/c)^2+V(\mathbf{r})\ . \tag{2.3}$$

To specify the final-state wave function $X_{\mathbf{p}'}^{(-)}$, along with the corresponding initial-state function $X_{\mathbf{p}}^{(+)}$, we first introduce the modified plane waves that would be appropriate in the treatment of the field-free scattering problem. These functions are characterized by the asymptotic behavior

$$\chi_{\mathbf{p}}^{(\pm)}(\mathbf{r},t)\sim(2\pi)^{-3/2}\exp[i\mathbf{p}\cdot\mathbf{r}-iE_{\mathbf{p}}t+is_{\mathbf{p}}^{(\pm)}(\mathbf{r})]\ , \tag{2.4}$$

with $E_{\mathbf{p}}=p^2/2m$. The phase function

$$s_{\mathbf{p}}^{(\pm)}(\mathbf{r})=\pm\frac{Ze^2m}{p}\ln(pr\mp\mathbf{p}\cdot\mathbf{r}) \tag{2.5}$$

represents a distortion due to the long-range Coulomb tail. The effect of the field on the asymptotic motion of the projectile is correctly accounted for by the choice

$$X_{\mathbf{p}}^{(\pm)}(\mathbf{r},t)=\chi_{\mathbf{p}}^{(\pm)}[\mathbf{r}-\boldsymbol{\alpha}(t),t]\times\exp\left[-i\int_0^t dt'\frac{e^2A^2(t')}{2mc^2}\right]\ , \tag{2.6a}$$

where

$$\boldsymbol{\alpha}(t)=-\frac{e}{mc}\int_0^t dt'\,\mathbf{A}(t')\ . \tag{2.6b}$$

The Kramers-Henneberger space translation has been introduced in Eq. (2.6a) and this will play an important role in all that follows. (Note that in the absence of a Coulomb tail the Volkov solutions are reproduced by the above construction.) To see the extent to which the effect of the Coulomb tail is included in Eq. (2.6) we first observe that

$$\left[H-i\frac{\partial}{\partial t}\right]X_{\mathbf{p}}^{(\pm)}(\mathbf{r},t)=(h-E_{\mathbf{p}})X_{\mathbf{p}}^{(\pm)}(\mathbf{r},t)\ , \tag{2.7}$$

where h is the Hamiltonian in the absence of the field. Now $(h-E_{\mathbf{p}})\chi_{\mathbf{p}}(\mathbf{r},t)$ is of order $1/r^2$ at great distances, as is the difference $V(\mathbf{r})-V(\mathbf{r}-\boldsymbol{\alpha}(t))$. It follows that the right-hand side of Eq. (2.7) falls off as $1/r^2$ asymptotically so that the accuracy of the asymptotic wave function in Eq. (2.6) is equivalent to that of the field-free version, in the sense that both functions, $X_{\mathbf{p}}(\mathbf{r},t)$ and $\chi_{\mathbf{p}}(\mathbf{r},t)$, satisfy their respective wave equations when terms of order $1/r^2$ are ignored. We note that wave packets constructed from these field-dependent asymptotic solutions are concentrated about the classical trajectories

$$\mathbf{r}(t)=\boldsymbol{\alpha}(t)+\frac{\mathbf{p}t}{m}\pm\frac{\mathbf{p}}{p^3}Ze^2m\ln\left[\frac{|t|}{t_0}\right] \tag{2.8}$$

for $t\to\mp\infty$ (t_0 is an arbitrary positive reference time), as required by physical considerations.

We represent the wave function describing the propagation of the system forward in time as

$$\Psi_{\mathbf{p}}^{(+)}(\mathbf{r},t)=i\lim_{t'\to-\infty}\int d^3r'G(\mathbf{r},t;\mathbf{r}',t')X_{\mathbf{p}}^{(+)}(\mathbf{r}',t')\ , \tag{2.9a}$$

and for later reference define a solution

$$\Psi_{\mathbf{p}}^{(-)*}(\mathbf{r},t)=i\lim_{t'\to\infty}\int d^3r'X_{\mathbf{p}}^{(-)*}(\mathbf{r}',t')G(\mathbf{r}',t';\mathbf{r},t) \tag{2.9b}$$

that evolves backward in time. The retarded Green's function $G(\mathbf{r},t;\mathbf{r}',t')$ is defined by the condition

$$G(\mathbf{r},t;\mathbf{r}',t')=0,\quad t<t'\ , \tag{2.10a}$$

and the differential equation

$$\left[H-i\frac{\partial}{\partial t}\right]G(\mathbf{r},t;\mathbf{r}',t')=-\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\ . \tag{2.10b}$$

Useful identities for the wave functions represented in Eqs. (2.9) may be derived [5] in the form

$$\Psi_{\mathbf{p}}^{(+)}(\mathbf{r},t)=\Psi_{\mathbf{p}\tau}^{(+)}(\mathbf{r},t)+\int_{-\infty}^{\infty}dt'\int d^3r'G(\mathbf{r},t;\mathbf{r}',t')\left[H'-i\frac{\partial}{\partial t'}\right]\times\Psi_{\mathbf{p}\tau}^{(+)}(\mathbf{r}',t') \tag{2.11a}$$

and

$$\Psi_{\mathbf{p}}^{(-)*}(\mathbf{r},t)=\Psi_{\mathbf{p}\tau}^{(-)*}(\mathbf{r},t) + \int_{-\infty}^{\infty} dt' \int d^3r' \left[\left[H'-i\frac{\partial}{\partial t'} \right] \times \Psi_{\mathbf{p}'\tau}^{(-)}(\mathbf{r}',t') \right]^* \times G(\mathbf{r}',t';\mathbf{r},t) , \quad (2.11b)$$

where the $\Psi_{\mathbf{p}\tau}^{(\pm)}$ are trial wave functions that have the correct asymptotic behavior but that need not satisfy the wave equation (2.2). These identities provide the basis for the construction of variational principles for transition amplitudes, as shown below [11].

A convenient representation of the S matrix defined in Eq. (2.1) may be obtained using the identity (2.11a) for the wave function, with the trial function $\Psi_{\mathbf{p}\tau}^{(+)}$ chosen to be the modified plane wave $X_{\mathbf{p}}^{(+)}$. This incident wave makes no contribution to the S matrix in Eq. (2.1) owing to the presence of a rapidly oscillating logarithmic phase factor in the integrand (See Sec. II B of Ref. [8] for a more detailed discussion of a very similar calculation.) The contribution from the second term on the right in Eq. (2.11a) is determined using Eq. (2.9b) to evaluate the infinite-time limit. One finds in this way that

$$S(\mathbf{p}',\mathbf{p})=-iT(\mathbf{p}',\mathbf{p}) , \quad (2.12)$$

where

$$T(\mathbf{p}',\mathbf{p})=\int_{-\infty}^{\infty} dt \int d^3r\, \Psi_{\mathbf{p}'}^{(-)*}(\mathbf{r},t) \left[H-i\frac{\partial}{\partial t} \right] \times X_{\mathbf{p}}^{(+)}(\mathbf{r},t) . \quad (2.13)$$

A variational generalization of this T-matrix identity is derived below.

The asymptotic form taken on by the scattered wave $\tilde{\Psi}_{\mathbf{p}}^{(+)} \equiv \Psi_{\mathbf{p}}^{(+)} - X_{\mathbf{p}}^{(+)}$ may now be determined following standard methods [12] suitably modified to allow for the Coulomb boundary conditions [13]. Omitting details, we find the form [14]

$$\tilde{\Psi}_{\mathbf{p}}^{(+)}(\mathbf{r},t) \sim (2\pi)^{-1} \int_0^{\infty} dE_{\mathbf{q}} \exp[-iE_{\mathbf{q}}t]\, B(\mathbf{q},\mathbf{p};\mathbf{r}-\boldsymbol{\alpha}(t)) . \quad (2.14)$$

The projectile emerges from the collision region with a distribution of possible energies since it can exchange energy with the external field; this is reflected in the presence of the energy integral in Eq. (2.14). The function $B(\mathbf{q},\mathbf{p};\mathbf{r})$ is a (Coulomb-distorted) outgoing-wave solution of the time-independent Schrödinger equation. The explicit form is found to be

$$B(\mathbf{q},\mathbf{p};\mathbf{r}) \sim [(2\pi)^{-3/2}(-4\pi^2 m)T(q\mathbf{r}/r,\mathbf{p})] \times \exp\left[iqr - i\frac{Ze^2 m}{q} \ln(2qr) \right] \Big/ r , \quad (2.15)$$

where the amplitude T is identical to the T matrix defined previously in Eq. (2.13). The inclusion of the space translation in Eq. (2.14) accounts for the fact that the projectile interacts with the time-dependent external field as well as the Coulomb potential.

Either of the wave-function identities shown in Eq. (2.11) may be used as the basis for establishing a variational principle for the S matrix. We begin by introducing a trial wave function $\Psi_{\mathbf{p}\tau}^{(+)} = X_{\mathbf{p}}^{(+)} + \tilde{\Psi}_{\mathbf{p}\tau}^{(+)}$ whose scattered-wave component has the asymptotic form shown in Eqs. (2.14) and (2.15), with the exact T matrix replaced by a trial value T_τ. Then, with

$$S_\tau(\mathbf{p}',\mathbf{p}) \equiv \lim_{t\to\infty} \int d^3r\, X_{\mathbf{p}'}^{(-)*}(\mathbf{r},t) \tilde{\Psi}_{\mathbf{p}\tau}^{(+)}(\mathbf{r},t) , \quad (2.16)$$

one finds that $S_\tau = -iT_\tau$, as may be verified [8] by evaluating the limit in Eq. (2.16) with the trial function replaced by its asymptotic form. [In taking the limit $t\to\infty$ we may also let $r\to\infty$ since all results obtained here are interpreted in terms of wave packets following the classical trajectories (2.8).] We now introduce the identity (2.11a) for the wave function into the defining relation (2.1) for the S matrix. With the aid of Eq. (2.9b) we arrive at the identity

$$S(\mathbf{p}',\mathbf{p})=S_\tau(\mathbf{p}',\mathbf{p}) - i\int_{-\infty}^{\infty} dt \int d^3r\, \Psi_{\mathbf{p}'}^{(-)*}(\mathbf{r},t) \left[H-i\frac{\partial}{\partial t} \right] \times \Psi_{\mathbf{p}\tau}^{(+)}(\mathbf{r},t) . \quad (2.17)$$

Finally, we replace the exact wave function in the integrand by the right-hand side of the identity (2.11b) and make use of Eq. (2.12) to arrive at the decomposition

$$T=T_{\text{var}}+\Delta T . \quad (2.18)$$

The variational approximation for the T matrix is

$$T_{\text{var}}(\mathbf{p}',\mathbf{p})=T_\tau(\mathbf{p}',\mathbf{p}) + \int_{-\infty}^{\infty} dt \int d^3r\, \Psi_{\mathbf{p}'\tau}^{(-)*}(\mathbf{r},t) \left[H-i\frac{\partial}{\partial t} \right] \times \Psi_{\mathbf{p}\tau}^{(+)}(\mathbf{r},t) . \quad (2.19)$$

The remainder, ΔT, is given explicitly as

$$\Delta T(\mathbf{p}',\mathbf{p})=\int_{-\infty}^{\infty} dt \int d^3r \int_{-\infty}^{\infty} dt' \int d^3r' \left[\left[H-i\frac{\partial}{\partial t} \right] \Psi_{\mathbf{p}'\tau}^{(-)}(\mathbf{r},t) \right]^* G(\mathbf{r},t;\mathbf{r}',t') \left[H'-i\frac{\partial}{\partial t'} \right] \Psi_{\mathbf{p}\tau}^{(+)}(\mathbf{r}',t') . \quad (2.20)$$

In the standard variational approach one introduces trial functions sufficiently accurate so that the remainder ΔT, of second order in the error in the trial functions, may be neglected. Here we shall proceed somewhat differently, seeking estimates of the remainder term by inserting suitable approximations for the unknown Green's function in Eq. (2.20). If the trial functions are accurate at great distances the integral will converge rapidly and the approximation for Green's function need only be accurate in the region near the target. This procedure will be illustrated, in Sec. III A, in a derivation of a low-frequency approximation for the T matrix. The simplifying feature of this approach to scattering in a low-frequency field lies in the possibility of using (space-translated) field-free wave functions—either exact solutions or approximations—as trial functions in the external-field scattering problem. Trial functions of this type are appropriate since they are accurate in the asymptotic domain, where the low-frequency field has its greatest effect on the motion of the projectile. For the same reason the choice of a Green's function which provides a good approximation in the neighborhood of the target is simplified; in the electric-field gauge it is the field-free Green's function that provides a first approximation, with corrections of higher order determined by an expansion in powers of the electric-dipole interaction. (Thus the trial functions play the role of distorted waves in a modified perturbation theory.) The applicability of the variational method is, of course, not restricted to any specific dynamical assumptions. A knowledge of the boundary conditions is the basic requirement, and we have seen that if the boundary conditions for the field-free scattering process are known they may be modified to apply to the external-field problem through the introduction of a space translation, in the manner discussed above.

III. LOW-FREQUENCY APPROXIMATIONS

A. Potential scattering

By way of introduction, and to establish notation, we first consider the variational formulation of the field-free scattering problem and apply the treatment developed in Sec. II to this situation. Time-dependent trial functions are to be chosen in the form

$$\Psi_{\mathbf{p}\tau}^{(\pm)}(\mathbf{r},t)=\exp(-iE_{\mathbf{p}}t)u_{\mathbf{p}\tau}^{(\pm)}(\mathbf{r}) , \tag{3.1}$$

where the $u_{\mathbf{p}\tau}^{(\pm)}$ are trial solutions of the field-free wave equation $(h-E_{\mathbf{p}})u_{\mathbf{p}}^{(\pm)}=0$. The transition amplitude is given in terms of the exact t matrix of the field-free problem by

$$T(\mathbf{p}',\mathbf{p})=2\pi\delta(E_{\mathbf{p}'}-E_{\mathbf{p}})t(\mathbf{p}',\mathbf{p}) , \tag{3.2}$$

a relation that expresses energy conservation in the absence of the external field. A variational identity for the t matrix is provided by the expression

$$t(\mathbf{p}',\mathbf{p})=t_{\mathrm{var}}(\mathbf{p}',\mathbf{p})+\Delta t(\mathbf{p}',\mathbf{p}) , \tag{3.3a}$$

where the variational approximation is

$$t_{\mathrm{var}}(\mathbf{p}',\mathbf{p})=t_{\tau}(\mathbf{p}',\mathbf{p}) + \int d^3r\, u_{\mathbf{p}'\tau}^{(-)*}(\mathbf{r})(h-E_{\mathbf{p}})u_{\mathbf{p}\tau}^{(+)}(\mathbf{r}) , \tag{3.3b}$$

with the remainder given formally by

$$\Delta t(\mathbf{p}',\mathbf{p})=\int d^3r\int d^3r'[(h-E_{\mathbf{p}})u_{\mathbf{p}'\tau}^{(-)}(\mathbf{r})]^*g(\mathbf{r},\mathbf{r}';E_{\mathbf{p}}) \times(h'-E_{\mathbf{p}})u_{\mathbf{p}\tau}^{(+)}(\mathbf{r}') . \tag{3.3c}$$

In arriving at this form for the remainder we have replaced the Green's function in Eq. (2.20) by $g(\mathbf{r},t;\mathbf{r}',t')$, the field-free version, which may be represented as

$$g(\mathbf{r},t;\mathbf{r}',t')=(2\pi)^{-1}\int_{-\infty}^{\infty}dW\exp[-iW(t-t')] \times g(\mathbf{r},\mathbf{r}';W) . \tag{3.4}$$

Here $g(\mathbf{r},\mathbf{r}';W)=\langle\mathbf{r}|(W+i\varepsilon-h)^{-1}|\mathbf{r}'\rangle$, the coordinate representation of the time-independent resolvent operator, with ε a positive infinitesimal parameter. With the scattered part of the trial wave function at great distances expressed as

$$\tilde{u}_{\mathbf{p}\tau}^{(+)}(\mathbf{r})\sim(2\pi)^{-3/2}f_{\tau}(p\mathbf{r}/r,\mathbf{p}) \times\exp\left[ipr-i\frac{Ze^2m}{p}\ln(2pr)\right]\Big/r , \tag{3.5}$$

we find, from the field-free versions of Eqs. (2.14) and (2.15), along with Eq. (3.2), that the relation between the trial t matrix and the conventionally defined scattering amplitude is $t_{\tau}(\mathbf{p}',\mathbf{p})=-[4\pi^2m]^{-1}f_{\tau}(\mathbf{p}',\mathbf{p})$.

It is sometimes convenient to adopt a simple form for the trial function; the identity remains valid though the term shown in Eq. (3.3b) need not be an accurate approximation. We may choose, for example,

$$u_{\mathbf{p}\tau}^{(\pm)}(\mathbf{r})=(2\pi)^{-3/2}\exp(i\mathbf{p}\cdot\mathbf{r})F_{\mathbf{p}}^{(\pm)}(\mathbf{r}) , \tag{3.6a}$$

where

$$F_{\mathbf{p}}^{(\pm)}(\mathbf{r})=\exp[is_{\mathbf{p}}^{(\pm)}(\mathbf{r})]\{1-\exp[-\beta(pr\mp\mathbf{p}\cdot\mathbf{r})^2]\} , \tag{3.6b}$$

and where β is an arbitrary positive constant. This function provides, asymptotically, the required logarithmic phase shown in Eq. (2.4). By including a factor that vanishes sufficiently rapidly for $|pr\mp\mathbf{p}\cdot\mathbf{r}|\to0$ we have cancelled the logarithmic singularity in the phase factor, leaving a function whose value and first two derivatives are finite everywhere. It is therefore an allowable trial function, with $t_{\tau}(\mathbf{p}',\mathbf{p})=0$. The identity given in Eqs. (3.3) becomes, with trial functions chosen as in (3.6),

$$t(\mathbf{p}',\mathbf{p})=(2\pi)^{-3}\left\{\int d^3r[e^{i\mathbf{p}'\cdot\mathbf{r}}F_{\mathbf{p}'}^{(-)}(\mathbf{r})]^*(h-E_{\mathbf{p}})e^{i\mathbf{p}\cdot\mathbf{r}}F_{\mathbf{p}}^{(+)}(\mathbf{r}) + \int d^3r\int d^3r'[(h'-E_{\mathbf{p}})e^{i\mathbf{p}'\cdot\mathbf{r}'}F_{\mathbf{p}'}^{(-)}(\mathbf{r}')]^*g(\mathbf{r}',\mathbf{r};E_{\mathbf{p}})(h-E_{\mathbf{p}})e^{i\mathbf{p}\cdot\mathbf{r}}F_{\mathbf{p}}^{(+)}(\mathbf{r})\right\} . \tag{3.7}$$

This result, a time-independent analog of the identity shown in Eqs. (2.18)–(2.20), will be referred to later on.

With the external field now assumed to be present we choose the trial functions [14]

$$\Psi_{\mathbf{p}\tau}^{(\pm)}(\mathbf{r},t)=\exp(-iE_{\mathbf{p}}t)u_{\mathbf{p}\tau}^{(\pm)}[\mathbf{r}-\boldsymbol{\alpha}(t)] . \tag{3.8}$$

[Here the $u_{\mathbf{p}\tau}^{(\pm)}$ are arbitrary field-free trial functions, not restricted to the form shown in Eq. (3.6).] The function defined in Eq. (3.8) is an allowable choice since it satisfies the boundary conditions; from an examination of the amplitude of the outgoing wave at infinity we find that the trial scattering amplitude, entering into the variational expression (2.19), is given by

$$T_{\tau}(\mathbf{p}',\mathbf{p})=2\pi\delta(E_{\mathbf{p}'}-E_{\mathbf{p}})t_{\tau}(\mathbf{p}',\mathbf{p}) . \tag{3.9}$$

We remark that in the weak-field limit, and with the functions $u_{\mathbf{p}\tau}^{(\pm)}(\mathbf{r})$ taken to be accurate approximations to the field-free solutions, the trial functions (3.8) will be accurate everywhere, not only asymptotically, and the variational approximation (2.19) will introduce an error of second order in the error in the trial functions. (If the functions $u_{\mathbf{p}\tau}^{(\pm)}$ are *exact* field-free scattering solutions, the variational approximation, evaluated to first order in the charge of the electron, reproduces the exact single-photon spontaneous bremsstrahlung amplitude, as is readily verified.) As the field intensity is increased the trial function (3.8) will not, in general, remain a good approximation. An estimate of the correction term ΔT, given in Eq. (2.20), should then be included to retain an accurate representation of the effect of the particle-field interaction in the neighborhood of the target. This may be accomplished using the modified perturbation expansion reviewed below. The validity of such an approximation procedure depends, firstly, on the special properties assumed for the field, namely, low frequency and moderate intensity. One also requires the use of sufficiently accurate "distorted waves," such as those introduced here, to properly account for the asymptotic interactions. Otherwise, spurious near singularities arising from poorly convergent spatial integrations may appear, as demonstrated earlier [5].

In evaluating the variational approximation (2.19) we adopt the simple form $\mathbf{A}(t)=\mathbf{a}\cos\omega t$ for the vector potential, so that, from Eq. (2.6b), $\boldsymbol{\alpha}(t)=\boldsymbol{\alpha}_0\sin\omega t$, with

$$\boldsymbol{\alpha}_0=-\left[\frac{e}{\omega mc}\right]\mathbf{a} . \tag{3.10}$$

The trial functions may then be expanded in Fourier series. Consider the second term on the right in Eq. (2.19), namely, the contribution

$$T^{(1)}(\mathbf{p}',\mathbf{p})=\int_{-\infty}^{\infty}dt\int d^3r\,\Psi_{\mathbf{p}'\tau}^{(-)*}(\mathbf{r},t)\left[H-i\frac{\partial}{\partial t}\right]\times\Psi_{\mathbf{p}\tau}^{(+)}(\mathbf{r},t) . \tag{3.11}$$

After the Fourier transformation is introduced this term becomes

$$T^{(1)}(\mathbf{p}',\mathbf{p})=\sum_{n=-\infty}^{\infty}2\pi\delta(E_{\mathbf{p}'}-E_{\mathbf{p}}-n\omega)T_n^{(1)}(\mathbf{p}',\mathbf{p}) , \tag{3.12a}$$

with

$$T_n^{(1)}(\mathbf{p}',\mathbf{p})=\frac{1}{2\pi}\int_0^{2\pi}d\theta\,e^{in\theta}\int d^3r\,u_{\mathbf{p}'\tau}^{(-)*}(\mathbf{r}-\boldsymbol{\alpha}_0\sin\theta)\times(h-E_{\mathbf{p}})u_{\mathbf{p}\tau}^{(+)}(\mathbf{r}-\boldsymbol{\alpha}_0\sin\theta) . \tag{3.12b}$$

The index n is interpreted as the net number of photons absorbed ($n>0$) or emitted ($n<0$) during the collision. To obtain an estimate of the error term shown in Eq. (2.20) we first introduce a gauge transformation of the Green's function that provides the basis for a modified perturbation theory, as described below. Thus we write

$$G(\mathbf{r},t;\mathbf{r}',t')=e^{i(e/c)\mathbf{A}(t)\cdot\mathbf{r}}\bar{G}(\mathbf{r},t;\mathbf{r}',t')e^{-i(e/c)\mathbf{A}(t')\cdot\mathbf{r}'} . \tag{3.13}$$

The transformed Green's function satisfies

$$\left[\bar{H}-\frac{\partial}{\partial t}\right]\bar{G}(\mathbf{r},t;\mathbf{r}',t')=-\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') , \tag{3.14}$$

with

$$\bar{H}=-\frac{\nabla^2}{2m}+V(\mathbf{r})-e\mathbf{E}(t)\cdot\mathbf{r} , \tag{3.15}$$

and

$$\mathbf{E}(t)=-\frac{1}{c}\frac{d\mathbf{A}}{dt}=\frac{\omega}{c}\mathbf{a}\sin\omega t . \tag{3.16}$$

The solution of Eq. (3.14) may be expanded (with integration variables suppressed) as

$$\bar{G}=g+g(-i\mathbf{E}\cdot\mathbf{r})g+\cdots , \tag{3.17}$$

where $g(\mathbf{r},t;\mathbf{r}',t')$ is the field-free Green's function introduced earlier in Eq. (3.4). For fields of moderate intensity, and in the absence of resonances, we may treat the electric-dipole interaction as a perturbation in this expansion since successive terms bring in higher powers of the frequency. More specifically, we introduce the small parameters [15]

$$\epsilon_1=\frac{\omega}{p^2/2m},\quad \epsilon_2=\frac{ea}{cp} ; \tag{3.18}$$

each parameter is treated as a quantity of first order, with the electric-dipole interaction (suitably scaled) taken to be of second order. It is one of the advantages of this method that higher-order corrections may be introduced in a systematic way. However, to simplify the present

discussion, we shall work to first order in these small parameters. The term ΔT, defined in Eq. (2.20), may be expanded as

$$\Delta T \cong \sum_{n=-\infty}^{\infty} 2\pi\delta(E_{\mathbf{p}'}-E_{\mathbf{p}}-n\omega)T_n^{(2)}(\mathbf{p}',\mathbf{p}) . \quad (3.19)$$

With the definitions

$$K_{\mathbf{p}}^{(\pm)}(\mathbf{r},\theta)=\exp\left[-i\frac{e}{c}\mathbf{a}\cdot\mathbf{r}\cos\theta\right](h-E_{\mathbf{p}}) \times u_{\mathbf{p}\tau}^{(\pm)}(\mathbf{r}-\boldsymbol{\alpha}_0\sin\theta) , \quad (3.20)$$

and with only the first term in the expansion (3.17) retained, we arrive at the approximation

$$T_n^{(2)}(\mathbf{p}',\mathbf{p})\cong\sum_{n'=-\infty}^{\infty}\int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^{2\pi}\frac{d\theta'}{2\pi}\exp[-i(n'-n)\theta'+in'\theta]\int d^3r\int d^3r' K_{\mathbf{p}'}^{(-)*}(\mathbf{r}',\theta')g(\mathbf{r}',\mathbf{r},;E_{\mathbf{p}}+n'\omega)K_{\mathbf{p}}^{(+)}(\mathbf{r},\theta) . \quad (3.21)$$

As a first step in carrying out the sum over n', while maintaining first-order accuracy, we expand the time-independent Green's function as

$$g(E_{\mathbf{p}}+n'\omega)\cong g(E_{\mathbf{p}})+g(E_{\mathbf{p}})(-n'\omega)g(E_{\mathbf{p}}) . \quad (3.22)$$

The contribution from the second term is transformed through an integration-by-parts procedure, resulting in the replacement

$$n'\omega e^{in'\theta}K_{\mathbf{p}}^{(+)}(\mathbf{r},\theta)\rightarrow e^{in'\theta}\left[i\omega\frac{\partial}{\partial\theta}\right]K_{\mathbf{p}}^{(+)}(\mathbf{r},\theta) .$$

The required sums may now be performed using the relation

$$\frac{1}{2\pi}\sum_{n'=-\infty}^{\infty}e^{in'(\theta-\theta')}=\delta(\theta-\theta') . \quad (3.23)$$

We find in this way that, to first order in the small parameters introduced in Eq. (3.18),

$$T_n^{(2)}(\mathbf{p}',\mathbf{p})=\int_0^{2\pi}\frac{d\theta}{2\pi}\int d^3r\int d^3r' K_{\mathbf{p}'}^{(-)*}(\mathbf{r}',\theta) \times g_\theta(\mathbf{r}',\mathbf{r};E_{\mathbf{p}})K_{\mathbf{p}}^{(+)}(\mathbf{r},\theta) , \quad (3.24)$$

where a modified Green's function has been introduced of the form

$$g_\theta(\mathbf{r}',\mathbf{r};E_{\mathbf{p}})\equiv\left[E_{\mathbf{p}}+i\varepsilon+i\omega\frac{\partial}{\partial\theta}-h\right]^{-1} . \quad (3.25)$$

To summarize, we have obtained a first-order approximation for the amplitude T_n for scattering with the exchange of n photons with the field. Combining Eqs. (3.12) and (3.24) we have

$$T_n(\mathbf{p}',\mathbf{p})\cong t_\tau(\mathbf{p}',\mathbf{p})\delta_{n0}+\int_0^{2\pi}\frac{d\theta}{2\pi}e^{in\theta}C(\mathbf{p}',\mathbf{p};\theta) , \quad (3.26a)$$

with

$$C(\mathbf{p}',\mathbf{p};\theta)=\int d^3r\, u_{\mathbf{p}'\tau}^{(-)*}(\mathbf{r}-\boldsymbol{\alpha}_0\sin\theta)(h-E_{\mathbf{p}}) \times u_{\mathbf{p}\tau}^{(+)}(\mathbf{r}-\boldsymbol{\alpha}_0\sin\theta) + \int d^3r\int d^3r' K_{\mathbf{p}'}^{(-)*}(\mathbf{r}',\theta) \times g_\theta(\mathbf{r}',\mathbf{r};E_{\mathbf{p}})K_{\mathbf{p}}^{(+)}(\mathbf{r},\theta) . \quad (3.26b)$$

The first term on the right in Eq. (3.26a) represents the contribution from the trial transition amplitude, as determined from Eq. (3.9).

The differential cross section for scattering, summed over all final states of the field, is given by the expression

$$\frac{d\sigma}{d\Omega}=\frac{(2\pi)^4 m}{p}\sum_{n=-\infty}^{\infty}\int_0^\infty p'^2 dp'\delta(E_{\mathbf{p}'}-E_{\mathbf{p}}-n\omega) \times|T_n(\mathbf{p}',\mathbf{p})|^2 . \quad (3.27)$$

The approximation (3.26) is now adopted for T_n. The sum over terms proportional to δ_{n0} may of course be evaluated trivially and we focus our attention on the remainder. With the introduction of the δ-function representation

$$\delta(E_{\mathbf{p}'}-E_{\mathbf{p}}-n\omega)=\int_{-\infty}^{\infty}\frac{ds}{2\pi}\exp[i(E_{\mathbf{p}'}-E_{\mathbf{p}}-n\omega)s] , \quad (3.28)$$

the sum encountered in the evaluation of this remainder may be performed using the identity (3.23). The contribution to the cross section of the remainder term then takes the form of an integral over final states of the expression

$$\int_{-\infty}^{\infty}\frac{ds}{2\pi}\exp[i(E_{\mathbf{p}'}-E_{\mathbf{p}})s] \times\int_0^{2\pi}\frac{d\theta}{2\pi}C^*(\mathbf{p}',\mathbf{p};\theta)C(\mathbf{p}',\mathbf{p};\theta+\omega s) . \quad (3.29)$$

With the quantity ωs appearing in the argument of the function C treated as a first-order term, the parameter s being effectively limited in magnitude by the convergence of the integral in which it appears, we make the replacement

$$C(\mathbf{p}',\mathbf{p};\theta+\omega s)\cong C(\mathbf{p}',\mathbf{p};\theta)+\omega s\frac{\partial}{\partial\theta}C(\mathbf{p}',\mathbf{p};\theta) . \quad (3.30)$$

The second term on the right makes no contribution to the integral over θ in the expression (3.29) since that contribution, to first order in the small parameters, is of the form

$$\int_0^{2\pi} d\theta F(\sin\theta)\cos\theta = 0 \ . \tag{3.31}$$

In this way we arrive at the sum rule

$$\frac{d\sigma}{d\Omega} = \frac{(2\pi)^4 m}{p} \int_0^{\infty} p'^2 dp' \delta(E_{p'} - E_p) \times \int_0^{2\pi} \frac{d\theta}{2\pi} |t_\tau(\mathbf{p}',\mathbf{p}) + C(\mathbf{p}',\mathbf{p};\theta)|^2 \ , \tag{3.32}$$

accurate to first order. As a consistency check on this approximation we note that the exact field-free cross section is recovered in the limit of vanishing field intensity. This may be verified by referring back to the identity $t = t_{\rm var} + \Delta t$ given in Eq. (3.3).

To see more clearly the relationship between the present formulation of the low-frequency approximation for Coulomb scattering and that obtained by Kroll and Watson [9] for the case of scattering by a short-range potential we apply to the latter case the approximation procedure developed above, leading to Eq. (3.26). For the trial function $u_{p\tau}(\mathbf{r})$ we choose the plane wave $(2\pi)^{-3/2}\exp(i\mathbf{p}\cdot\mathbf{r})$; this is acceptable since the boundary conditions are thereby satisfied, with $t_\tau(\mathbf{p}',\mathbf{p}) = 0$. With g_θ in Eq. (3.25) expanded out to first order in the frequency, the approximation (3.26) is readily evaluated as

$$T_n(\mathbf{p}',\mathbf{p}) \cong \int_0^{2\pi} \frac{d\theta}{2\pi} \exp[in\theta + i(\mathbf{p}'-\mathbf{p})\cdot\boldsymbol{\alpha}_0 \sin\theta] \times t(\mathbf{p}'(\theta),\mathbf{p}(\theta);E_{\mathbf{p}(\theta)}) \ . \tag{3.33}$$

Here $\mathbf{p}(\theta) \equiv \mathbf{p} - (e/c)\mathbf{a}\cos\theta$ and $t(\mathbf{p}',\mathbf{p};W)$, the off-shell scattering amplitude, is defined as the momentum representation of the scattering operator $[V + V(W + i\varepsilon - h)^{-1}V]$. The approximation (3.33) was obtained, in a different manner, by Kroll and Watson, who went on to reduce it to a form involving the on-shell t matrix. (An analysis of the domain of validity of such a reduction was given in Ref. [15].) The approximation (3.26) may then be understood to be a generalization of the Kroll-Watson low-frequency approximation, allowing for scattering from a potential with a long-range Coulomb tail.

To lowest order, that is, with $p' = p$ and with the replacement $\mathbf{p}(\theta) \to \mathbf{p}$ in the t-matrix arguments of Eq. (3.33), that expression simplifies to

$$T_n(\mathbf{p}',\mathbf{p}) \cong J_{-n}((\mathbf{p}'-\mathbf{p})\cdot\boldsymbol{\alpha}_0) t(\mathbf{p}',\mathbf{p};E_p) \ , \tag{3.34}$$

involving the on-shell t matrix and the regular Bessel function

$$J_{-n}((\mathbf{p}'-\mathbf{p})\cdot\boldsymbol{\alpha}_0) = \int_0^{2\pi} \frac{d\theta}{2\pi} \exp[in\theta + i(\mathbf{p}'-\mathbf{p})\cdot\boldsymbol{\alpha}_0 \sin\theta] \ . \tag{3.35}$$

Equation (3.34) remains valid, to lowest order, in the case where the potential has a Coulomb tail [16]. This may be verified by choosing the modified plane wave shown in Eq. (3.6a) as trial function. Then Eq. (3.26), with $t_\tau(\mathbf{p}',\mathbf{p}) = 0$ and with terms of first order neglected, leads directly to Eq. (3.34). We arrive at this result with the exact field-free t matrix represented by a version of the identity (3.7) in which the replacement

$$F_{\mathbf{p}}^{(\pm)}(\mathbf{r}) \to F_{\mathbf{p}}^{(\pm)}(\mathbf{r} - \boldsymbol{\alpha}_0 \sin\theta) \tag{3.36}$$

has been made. This replacement leaves the asymptotic behavior of the trial functions, and hence the validity of the identity, unaltered.

B. Laser-assisted $(e,2e)$ reaction

The procedure described for generating a low-frequency approximation for scattering by a center of force in the presence of a laser field may be generalized to apply to a wider class of problems. As an illustration, we consider the process of electron-impact ionization of an atom, taken here to be hydrogen for simplicity. Given sufficiently accurate trial functions appropriate to scattering in the absence of the field, they may be adapted, by means of a coordinate translation, for use in a variational calculation of the laser-assisted transition amplitude. Fortunately, a trial function is available that has a relatively simple form incorporating the correct Coulomb boundary conditions. That function, when used in a nonvariational calculation of the field-free ionization amplitude [10], leads to impressive agreement with experiment in a range of scattering energies above 100 eV, yet below values at which the Born approximation is adequate. One has reason to expect, therefore, that the approach described below may provide some improvement over methods proposed earlier [4,17], based on the Coulomb-Born approximation, with field effects incorporated through the introduction of a Volkov phase factor.

In the scattering problem under consideration an electron with initial momentum $\mathbf{p}$ strikes a hydrogen atom initially in its ground state, with wave function and energy denoted as $\phi(\mathbf{r}_2)$ and E_B, respectively, in the presence of a low-frequency laser field. We look for the amplitude $T(\mathbf{p}',\mathbf{q}';\mathbf{p})$ for the transition in which the atom is ionized, with the two electrons emerging in the final state with momenta $\mathbf{p}'$ and $\mathbf{q}'$. The formulation of the external-field scattering problem given in Sec. II is readily extended to allow for a composite target. The initial-state trial function is chosen as [14]

$$\Psi_{i\tau}^{(+)}(\mathbf{r}_1,\mathbf{r}_2,t) = (2\pi)^{-3/2} \exp\{i\mathbf{p}\cdot[\mathbf{r}_1 - \boldsymbol{\alpha}(t)]\} \times \exp\left[i\frac{e}{c}\mathbf{A}(t)\cdot\mathbf{r}_2 - iE_i t\right] \phi(\mathbf{r}_2) \ , \tag{3.37}$$

with $E_i = E_p + E_B$. Here we have introduced an approximation for the target wave function in the presence of the field, arrived at by transforming to the electric-field gauge and retaining only the leading term in the perturbation expansion of the wave function in powers of the electric-dipole interaction. This approximation procedure, which is in close formal analogy with our earlier treatment of the Green's function in Eq. (3.17), will be accurate for

photon energies small compared with characteristic target excitation energies. For simplicity the effect of the projectile-target interaction is ignored in the construction of the initial-state trial function. (In doing so we lose the variational property, so that a first-order error in the final-state trial function will introduce a *first*-order error in the calculated transition amplitude, even in the absence of the external field.)

In preparation for the introduction of our choice of final-state trial function we first consider the field-free problem. A form that satisfies the Coulomb boundary conditions and accounts in an approximate way for the long-range pairwise Coulomb interactions can be taken as [18]

$$\Psi_{f\tau}^{(-)}(\mathbf{r}_1,\mathbf{r}_2,t)=(2\pi)^{-3}\exp[i(\mathbf{p}'\cdot\mathbf{r}_1+\mathbf{q}'\cdot\mathbf{r}_2)-iE_f t] \times\eta_1^{(-)}(\mathbf{r}_1)\eta_2^{(-)}(\mathbf{r}_2)\eta_{12}^{(-)}(\mathbf{r}_{12}) . \quad (3.38)$$

Here the function $\eta_1^{(-)}(\mathbf{r}_1)$, for example, is defined by expressing the continuum wave function for electron 1 in the Coulomb field of the proton as

$$\Psi_1^{(-)}(\mathbf{r}_1)=(2\pi)^{-3/2}\exp(i\mathbf{p}'\cdot\mathbf{r}_1)\eta_1^{(-)}(\mathbf{r}_1) . \quad (3.39)$$

The function $\eta_2^{(-)}$ is similarly defined, as is $\eta_{12}^{(-)}$, the latter expressed in terms of the relative momentum of the two-electron system in its center-of-mass frame. [The required antisymmetrization is not explicitly indicated in Eq. (3.38) and in the following; in fact exchange effects are expected to be small in the range of scattering energies appropriate to this approximation.]

Returning now to the laser-assisted scattering problem we introduce an identity for the transition amplitude, given in terms of the exact final-state wave function, as

$$T(\mathbf{p}',\mathbf{q}';\mathbf{p})=\int_{-\infty}^{\infty}dt\int d^3r_1\int d^3r_2\Psi_f^{(-)*}(\mathbf{r}_1,\mathbf{r}_2,t) \times\left[H-i\frac{\partial}{\partial t}\right] \times\Psi_{i\tau}^{(+)}(\mathbf{r}_1,\mathbf{r}_2,t) , \quad (3.40)$$

which is just the three-particle generalization of the identity given in Eq. (2.17), with the trial T matrix taken to be zero as required by the form of Eq. (3.37). An approximate final-state trial function may be obtained from the function shown in Eq. (3.38) by making the replacements $\mathbf{r}_j\rightarrow\mathbf{r}_j-\boldsymbol{\alpha}(t)$, $j=1,2$. The field-modified boundary conditions are thereby satisfied and wave packets constructed from functions of this form will be concentrated about the classical trajectories. The explicit form for the approximate T matrix resulting from this choice is

$$T(\mathbf{p}',\mathbf{q}';\mathbf{p})\cong\int_{-\infty}^{\infty}dt\int d^3r_1 \times\int d^3r_2\Psi_{f\tau}^{(-)*}(\mathbf{r}_1-\boldsymbol{\alpha}(t),\mathbf{r}_2-\boldsymbol{\alpha}(t),t) \times V_i\Psi_{i\tau}^{(+)}(\mathbf{r}_1,\mathbf{r}_2,t) , \quad (3.41)$$

with $V_i=-e^2/r_1+e^2/r_{12}-e\mathbf{E}(t)\cdot\mathbf{r}_2$. The (second-order) electric-dipole interaction term shown here will be dropped in the following. The generalization of this approximation to allow for a nucleus of charge $Z>1$ is obtained, in the obvious way, by replacing the plane wave in the initial-state trial function by a Coulomb-modified plane wave.

The integrand in Eq. (3.41) may be expanded in a Fourier series. This leads to the representation

$$T(\mathbf{p}',\mathbf{q}';\mathbf{p})\cong\sum_{n=-\infty}^{\infty}2\pi\delta(E_f-E_i-n\omega)T_n , \quad (3.42)$$

with

$$T_n=(2\pi)^{-9/2}\frac{1}{2\pi}\int_0^{2\pi}d\theta\int d^3r_1\int d^3r_2\exp(i[n\theta-\mathbf{p}\cdot\boldsymbol{\alpha}_0\sin\theta+\frac{e}{c}\mathbf{a}\cdot\mathbf{r}_2\cos\theta]) \times\exp\{-i[\mathbf{p}'\cdot(\mathbf{r}_1-\boldsymbol{\alpha}_0\sin\theta)+\mathbf{q}'\cdot(\mathbf{r}_2-\boldsymbol{\alpha}_0\sin\theta)]\} \times\eta_1^{(-)*}(\mathbf{r}_1-\boldsymbol{\alpha}_0\sin\theta)\eta_2^{(-)*}(\mathbf{r}_2-\boldsymbol{\alpha}_0\sin\theta)\eta_{12}^{(-)*}(\mathbf{r}_{12})[-e^2/r_1+e^2/r_{12}]e^{i\mathbf{p}\cdot\mathbf{r}_1}\phi(\mathbf{r}_2) . \quad (3.43)$$

The spatial integral appearing in Eq. (3.43) is similar in form to that evaluated in Ref. [10], reducing to it in the limit of vanishing field intensity. The calculation is complicated by the presence of an additional integral over the angle θ. That integration may be interpreted as an averaging of the transition amplitude over the phase of the field, with the scattering assumed to be completed in a time short enough compared with the period of the field that it may be thought of as taking place instantaneously. The effect of the field, in this picture, is to shift the electronic configuration by an amount determined by the classical displacement of the particle in the presence of the electric field alone. The time plays the role of a parameter here, rather than of a dynamical variable, and this is the essential simplification achieved by the introduction of the low-frequency approximation. A similar interpretation may be given to the low-frequency approximation for potential scattering shown in Eq. (3.26). This latter result includes a correction term, the second term on the right in Eq. (3.26b), that accounts, approximately, for the error in the final-state trial function. The corresponding term for the $(e,2e)$ reaction could be included formally but would be more difficult to evaluate. In omitting this correction term we have assumed that the postcollision interactions play a dominant role in the field-free scattering problem and that they are adequately

described by the final-state trial function shown in Eq. (3.38), as may be inferred from the calculations of Ref. [10]. It has also been assumed that for moderate intensities the low-frequency field has its greatest effect on the asymptotic motion of the electrons (as a result of the near degeneracies that exist between adjacent states of the system differing in the number of soft photons present in the field), and that these asymptotic interactions are accurately accounted for by the space-translation procedure. We note, finally, that a sum rule for the ionization cross section, based on the approximation (3.43), is readily obtained by following the procedure outlined earlier in the derivation of the analogous sum rule for potential scattering shown in Eq. (3.32).

IV. DISCUSSION

The problem of scattering by a potential with a long-range Coulomb tail, in the absence of an external field, is usually treated by introducing distorted waves in the form of exact solutions for scattering by the unmodified Coulomb potential. Such a procedure is not generally applicable; for example, exact final-state Coulomb solutions are not available for the calculation of electron-impact ionization amplitudes. Coulomb scattering in an external field provides another example in which exact distorted-wave solutions are unknown. The variational method, based on the introduction of trial functions having the correct asymptotic form, provides a convenient procedure for dealing systematically with such problems. Given the correct boundary conditions appropriate to scattering in the absence of the field, the modification required to include the effect of the field may be introduced by means of a simple (Kramers-Henneberger) coordinate translation. This procedure is readily incorporated into standard time-dependent scattering theory, as shown in Sec. II. Derivations of low-frequency approximations were given in Sec. III as illustrations of the applicability of the method to problems of current interest. The approximation obtained for potential scattering reduces, to lowest order in an expansion in the small parameters defined in Eq. (3.18), to the same form as that obtained for scattering by a short-range potential. That result, shown in Eq. (3.34), involves the physical (on-shell) field-free scattering amplitude. The more accurate approximation given by Eq. (3.26) cannot be evaluated in terms of on-shell parameters alone. In contrast, the scattering amplitude appearing in the Kroll-Watson approximation for scattering by a short-range potential, *is* on the energy shell. This distinction reflects the fact that the two long-range interactions—arising from the presence of both the laser field and the Coulomb tail—cannot be treated independently. The field induces a distortion of the Coulomb wave function at great distances, taking the form of the space translation shown in Eq. (2.6). Failure to account for this effect in calculations can lead to numerical inaccuracies.

The procedure leading to the low-frequency approximation for the laser-assisted $(e,2e)$ reaction could easily be modified to apply to the (computationally simpler) process in which a negative ion is placed in a superposition of two laser fields. One field is taken to be weak and of frequency high enough so that two electrons may be ejected by single-photon absorption; this final state is then "dressed" by the presence of an intense low-frequency field. A variational formulation of a two-color ionization processes of this type can be developed in analogy with the treatment given earlier of the process in which a single electron is ejected [19]. The final-state trial function shown in Eq. (3.38) and adopted, after application of the space-translation method, in the derivation of Eq. (3.43) would be appropriate for use in such a calculation.

An important feature of the variational method lies in the fact that it provides an explicit expression for the error in the approximation. In the present formulation of the low-frequency approximation this expression is proportional to an integral that converges rapidly enough so that it is nonsingular in the zero-frequency limit. One can then be confident in assigning an estimate to the order of the error term. The accuracy of the asymptotic solutions is crucial in this regard; with less accurate solutions one must account for the possible appearance of near singularities [5] that can complicate the task of estimating the order of the error integral.

The methods described here are applicable, in principle, to the treatment of the multiphoton ionization of a neutral atom. The long-range nature of the Coulomb potential manifests itself here not only in the asymptotic form of the final-state wave function but also in the appearance of intermediate-state resonances, and special care must be taken to properly include such resonance effects. Isolated resonances can be accounted for by standard methods, but the close spacing of bound states below the ionization threshold may require the introduction of new techniques. It seems likely that variational methods, such as that described here, can provide an appropriate framework for the development of such techniques.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant No. PHY-9019745. One of us (L.R.) wishes to thank Dr. M. Gavrila for a very helpful remark.

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