Quantum deformations of the discrete nonlinear Schrödinger equation

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A quantum system with the Hamiltonian and commutation relations depending on a deformation parameter ϵ is introduced. When $\epsilon=0$ the system reduces to the quantum Ablowitz-Ladik (QAL) equation, for $\epsilon=\gamma/3$ it represents a quantum discrete nonlinear Schrödinger (QDNLS) system, and for $\epsilon=\gamma$ the system reduces to the usual QDNLS equation. We show that the energy levels of this system can be continuously deformed into the corresponding ones of the QAL and QDNLS equations. The physical significance of this system is also discussed.

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There are two simple discrete versions of the nonlinear Schrödinger equation: the discrete nonlinear Schrödinger (DNLS) equation

$$i\dot{A}_i+A_{i+1}-2A_i+A_{i-1}+\gamma|A_i|^2A_i-\omega_iA_i=0\ , \tag{1}$$

which preserves the standard norm [1], and the Ablowitz-Ladik (AL) system [2]

$$i\dot{A}_i+A_{i+1}-2A_i+A_{i-1}+(\gamma/2)|A_i|^2(A_{i+1}+A_{i-1})-\omega_iA_i=0\ , \tag{2}$$

which is completely integrable by the inverse scattering method [3]. The quantum problem of these two systems can be solved by different methods, which are based upon different properties of these equations. The AL system can be solved by the quantum inverse scattering method (QISM) [4–6], which is based on the complete integrability of the classical system. On the contrary, the QISM is of no use in the solution of the quantum DNLS (QDNLS) equation since this equation, except for two degrees of freedom, is not integrable. To solve the quantum problem of the DNLS equation, one can use an alternate method, which is based upon the conservation of the standard norm. This leads to the existence of an invariant operator with finite-dimensional eigenspaces (the number operator), which allows one to reduce the infinite-dimensional eigenvalue problem for the Hamiltonian to the diagonalization of finite matrices [7,8]. In a recent paper it has been shown that this method is also effective in solving the quantum problem of the Ablowitz-Ladik system [9]. The aim of the present paper is to introduce a DNLS equation, which has the following properties.

(i) Its quantum version is exactly solvable by the above alternate method.

(ii) The Hamiltonian, as well as the Poisson bracket, continuously depends on a deformation parameter ϵ so that when $\epsilon=0$ the system reduces to the AL system, which has a nonstandard Poisson bracket; for $\epsilon=\gamma/3$ the system represents a new discrete quantum version of the nonlinear Schrödinger equation, and in the limit ϵ going to γ the system reduces to the QDNLS system, which has the standard Poisson bracket. These properties hold true also for the corresponding quantum systems (by replacing Poisson brackets with commutators).

(iii) Except for the case $\epsilon=0$, this system is nonintegrable so that the QISM is of no use in its quantization.

From a physical point of view this system describes the propagation of molecular excitations in the presence of both resonance interaction and molecular vibrations coupled with low-frequency phonons. For the above properties we call this system the general discrete nonlinear Schrödinger (GDNLS) equation. In this paper we concentrate only on the quantum GDNLS (QGDNLS) equation, leaving the classical analysis to a planned forthcoming paper. In the following we show that the QGDNLS equation is a q deformation of the quantum DNLS (QDNLS) equation, the corresponding quantum group being the q-Heisenberg group. Quantum groups have been shown to be of importance in physics, especially in the fields of statistical mechanics, quantum field theory, and quantum optics [10–12]. From a mathematical point of view, quantum groups are usually introduced in connection with integrable systems. In the case of the QGDNLS system, the q-Heisenberg group, it is shown to characterize both integrable and nonintegrable fields. To this end, we derive in the simplest case of a two-particle chain some explicit formulas for the first excited levels of the QGDNLS equation, and we show that they can be continuously deformed into the corresponding ones of the QDNLS and of the quantum AL (QAL) systems. Thus although the classical behaviors of the AL and of the DNLS systems are quite different (one is integrable, the other is chaotic), their quantum problems are both exactly solvable.

We start by introducing the Hamiltonian operator of the QGDNLS system

$$H=\sum_{i=1}^{f}\left[-A_i^\dagger(A_{i+1}+A_{i-1})+\frac{(2+\omega_i+\eta)}{\ln(1+\epsilon/\eta)}\times\ln\left[1+\frac{\epsilon}{\eta}A_i^\dagger A_i\right]-\eta A_i^\dagger A_i\right]\ , \tag{3}$$

where f denotes the number of degrees of freedom and periodic boundary conditions $A_{f+i}=A_i$ are assumed.

The operators $A_i^\dagger$ and A_i are creation and annihilation operators satisfying the following nonstandard commutation relations

$$[A_i^\dagger, A_j^\dagger]=[A_i, A_j]=0, \quad [A_i, A_j^\dagger]=[1+(\epsilon/\eta)A_i^\dagger A_j]\delta_{ij} , \tag{4}$$

with η depending on ϵ and on a free parameter γ in the following manner:

$$\eta=\frac{2\epsilon}{\gamma-\epsilon} . \tag{5}$$

Note that commutation relations (4) can be seen as a deformation of the usual ones for the boson creation and annihilation operators and are related to the ones defying the quantum Heisenberg group of the q oscillator [13]. By using Eqs. (3) and (4), one gets from the Heisenberg equations of motion

$$i\dot{A}_i=[A_i,H] , \tag{6}$$

the QGDNLS system

$$i\dot{A}_i-(\omega_i+2-\epsilon A_i^\dagger A_i)A_i + [1+(\epsilon/\eta)A_i^\dagger A_i](A_{i+1}+A_{i-1})=0 , \tag{7}$$

with $i=1,2,..,f$. By properly choosing the deformation parameter ϵ, we can obtain from system (7) more conventional quantum nonlinear Schrödinger equations on the lattice. Indeed, when $\epsilon=0$ we have from Eq. (5) that system (7) reduces to the quantum Ablowitz-Ladik system [6,9]

$$i\dot{A}_i-(\omega_i+2)A_i+[1+(\gamma/2)A_i^\dagger A_i](A_{i+1}+A_{i-1})=0 , \tag{8}$$

with the Hamiltonian and commutation relations respectively obtained from Eqs. (3) and (4) with $\epsilon=\eta=0$, and from Eq. (5), $\epsilon/\eta=\gamma/2$. When $\epsilon=\gamma/3$ the QGDNLS system represents a new discretization of the quantum nonlinear Schrödinger equation

$$i\dot{A}_i-[\omega_i+2-(\gamma/3)A_i^\dagger A_i]A_i + [1+(\gamma/3)A_i^\dagger A_i](A_{i+1}+A_{i-1})=0 , \tag{9}$$

while in the limit $\epsilon\rightarrow\gamma$, we have from Eq. (5) that system (7) becomes the QDNLS equation

$$i\dot{A}_i+A_{i+1}-2A_i+A_{i-1}+\gamma A_i^\dagger A_i A_i-\omega_i A_i=0 . \tag{10}$$

Note that η diverges as $\epsilon\rightarrow\gamma$, but $\epsilon/\eta\rightarrow 0$ so that the commutation relations in Eq. (4) reduce to the usual ones for bosonic creation and annihilation operators. By using Eq. (5), one easily verifies that

$$\lim_{\eta\rightarrow\infty}\left[\frac{\eta}{\ln(1+\epsilon/\eta)}\ln\left[1+\frac{\epsilon}{\eta}A_i^\dagger A_i\right]-\eta A_i^\dagger A_i\right]_{\epsilon=\gamma} = -(\gamma/2)(A_i^\dagger A_i)^2+(\gamma/2)A_i^\dagger A_i , \tag{11}$$

so that in the limit $\epsilon\rightarrow\gamma$ the Hamiltonian (3) approaches the corresponding Hamiltonian of the QDNLS system

$$H=\sum_{i=1}^{f}[-A_i^\dagger(A_{i+1}+A_{i-1})-\gamma/2(A_i^\dagger A_i)^2 + (\gamma/2+2+\omega_i)A_i^\dagger A_i] . \tag{12}$$

The physical meaning of the above inclusion of the AL system into a family of nonintegrable DNLS equations is readily seen by comparing the corresponding classical equation of (7) (GDNLS equation) with the simplest classical equation describing the propagation of a molecular excitation [14,15], namely,

$$i\dot{A}_i-\omega_i A_i+J(A_{i+1}+A_{i-1})=0 . \tag{13}$$

In this last equation A_i is the complex mode amplitude of a particular molecular vibration, ω_i is the site frequency of this vibration, and J is the next-neighbor resonance interaction energy. Assuming coupling of the ω_i's to low-frequency phonons (lattice distortion) leads, in an adiabatic and small-field approximation, to a dependence upon local energy as

$$\omega_i=\omega_{0i}+\omega_{1i}|A_i|^2 . \tag{14}$$

This is the anharmonicity of a standard polaron and it corresponds to the second term of Eq. (7). Similarly, coupling of the resonance interaction to low-frequency phonons leads to

$$J=J_0+J_1|A_i|^2 , \tag{15}$$

which is represented by the third term of Eq. (7). In general one expects both these terms to be present and this is just what the present extended system realizes.

The energy levels of the above quantum discretizations of the nonlinear Schrödinger equation can be obtained as deformations of the corresponding levels of system (7) in the following manner. In order to solve the quantum problem of the QGDNLS system, we introduce the Fock space $\mathcal{F}_k$ corresponding to the creation operator $A_k^\dagger$ and its usual basis $|0\rangle_k, |1\rangle_k, \ldots, |j\rangle_k, \ldots$, defined as

$$\begin{aligned} &A_k|0\rangle_k=0 , \\ &A_k^\dagger|n\rangle_k=\sqrt{\beta_n}|n+1\rangle_k , \\ &A_k|n\rangle_k=\sqrt{\alpha_n}|n-1\rangle_k . \end{aligned} \tag{16}$$

By using the commutation relations (4) and requiring the orthonormality of the above basis, we easily obtain the action of the $A_k^\dagger$, A_k operators on the basis states

$$\begin{aligned} &A_k^\dagger|n\rangle_k=\{\eta[(1+\epsilon/\eta)^{n+1}-1]/\epsilon\}^{1/2}|n+1\rangle_k , \\ &A_k|n\rangle_k=\{\eta[(1+\epsilon/\eta)^{n}-1]/\epsilon\}^{1/2}|n-1\rangle_k . \end{aligned} \tag{17}$$

Note that these relations reduce to the usual ones for bosonic operators when the deformation parameter ϵ goes to γ and they give an explicit representation of the quantum Heisenberg group [10]. In terms of q numbers, notation equations (17) read as

$$\begin{aligned} &A_k^\dagger|n\rangle_k=\sqrt{[n+1]_q}|n+1\rangle_k , \\ &A_k|n\rangle_k=\sqrt{[n]_q}|n-1\rangle_k , \end{aligned} \tag{18}$$

where $q=1+\epsilon/\gamma$ and $[n]_q$ is defined as

$[n]_q=(q^n-1)/(q-1)$. As a consequence we see that the number operator of the QGDNLS system is the same as the one of the q oscillator, i.e.,

$$N=\frac{1}{\ln(1+\epsilon/\eta)}\sum_{j=1}^{f}\ln\left[1+\frac{\epsilon}{\eta}A_j^{\dagger}A_j\right], \tag{19}$$

as it is easily verified using relations (17). An important property is that the number (19) commutes with the Hamiltonian (3)

$$[N,H]=0, \tag{20}$$

so that one can decompose the Hilbert space of quantum states into the direct sum

$$\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n\oplus\cdots, \tag{21}$$

where $\mathcal{H}_n$ denotes the eigenspaces of N with n as an eigenvalue whose dimension is

$$\dim(\mathcal{H}_n)=\frac{(n+f-1)!}{n!(f-1)!}. \tag{22}$$

Due to this decomposition, the Hamiltonian can be separately diagonalized in the finite-dimensional eigenspaces $\mathcal{H}_n$, thus reducing the infinite-dimensional eigenvalue problem for H to a simple algebraic problem. For a finite number of degrees of freedom f, the Hilbert space $\mathcal{H}$ of the quantum states is identified with the tensor product

$$\mathcal{H}=\mathcal{F}_1\otimes\mathcal{F}_2\otimes\cdots\otimes\mathcal{F}_f, \tag{23}$$

with the generic element of the corresponding product basis given by

$$|j_1j_2\cdots j_f\rangle=|j_1\rangle_1\otimes|j_2\rangle_2\otimes\cdots\otimes|j_f\rangle_f. \tag{24}$$

In this representation the matrix elements of the restriction of the Hamiltonian H to $\mathcal{H}_n$ are given by

$$\begin{aligned}\langle i_1i_2\cdots i_f|H|j_1j_2\cdots j_f\rangle=&\sum_k\left[(\omega_k+\eta)j_k-\frac{\eta^2}{\epsilon}[(1+\epsilon/\eta)^{j_k}-1]+2n\right]\delta_{i_1j_1}\delta_{i_2j_2}\cdots\delta_{i_fj_f}\\&-\sum_k\eta/\epsilon(\{[(1+\epsilon/\eta)^{j_{k-1}+1}-1][(1+\epsilon/\eta)^{j_k}-1]\}^{1/2}\delta_{i_{k-1}j_{k-1}+1}\delta_{i_kj_k-1}\\&+\{[(1+\epsilon/\eta)^{j_{k+1}+1}-1][(1+\epsilon/\eta)^{j_k}-1]\}^{1/2}\delta_{i_{k+1}j_{k+1}+1}\delta_{i_kj_{k+1}-1}),\end{aligned} \tag{25}$$

where $i_1+i_2+\cdots+i_f=j_1+j_2+\cdots+j_f=n$. From Eq. (25) the eigenvalues and eigenvectors of H are readily obtained by diagonalizing the corresponding $(f+n-1)!/[(f-1)!n!]$ matrix representation. For example, let us compute the first quantum levels for the two degrees of freedom $(f=2)$ QGDNLS system in the resonant case $\omega_1=\omega_2=\Omega$.

For $n=1$ the secular equation corresponding to the 2×2 matrix (25) is readily written as

$$\det\begin{vmatrix}2-E & -2\\ -2 & 2-E\end{vmatrix}=0, \tag{26}$$

from which we have $E_1=0$, $E_2=4$; i.e., classical anharmonicity does not enter in the determination of the quantum levels. For $n=2$ anharmonicity enters. Indeed the secular equation for the 3×3 matrix (25) is

$$\det\begin{vmatrix}4+2\Omega-\epsilon-E & -2\sqrt{2+\epsilon/\eta} & 0\\ -2\sqrt{2+\epsilon/\eta} & 4+2\Omega-E & -2\sqrt{2+\epsilon/\eta}\\ 0 & -2\sqrt{2+\epsilon/\eta} & 4+2\Omega-\epsilon-E\end{vmatrix}=0, \tag{27}$$

from which we get

$$\begin{aligned}&E_1=4+\Omega-\epsilon,\\&E_{2,3}=4+\Omega-\epsilon/2\pm[(\epsilon/2)^2+8(2+\epsilon/\eta)]^{1/2}.\end{aligned} \tag{28}$$

We see that for $\epsilon=0$ these levels coincide with those of the QAL system $E_1=4+\Omega$, $E_{2,3}=4+\Omega\pm2\sqrt{4+\gamma}$ derived in Ref. [9], while for $\epsilon=\gamma$ they reduce to the levels of the QDNLS system $E_1=4+\Omega-\gamma$, $E_{2,3}=4+\Omega-\gamma/2\pm[(\gamma/2)^2+16]^{1/2}$. In Fig. 1 we have plotted the quantum levels (28) as a function of the deformation parameter ϵ, with η given by Eq. (5), $\Omega=0$ and $\gamma=2$. From this figure we see that the quantum levels of

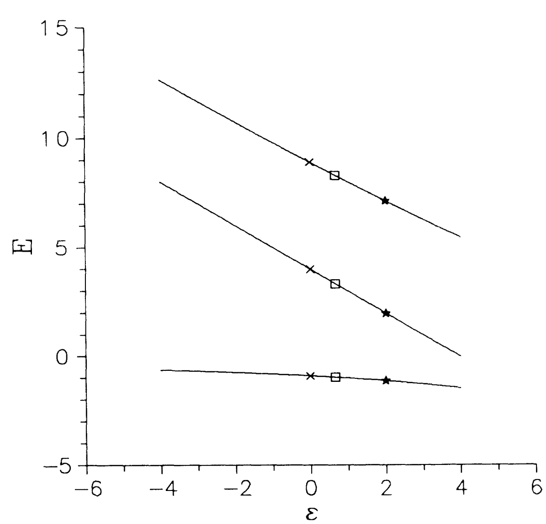

FIG. 1. The quantum levels of the two-degrees-of-freedom QGDNLS equation vs ϵ with η given by Eq. (5), $\Omega=0$ and $\gamma=2$. The crosses refer to the levels of the QAL system, the squares to the ones of system (9), and the stars to those of the QDNLS system.

the QAL system (in the figure denoted by crosses) are continuously deformed into the levels of the QDNLS equation (denoted by stars) as ϵ is increased from zero to γ.

This fact holds true for arbitrary n and for any number of degrees of freedom, having in general matrix (25) a band structure that makes the problem of computing the spectrum particularly suitable from a numerical point of view. Finally we remark that, except the case $\epsilon=0$, the QISM approach is of no use to solve the quantum problem of the QGDNLS equation with more than two freedoms ($f>2$), because this system is classically nonintegrable.

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