Perturbed factorization of the symmetric-anharmonic-oscillator eigenequation

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(Received 27 April 1992)

The perturbed-ladder-operator method is applied to the analytical solution of the harmonic-oscillator eigenequation perturbed by a symmetric potential $V(x)$. This method, well adapted for computer algebra, is an extension of the original Schrödinger-Infeld-Hull factorization method within the perturbative scheme and allows an analytical solution of nonfactorizable Sturm-Liouville eigenequations in almost the same way as factorizable ones. Closed-form expressions of the perturbed harmonic-oscillator eigenvalues are obtained by means of a few algebraic manipulations, either in a series of binomial functions $\binom{v}{u}$ or in a series of powers $(v+\frac{1}{2})^u$. Alternative expansions of the perturbed potential $V(x)$ in a series of Hermite polynomials $\mathcal{H}_{2s}(x)$ or in a series of x^{2s} are considered and some illustrative examples demonstrating the capabilities of the method are given. Particularly, analytical expressions of the x^4-perturbed harmonic-oscillator energies, as well as analytical approximations of the eigenenergies for the $x^2+\lambda x^2/(1+gx^2)$ interaction, are quickly derived.

PACS number(s): 03.65.Fd, 31.30.−i

I. INTRODUCTION

In two previous papers [1,2] (hereafter referred to as paper I and paper II, respectively), the "perturbed-ladder-operator" method has been proposed for providing an algebraic recursive solution of perturbed wave equations. This method is an extension of the Schrödinger-Infeld-Hull factorization method [3,4] within the perturbative scheme. It is particularly well adapted for treating problems which can be conveniently described by a "kernel potential," leading to an Infeld-Hull factorizable equation, together with an additional perturbation. As a matter of fact, such kernel potentials, and therefore solutions of factorizable equations (factorizable types A to E, within the Infeld-Hull nomenclature), are involved in many physical models. Among solutions of factorizable equations of fundamental interest in atomic and molecular physics applications, let us quote, for instance, the spherical harmonic or symmetric top functions (factorizable type A), the Morse-oscillator functions (factorizable type B), the nonrotating or rotating harmonic-oscillator functions (factorizable type D or C), the Schrödinger (and Dirac) hydrogenic radial functions in the usual Euclidean flat space (factorizable type F) or in a space of constant curvature (factorizable type E) and, more generally, the Gauss or confluent hypergeometric functions (general factorizable type A or B). Briefly stated, many equations of current interest in physics can be viewed as "perturbed factorizable" equations and are relevant to the perturbed-ladder-operator method.

Summarizing *grosso modo* the principle of the "perturbed factorization" technique, one assumes that the perturbed potential function, as well as the ladder and factorization functions, can be expanded in a perturbation series. Then, one tries to build up the required perturbed ladder operators and the perturbed factorization functions allowing the factorization of the perturbed equation at any rank N of the perturbation. The efficiency of the procedure mainly relies on the use of suitable associated basis functions $y_s(x)$ and $Y_s(x)$ for expanding the perturbation and the perturbed part of the ladder function, respectively. These basis functions have to satisfy selective ladderlike properties and, also, have to lead to a manageable finite-difference solution of the factorizability condition giving the required perturbed functions a suitable dependence on the quantum number. Once the perturbed ladder and factorization functions have been found, the perturbed problem may be handled in the same way as the exact factorizable (unperturbed) problem: analytical expressions of the perturbed eigenvalues in terms of the quantum numbers are readily obtained from the knowledge of the perturbed factorization function and the complete set of the perturbed eigenfunctions can be generated by repeated application of the ladder operator on the perturbed "key function," which is a solution of a first-order differential equation.

In paper I, the main features of the perturbed-ladder-operator method have been given and general formulas have been derived allowing the "perturbed factorization" of eigenequations which correspond to unperturbed ladder operators which are linear functions of the quantum number (factorizable types A and D). Although valid for any factorizable type, these formulas, when applied to the last two factorization types E and F, where the unperturbed ladder function is not a linear function of the quantum number, lead to rather lengthy and intricate calculations. Therefore, in paper II, the method has been reformulated for analytically solving the perturbed Coulomb (type-F) eigenequation and, by the way, it has been found that the introduction of specific basis functions, instead of a standard Newton's expansion, greatly simplifies the finite-difference solution of the factorizability condition. Thus, it appears that the capabilities of the perturbed factorization scheme have not yet been completely exploited, even for the cases where the unperturbed ladder function is a linear function of the quantum number.

In the present paper, special attention is paid to the symmetric-anharmonic-oscillator eigenequation (perturbed type-D factorization). There has always been a great deal of interest in the analytical solution of this eigenequation which, apart from certain very particular cases [5], is not exactly solvable analytically while, also, it is the simplest case of a "perturbed factorizable" eigenequation. Moreover, it seems well suited for giving a deeper insight into the perturbed factorization procedure. After a necessary and brief reminder of the exact and perturbed factorization schemes (Sec. II), we focus on perturbed type D. Alternative solutions of the factorizability condition, which lead to closed-form expressions of the perturbed harmonic-oscillator energies either in a series of binomial functions $\binom{v}{u}$ or, more classically, in a series of power functions $(v+\frac{1}{2})^u$, are carried out. Suitable expansions of the perturbation either in a series of Hermite polynomials $\mathcal{H}_{2s}(x)$ or in power series x^{2s} have been considered (Sec. III). In Sec. IV, illustrative applications are given. General expressions for the perturbed eigenvalues have been written down, up to the third order of the perturbation. The solution of the x^4-perturbed harmonic oscillator, which has been studied many times and by various methods (WKB, Padé approximant, Hill determinant, hypervirial, perturbative variational methods, etc.), is chosen as a test example of the capabilities of the perturbed factorization procedure. Finally, the solution of the Schrödinger equation with a potential function $x^2+\lambda x^2/(1+gx^2)$, which is of interest in several areas of physics, is considered.

II. FACTORIZATION SCHEME

In order to set up the definitions and notations, it is first necessary to briefly recall the main features of the exact and perturbed factorization schemes.

A. Exact factorization

After exact or approximate separation of variables and appropriate transformations of variable and function, many eigenequations of current interest in quantum mechanics can be reduced to the standard form

$$\left[\frac{d^2}{dx^2}+U(x,m)+\Lambda_j\right]\Psi_{jm}(x)=0 \tag{2.1}$$

associated with the boundary conditions $(x_1 \le x \le x_2)$

$$|\Psi(x_1)|^2=|\Psi(x_2)|^2=0, \quad \int_{x_1}^{x_2}|\Psi(x)|^2dx=1 , \tag{2.2}$$

where $m=m_0,\ m_0+1, m_0+2, \ldots$ is a quantum number which takes successive discrete values labeling the eigenfunctions.

Such an equation (2.1) is factorizable when it can be replaced by each of the following two difference-differential equations:

$$\begin{aligned} H^-_{m+1}H^+_{m+1}\Psi_{jm}&=[\Lambda_j-L(m+1)]\Psi_{jm} , \\ H^+_m H^-_m\Psi_{jm}&=[\Lambda_j-L(m)]\Psi_{jm} , \end{aligned} \tag{2.3}$$

where $L(m)$ is the factorization function, which does not depend on x, and $H^{\pm}_m$ are mutually adjoint ladder operators: $H^{\pm}_m=K(x,m)\mp(d/dx)$. Owing to the mutual adjointness of the ladder operators H^+_m and H^-_m, the necessary condition for the existence of quadratically integrable solutions of Eq. (2.1), i.e., the quantization condition, is

$$\varepsilon(j-m)=v=\text{integer}\ge 0 ,$$

where $\varepsilon=+1$ (or $\varepsilon=-1$) according to whether $L(m)$ is an increasing (or decreasing) function of m.

The interest and advantages of the factorization method are well known [4]:

(i) Closed-form expressions of the eigenvalues are readily obtainable from the knowledge of the factorization function $L(m)$:

$$\Lambda_j=L\left[j+\frac{\varepsilon}{2}+\frac{1}{2}\right] . \tag{2.4}$$

(ii) The normalized eigenfunctions are solutions of the following pair of difference-differential equations:

$$\begin{aligned} \left[K(x,m)+\frac{d}{dx}\right]\Psi_{jm}&=\mathcal{N}_j(m)\Psi_{jm-1} , \\ \left[K(x,m+1)-\frac{d}{dx}\right]\Psi_{jm}&=\mathcal{N}_j(m+1)\Psi_{jm+1} \end{aligned} \tag{2.5}$$

with $\mathcal{N}_j(m)=[\Lambda_j-L(m)]^{1/2}$.

These "ladder" equations allow the determination of any $\Psi_{jm}(x)$ function from the knowledge of any one of them, particularly from the knowledge of the normalized "key" function $\Psi_{jj}(x)$ which is the solution of the first-order differential equation

$$\left[K\left[x,j+\frac{\varepsilon}{2}+\frac{1}{2}\right]-\varepsilon\frac{d}{dx}\right]\Psi_{jj}(x)=0 . \tag{2.6}$$

In fact, when an eigenequation is exactly factorizable, closed-form expressions of the eigenfunctions involving classical orthogonal polynomials are known [6].

There are six fundamental types of potential functions $U^{(0)}(x,m)$ (denoted types A to F, within the Infeld-Hull nomenclature) leading to factorizable equations. Moreover, as pointed out by Infeld and Hull [4], when direct factorization is not possible solely because of the inadequate m dependence of the potential function $U(x,m)$ under consideration, one can resort to "artificial" factorization, i.e., one can consider $U(x,m)$ as "embedded" in a new potential function $u(x,m;\mu)$ which depends on a supplementary "artificial" parameter μ such that $u(x,m;\mu)$ can be identified in m with a factorizing potential $U^{(0)}(x,m)$ and that $u(x,m;\mu=m)=U(x,m)$. Then, Eq. (2.1) is factorized using $u(x,m;\mu)$, and the eigenvalues $\Lambda_j(\mu)=L(j+\varepsilon/2+\frac{1}{2};\mu)$ are determined as well as the eigenfunctions $\Psi_{jm}(x;\mu)$, both depending on the parameter μ. At the end of the ladder procedure (2.5), one merely sets $\mu=m$ and obtains the required eigenvalues

$\Lambda_j(m)=\Lambda_j(\mu=m)$ and eigenfunctions $\Psi_{jm}(x)=\Psi_{jm}(x;\mu=m)$. This "artificial" or "embedded" factorization device is widely used all along the "perturbed factorization" scheme.

B. Perturbed factorization

Let us now consider an eigenequation (2.1) where the potential function $U(x,m)$ does not belong to any of the six Infeld-Hull factorization types, and let us assume that this potential function, as well as the associated ladder and factorization functions $K(x,m)$ and $L(m)$ to be found, can be expanded in a perturbation series with a parameter η,

$$\begin{aligned} U(x,m)&=U^{(0)}(x,m)+\eta U^{(1)}(x,m)+\eta^2U^{(2)}(x,m)+\cdots ,\\ K(x,m)&=K^{(0)}(x,m)+\eta K^{(1)}(x,m)+\eta^2K^{(2)}(x,m)+\cdots , \\ L(m)&=L^{(0)}(m)+\eta L^{(1)}(m)+\eta^2L^{(2)}(m)+\cdots , \end{aligned} \tag{2.7}$$

where $K^{(0)}(x,m)$ and $L^{(0)}(m)$ are the ladder and factorization functions allowing an exact factorization of Eq. (2.1) with $U^{(0)}(x,m)$.

As it has been shown in paper I, the critical point of this extension of the factorization method within the perturbation scheme relies on the choice of suitable x-basis functions $y_s(x)$ and $Y_s(x)$ for expanding the required factorizing perturbations $U^{(N)}(x,m)$ and associated perturbed ladder functions $K^{(N)}(x,m)$, respectively. These basis functions, which are specific to each factorization type, have to satisfy the following "ladderlike" relations:

$$\begin{aligned} 2K^{(0)}(x,m)Y_s(x)&=A_s(m)y_s(x)+B_s(m)y_{s+1}(x) ,\\ \frac{dY_s}{dx}&=\alpha_s y_s(x)+\beta_s y_{s+1}(x) , \end{aligned} \tag{2.8}$$

$$Y_s(x)Y_t(x)=\sum_r h(s,t,r)y_r(x) . \tag{2.9}$$

We set

$$K^{(N)}(x,m)=\sum_{s=0}^{S_N} \gamma_s^{(N)}(m)Y_s(x) \tag{2.10}$$

and, as a consequence of Eq. (2.9), the potential-like function $\mathcal{W}^{(N)}(x,m)$, which at each order of the perturbation is generated from the preceding orders of the perturbation and is involved in the factorizability condition, can be expanded in a series of $y_s(x)$:

$$\mathcal{W}^{(N)}(x,m)=\sum_{\nu=1}^{N-1} K^{(\nu)}(x,m)K^{(N-\nu)}(x,m) =\sum_s W_s^{(N)}(m)y_s(x) . \tag{2.11}$$

Thus, using the artificial factorization device with an artificial parameter μ (see paper I), one can solve physicomathematical problems with a potential function $V(x,m)$ such as

$$V(x,m)=U^{(0)}(x,m)+\eta V^{(1)}(x)+\eta^2V^{(2)}(x)+\cdots , \tag{2.12}$$

where the $V^{(\nu)}(x)$ have the same dependence on x as the $U^{(\nu)}(x,m)$, i.e., $V^{(\nu)}(x)=U^{(\nu)}(x;m=\mu)$ and

$$V^{(N)}(x)=\sum_{s=1}^{S_N+1} b_s^{(N)}y_s(x) . \tag{2.13}$$

Hence, as it has been shown in paper II, we set

$$K^{(N)}(x,m)=\sum_{s=0}^{S_N} Y_s(x)Q_s(m)\{k_s^{(N)}+F_s^{(N)}(m)\} . \tag{2.14}$$

Then, solving the factorizability condition, one gets the following expressions of $Q_s(m)$ and $k_s^{(N)}$, involving the $A_s(m)$, $B_s(m)$, α_s, and β_s coefficients, which are specific to the factorization type and associated basis $y_s(x)$ and $Y_s(x)$ under consideration:

$$Q_s(m)=\prod_{j=1}^{m-1}\frac{[B_s(j)-\beta_s]}{[B_s(j+1)+\beta_s]} , \tag{2.15}$$

$$k_s^{(N)}=\sum_{u=s+1}^{S_N+1} d_{us}(\mu)\{b_u^{(N)}+W_u^{(N)}(\mu)\} , \tag{2.16}$$

where

$$d_{us}(\mu)=-\prod_{t=s+1}^{u-1}\{A_t(\mu)-\alpha_t\}\Big/Q_s(\mu)\prod_{t=s}^{u-1}\{B_t(\mu)-\beta_t\} .$$

The required $F_s^{(N)}(m)$ and factorization $L^{(N)}(m)$ functions are then solutions of the following finite-difference equations:

$$\Delta F_{s-1}^{(N)}=-\frac{\Delta W_s^{(N)}(m)+\Delta\{[A_s(m)+\alpha_s]\gamma_s^{(N)}(m)\}+2\alpha_s\gamma_s^{(N)}(m)}{Q_{s-1}(m+1)\{B_{s-1}(m+1)+\beta_{s-1}\}} , \tag{2.17}$$

$$\Delta L^{(N)}=-\Delta W_0^{(N)}(m)-\Delta\{[A_0(m)+\alpha_0]\gamma_0^{(N)}(m)\}-2\alpha_0\gamma_0^{(N)}(m) , \tag{2.18}$$

where $\Delta F(m)=F(m+1)-F(m)$ is the usual first difference Δ operator in m.

$$\gamma_s^{(N)}(m)=Q_s(m)\{k_s^{(N)}+F_s^{(N)}(m)\} ,$$

$$F_s(m)=0 \quad \text{for } s=s_N$$

and the following associated conditions have to be fulfilled:

$$\begin{aligned} &F_s(m=\mu)=0 ,\\ &L^{(N)}(m=\mu)=-W_0^{(N)}(\mu)-[A_0(\mu)-\alpha_0]\gamma_0^{(N)}(\mu) . \end{aligned} \tag{2.19}$$

At each order N of the perturbation, the finite-difference equations (2.17) and (2.18) will be solved recursively, starting from $s=S_N$ down to $s=1$.

The factorizability condition is solved recursively, i.e., when considering the determination of $K^{(N)}(x,m;\mu)$ and $L^{(N)}(m;\mu)$, both depending on the artificial parameter μ, it is assumed that all the $K^{(\nu)}(x,m;\mu)$ and $L^{(\nu)}(m;\mu)$ for $\nu=1,2,\ldots,N-1$ have already been found. Hence, the perturbed problem (up to the Nth order of the perturbation) can be handled in the same way as the exact factorizable (unperturbed) problem.

(i) The total perturbed eigenvalue and associated ladder function are

$$\Lambda_j(m)=L^{(0)}\left[j+\frac{\varepsilon}{2}+\frac{1}{2}\right] + \sum_{\nu=1}^{N}\eta^\nu L^{(\nu)}\left[m=j+\frac{\varepsilon}{2}+\frac{1}{2};\mu=m\right] , \quad (2.20)$$

$$K(x,m;\mu)=K^{(0)}(x,m)+\sum_{\nu=1}^{N}\eta^\nu K^{(\nu)}(x,m;\mu) , \quad (2.21)$$

where $\varepsilon=+1$ (or $\varepsilon=-1$) according to whether the unperturbed factorization function $L^{(0)}(m)$ is an increasing (or decreasing) function of m.

(ii) The ladder equations (2.5) and (2.6) hold with $K(x,m;\mu)$ for the determination of the perturbed eigenfunctions $\Psi_{jm}(x;\mu)$. Once the ladder process is achieved, one sets $\mu=m$ and obtains the required $\Psi_{jm}(x;m)$ perturbed eigenfunctions. One can also use an alternative procedure which provides the perturbed eigenfunctions as linear combinations of the unperturbed eigenfunctions [7,8].

Let us now apply these general results to the solution of the anharmonic-oscillator eigenequation.

III. PERTURBED FACTORIZATION OF THE ANHARMONIC-OSCILLATOR EIGENEQUATION

Let us consider the anharmonic-oscillator eigenequation:

$$\left[\frac{d^2}{dx^2}-b^2x^2+b(2m+1)+V(x)+\Lambda\right]\Psi_{jm}(x)=0 , \quad (3.1)$$

where $-\infty<x<\infty$ and $V(x)=\eta V^{(1)}(x)+\eta^2V^{(2)}(x)+\cdots$ is a perturbation.

A. Exact factorization of the unperturbed eigenequation

When $V(x)=0$, the eigenequation (3.1) reduces to an exact Infeld-Hull type-D factorizable equation with the following factorizing ladder and factorization functions:

$$K^{(0)}(x,m)=bx, \quad L^{(0)}(m)=-2bm . \quad (3.2)$$

Let us assume $b>0$ [9]. The factorization function $L^{(0)}(m)$ is a decreasing function of m. This is a class-II problem with $\varepsilon=-1$, $v=m-j$, $\Lambda^{(0)}=L^{(0)}(j)=-2bj$, and we get the following expression of the harmonic-oscillator energy:

$$E_v^{(0)}=\tfrac{1}{2}[\Lambda^{(0)}+b(2m+1)]=b(v+\tfrac{1}{2}) . \quad (3.3)$$

The normalized unperturbed eigenfunctions are [6]

$$\Psi_v^{(0)}(x)=(b/\pi)^{1/4}(1/2^v v!)^{1/2}\exp(-\tfrac{1}{2}bx^2)\mathcal{H}_v(b^{1/2}x) , \quad (3.4)$$

where $\mathcal{H}_v(x)$ is a Hermite polynomial of degree v.

When $V(x)\neq0$, the different possible choices of the x-basis functions $y_s(x)$ and $Y_s(x)$ satisfying the ladderlike properties (2.8) lead to different possible perturbed factorizations of the eigenequation (3.1) (see Table I).

B. $\chi_s\mathcal{H}_{2s}(b^{1/2}x)$ expansion of the perturbation associated with a $\binom{v}{u}$ expansion of the perturbed eigenvalue

As pointed out in paper I, when dealing with type-D factorization, it is rewarding to expand the perturbations $V^{(N)}(x)$ in a series of Hermite polynomials $\mathcal{H}_{2s}(b^{1/2}x)$ rather than in a series of the familiar x^{2s} basis. Moreover, the associated use of a Hermite polynomial x-basis and binomial functions, for the m dependence of the ladder and factorization functions leads to a compact expression of the first-order energy involving only one summation instead of two summations when using other basis functions (see, for instance, Ref. [8]). It is then rewarding to first work out the expressions of the perturbed harmonic-oscillator eigenvalues and associated ladder functions when expanding the perturbations $V^{(N)}(x)$ and eigenvalues $\Lambda_v^{(N)}$ in a series of Hermite polynomials $\mathcal{H}_{2s}(b^{1/2}x)$ and binomial coefficients $\binom{v}{u}$, respectively.

Let us choose the associated x-basis functions

$$y_s=\chi_s\mathcal{H}_{2s}(b^{1/2}x)$$

and

$$Y_s=b^{-1/2}\chi_{s+1}\mathcal{H}_{2s+1}(b^{1/2}x) ,$$

where the factor $\chi_s=s!b/(2s)!$ is introduced for computational convenience. This choice is convenient (see Table I) and we set

$$V^{(N)}(x)=\sum_{s=1}^{S_N+1}b_s^{(N)}\chi_s\mathcal{H}_{2s}(b^{1/2}x) , \quad (3.5)$$

TABLE I. Perturbed factorizations of the anharmonic-oscillator eigenequation (3.1). $(m)_t=m(m-1)\cdots(m-t+1)$ is a generalized factorial; $\delta_{u,s}$ is the Kronecker symbol.

y_s	x^{2s}	$\chi_s\mathcal{H}_{2s}(b^{1/2}x)$
Y_s	x^{2s+1}	$b^{-1/2}\chi_{s+1}\mathcal{H}_{2s+1}(b^{1/2}x)$
$A_s(m)$	0	1
$B_s(m)$	$2b$	1
α_s	$2s+1$	1
β_s	0	0
$Q_s(m)$	1	1
$d_{us}(\mu)$	$\frac{1}{2}\left[-\frac{1}{b}\right]^{u-s}(u-\frac{1}{2})_{u-s-1}$	$-\delta_{u,s+1}$

$$K^{(N)}(x,m)=b^{-1/2}\sum_{s=0}^{S_N}\chi_{s+1}\mathcal{H}_{2s+1}(b^{1/2}x)\gamma_s^{(N)}(m) , \quad (3.6)$$

where [see Eq. (2.14) and note that $Q_s(m)=1$]

$$\gamma_s^{(N)}(m)=k_s^{(N)}+F_s^{(N)}(m)=k_s^{(N)}+Z_s^{(N)}(m)-Z_s^{(N)}(\mu) .$$

The first condition (2.19), i.e., $F_s(m=\mu)=0$, is *ipso facto* fulfilled by setting $F_s^{(N)}(m)=Z_s^{(N)}(m)-Z_s^{(N)}(\mu)$.

In order to build up the potential-like function $\mathcal{W}^{(N)}(x,m)$, the following multiplication formulas will be used (see paper I):

$$Y_sY_t=\sum_{u=|s-t|}^{s+t+1}h(s,t,u)y_u , \quad (3.7)$$

where

$$h(s,t,u)=\frac{2^{s+t-u-1}(2u)!s!t!}{u!(s+u-t)!(t+u-s)!(s+t+1-u)!} .$$

Thus, the potential-like function can be written:

$$\mathcal{W}^{(N)}(x,m)=\sum_{\nu=1}^{N-1}K^{(\nu)}(x,m)K^{(N-\nu)}(x,m)$$
$$=\sum_{s=0}^{S_N}W_s^{(N)}(m)\chi_s\mathcal{H}_{2s}(b^{1/2}x) . \quad (3.8)$$

At the first-order $N=1$ of the perturbation, we have $\mathcal{W}^{(1)}(x,m)=0$ and the upper bound S_1 which is involved in $K^{(1)}(x,m)$ can be arbitrarily chosen. At the higher orders $N>1$, the highest power of x is already fixed as data following from the preceding orders and the relation $S_N=S_\nu+S_{N-\nu}+1$ must hold for any ν ($\nu=1$ to $N-1$): the value of S_N depends on S_1 and N. One finds the following necessary condition to be fulfilled:

$$S_N=NS_1+N-1 . \quad (3.9)$$

One has now to determine the ladder $Z_s^{(N)}(m)$ and the factorization $L^{(N)}(m)$ functions which are solutions of the finite-difference equations (2.17) and (2.18) and have to satisfy the conditions (2.19). Since $B_s(m)$ does not depend on m and $\beta_s=0$ (see Table I), the finite summation of Eqs. (2.17) and (2.18) can be partly carried out. One gets, within an arbitrary summation constant,

$$Z_{s-1}^{(N)}(m)=-\frac{1}{B_{s-1}}\{W_s^{(N)}(m)+[A_s(m)+\alpha_s]\gamma_s^{(N)}(m)$$
$$+2\alpha_s\Delta^{-1}\gamma_s^{(N)}\} , \quad (3.10)$$

$$L^{(N)}(m)=-\{W_0^{(N)}(m)+[A_0(m)+\alpha_0]\gamma_0^{(N)}(m)$$
$$+2\alpha_0\Delta^{-1}\gamma_0^{(N)}\} . \quad (3.11)$$

As a matter of fact, these equations will hold and serve for the three perturbed factorizations under consideration in the present paper.

1. Determination of the perturbed ladder function

The required $Z_s^{(N)}(m)$ function is the solution of the finite-difference equation (3.10) with $A_s(m)=B_s(m)=\alpha_s=1$ (see Table I), i.e.,

$$Z_{s-1}^{(N)}(m)=-\{W_s^{(N)}(m)+2[k_s^{(N)}-Z_s^{(N)}(\mu)+Z_s^{(N)}(m)]$$
$$+2\Delta^{-1}[k_s^{(N)}-Z_s^{(N)}(\mu)+Z_s^{(N)}(m)]\} . \quad (3.12)$$

Before tackling the solution of this equation, let us remind the reader that, at the final step of the artificial factorization process, in order to obtain the analytical expression of the perturbed eigenvalue $\Lambda_j^{(N)}(m)$ in terms of the quantum numbers j and m, we set $m=j$ and $\mu=m$ in the final expression of the perturbed factorization function $L^{(N)}(m;\mu)$. Since we require $\Lambda_j^{(N)}(m)$ in a series of binomial coefficients $\binom{v}{u}$, where $v=m-j$, we require the final expression of $L^{(N)}(m;\mu)$ in a series of $\binom{\mu-m}{u}$. Therefore, we set

$$Z_s^{(N)}(m)=\sum_{t=1}^{S_N-s}C_s^{(N)}(t)\begin{bmatrix}\mu-m\\ t\end{bmatrix} , \quad (3.13)$$

$$W_s^{(N)}(m)=\sum_{t=0}^{S_N-s}w_s^{(N)}(t)\begin{bmatrix}\mu-m\\ t\end{bmatrix} . \quad (3.14)$$

Using the relation [10]

$$\Delta^{-1}\begin{bmatrix}\mu-m\\ t\end{bmatrix}=-\begin{bmatrix}\mu-m\\ t+1\end{bmatrix}-\begin{bmatrix}\mu-m\\ t\end{bmatrix} ,$$

we get, after some rearrangements,

$$\Delta^{-1}Z_s^{(N)}(m)=-\sum_{t=1}^{S_N-s}\begin{bmatrix}\mu-m\\ t\end{bmatrix}C_s^{(N)}(t)$$
$$-\sum_{t=2}^{S_N-s+1}\begin{bmatrix}\mu-m\\ t\end{bmatrix}C_s^{(N)}(t-1) . \quad (3.15)$$

After choosing the arbitrary summation constant so that $Z_{s-1}^{(N)}(m)$ keeps the same form (3.13) as $Z_s^{(N)}(m)$, we obtain

$$Z_{s-1}^{(N)}(m)=-\sum_{t=1}^{S_N-s}w_s^{(N)}(t)\begin{bmatrix}\mu-m\\ t\end{bmatrix}+2k_s^{(N)}\begin{bmatrix}\mu-m\\ 1\end{bmatrix}$$
$$+2\sum_{t=2}^{S_N-s+1}C_s^{(N)}(t-1)\begin{bmatrix}\mu-m\\ t\end{bmatrix} . \quad (3.16)$$

Now, comparing this expression for $Z_{s-1}^{(N)}(m)$ with its standard expression (3.13) and setting $k_s^{(N)}=C_s^{(N)}(0)$, we get the following recurrence formula, which holds for $1\le u\le S_N-s+1$:

$$C_{s-1}^{(N)}(u)=-w_s^{(N)}(u)+2C_s^{(N)}(u-1) . \quad (3.17)$$

Starting from $s=S_N+1$ down to $s=1$ and from $u=S_N-s+1$ down to $u=1$, this relation allows a recursive determination of the $C_s^{(N)}(u)$ in terms of the $k_s^{(N)}$. One has now to obtain analytical expressions of the $k_s^{(N)}$ in terms of the data, i.e., in terms of the expansion coefficients $b_s^{(N)}$ of the given perturbation $V^{(N)}(x)$. For this purpose, we use Eq. (2.16) together with the expression $d_{us}=-\delta_{u,s+1}$ of Table I. We get

$$k_{s-1}^{(N)} = -[b_s^{(N)} + W_s^{(N)}(\mu)]$$

and we obtain the following single recurrence formula allowing the determination of the $C_s^{(N)}(k)$ for $0 \le k \le S_N - s + 1$:

$$C_{s-1}^{(N)}(k) = -b_s^{(N)}(k) + 2C_s^{(N)}(k-1) , \tag{3.18}$$

where, since $W_s^{(N)}(\mu) = w_s^{(N)}(0)$ [see Eq. (3.14)],

$$b_s^{(N)}(0) = b_s^{(N)} + w_s^{(N)}(0) , \quad b_s^{(N)}(k) = w_s^{(N)}(k) . \tag{3.19}$$

Using this recurrence formula successively from $s = S_N + 1$ down to $s = 1$, it is easily found that the following closed-form expression holds:

$$C_{S_N-\sigma}^{(N)}(k) = -\sum_{j=0}^{k} 2^j b_{S_N+1-\sigma+j}^{(N)}(k-j) . \tag{3.20}$$

Finally, at each order N of the perturbation, one has at one's disposal the following closed-form expression of the perturbed ladder function in terms of the "data coefficients" $b_s^{(N)}(j)$:

$$K^{(N)}(x,m;\mu) = -b^{-1/2} \sum_{s=0}^{S_N} \chi_{s+1} \mathcal{H}_{2s+1}(b^{1/2}x) \sum_{t=0}^{S_N-s} \begin{bmatrix} \mu - m \\ t \end{bmatrix} \sum_{j=0}^{t} 2^{t-j} b_{s+1+t-j}^{(N)}(j) . \tag{3.21}$$

2. Determination of the perturbed eigenvalue

The perturbed factorization function is a solution of the finite-difference equation (3.11) with $A_0(m) = \alpha_0 = 1$, and the associated condition (2.19) to be fulfilled reduces to $L^{(N)}(m = \mu) = -W_0^{(N)}(\mu)$. We get

$$L^{(N)}(m;\mu) = -W_0^{(N)}(m) + 2 \sum_{t=1}^{S_N+1} C_0^{(N)}(t-1) \begin{bmatrix} \mu - m \\ t \end{bmatrix} . \tag{3.22}$$

Consequently, after substituting v for $(\mu - m)$ into this expression of $L^{(N)}(m;\mu)$ and using the expression (3.20) of the $C_{S_N-\sigma}^{(N)}(k)$, the following expression of the perturbed eigenvalue is obtained:

$$\Lambda_v^{(N)} = -\sum_{j=0}^{S_N+1} \sum_{t=0}^{S_N+1-j} 2^t \begin{bmatrix} v \\ t+j \end{bmatrix} b_t^{(N)}(j) \tag{3.23}$$

or, alternatively,

$$\Lambda_v^{(N)} = -\sum_{t=0}^{S_N+1} \begin{bmatrix} v \\ t \end{bmatrix} \sum_{j=0}^{S_N+1-t} 2^{t-j} b_{t-j}^{(N)}(j) . \tag{3.24}$$

At each order N of the perturbation, the determination of the perturbed eigenvalue amounts to the determination of the data coefficients $b_s^{(N)}(j)$ which depend upon the particular problem under consideration [see Eq. (3.19)]. In fact, the $b_s^{(N)}(j)$ coefficients involve the expansion coefficients of the perturbation $V^{(N)}(x)$ in a series of $y_s(x) = \chi_s \mathcal{H}_{2s}(b^{1/2}x)$ together with the expansion coefficients $w_s^{(N)}(j)$ of the potential-like function

$$\mathcal{W}^{(N)}(x,m) = \sum_{v=1}^{N-1} K^{(v)} K^{(N-v)}$$

in a series of $\chi_s \mathcal{H}_{2s}(b^{1/2}x)\binom{\mu-m}{j}$ [see Eqs. (3.8) and (3.14)].

Let us note that, at the first order ($N = 1$) of the perturbation, since the data coefficients $b_s^{(1)}(j)$ reduce to the expansion coefficients $b_s^{(1)}$ of the perturbation $V^{(1)}(x)$ in a series of $\chi_s \mathcal{H}_{2s}(b^{1/2}x)$ we find again the already known [8] compact expression of the first-order perturbed eigenvalue:

$$\Lambda_v^{(1)} = -\sum_{s=1}^{S_1+1} 2^s \begin{bmatrix} v \\ s \end{bmatrix} b_s^{(1)} .$$

Before applying the general results of the present section to specific cases, let us investigate an alternative perturbed factorization of the anharmonic-oscillator eigenequation (3.1) when, as usually done, the perturbation is expanded in a series of powers of x.

C. x^{2s} expansion of the perturbation associated with a $\binom{v}{u}$ expansion of the perturbed eigenvalue

Let us now assume that the perturbations $V^{(N)}(x)$ can be expanded in a series of x^{2s} and choose the associated basis functions $y_s = x^{2s}$ and $Y_s = x^{2s+1}$. We set [see Eq. (2.13)]

$$V^{(N)}(x) = \sum_{s=1}^{S_N+1} b_s^{(N)} x^{2s} . \tag{3.25}$$

Since $Q_s(m) = 1$ (see Table I), the perturbed ladder function is [see Eqs. (2.14)]

$$K^{(N)}(x,m) = \sum_{s=0}^{S_N} x^{2s+1} \gamma_s^{(N)}(m) , \tag{3.26}$$

where

$$\gamma_s^{(N)}(m) = k_s^{(N)} + Z_s^{(N)}(m) - Z_s^{(N)}(\mu) .$$

The potential-like function is

$$\mathcal{W}^{(N)}(x,m) = \sum_{v=1}^{N-1} K^{(v)}(x,m) K^{(N-v)}(x,m)$$
$$= \sum_{s=1}^{S_N} W_s^{(N)}(m) x^{2s} , \tag{3.27}$$

where S_N is still defined by Eq. (3.9).

1. Determination of the perturbed ladder function

In order to obtain the perturbed eigenvalues $\Lambda_v^{(N)}$ in a series of binomial functions $\binom{v}{u}$, we assume that the $Z_s^{(N)}(m)$ and $W_s^{(N)}(m)$ functions are still given by Eqs.

(3.13) and (3.14). Since $B_s(m)=2b$ and $\beta_s=0$ (see Table I), the required $Z_s^{(N)}(m)$ function is the solution of the finite-difference equation (3.10). Noting that $Z_s^{(N)}(m=\mu)=0$, and using Eq. (3.10) together with Table I, we get, after choosing the arbitrary constant so that $Z_{s-1}^{(N)}(m)$ keeps the same form (3.13) as $Z_s^{(N)}(m)$,

$$Z_{s-1}^{(N)}(m)=-\frac{1}{2b}\left[W_s^{(N)}(m)-2\alpha_s\begin{bmatrix}\mu-m\\1\end{bmatrix}k_s^{(N)}+\alpha_s Z_s^{(N)}(m)+2\alpha_s\Delta^{-1}Z_s^{(N)}(m)\right], \tag{3.28}$$

where $Z_s^{(N)}(m)$, $W_s^{(N)}(m)$, and $\Delta^{-1}Z_s^{(N)}(m)$ are given by Eqs. (3.13), (3.14), and (3.15), respectively.

From the comparison of this expression (3.28) of $Z_{s-1}^{(N)}(m)$ with its standard expression (3.13), we get

$$C_{s-1}^{(N)}(1)=-\frac{1}{2b}\{w_s^{(N)}(1)-2\alpha_s k_s^{(N)}-\alpha_s C_s^{(N)}(1)\} \tag{3.29}$$

and, for $2\le t\le S_N-s+1$,

$$C_{s-1}^{(N)}(t)=-\frac{1}{2b}\{w_s^{(N)}(t)-2\alpha_s C_s^{(N)}(t-1)-\alpha_s C_s^{(N)}(t)\}\ . \tag{3.30}$$

Starting from $s=S_N+1$ down to $s=1$ and from $u=S_N-s+1$ down to $u=1$, these relations allow a recursive determination of the $C_s^{(N)}(u)$ in terms of the $k_s^{(N)}$. In order to obtain analytical expressions of the $k_s^{(N)}$ in terms of the expansion coefficients $b_s^{(N)}$ of the given perturbation $V^{(N)}(x)$, we use Eq. (2.16) together with the expression of the $d_{us}(\mu)$ coefficients of Table I and we get

$$k_s^{(N)}=\tfrac{1}{2}\sum_{u=s+1}^{S_N+1}\left[-\frac{1}{b}\right]^{u-s}(u-\tfrac{1}{2})_{u-s-1}\{b_u^{(N)}+W_u^{(N)}(\mu)\}\ . \tag{3.31}$$

Consequently, we have

$$k_{s-1}^{(N)}=-\frac{1}{2b}\{b_s^{(N)}+W_s^{(N)}(\mu)-\alpha_s k_s^{(N)}\}\ . \tag{3.32}$$

Now, introducing the notation $k_s^{(N)}=C_s^{(N)}(0)$ and setting $\alpha_s=2s+1$ (see Table I), it is easily found that the following single recurrence relation holds ($0\le k\le S_N-s+1$):

$$C_{s-1}^{(N)}(k)=-\frac{1}{2b}\{b_s^{(N)}(k)-(2s+1)[2C_s^{(N)}(k-1)+C_s^{(N)}(k)]\}\ , \tag{3.33}$$

where, since $W_s^{(N)}(\mu)=w_s^{(N)}(0)$ [see, Eq. (3.14)], the $b_s^{(N)}(k)$ are still given by Eq. (3.19).

Using this recurrence relation for $s=S_N+1,S_N,S_N-1,\ldots$ it can be inferred that one can write (see Appendix A)

$$C_{S_N-\sigma}^{(N)}(k)=-\sum_{j=0}^{k}\sum_{u=j}^{j+\sigma-k}\left[\frac{1}{2b}\right]^{\sigma+1-u}\times d_j(\sigma-u,k,S_N-u)\times b_{S_N+1-u}^{(N)}(j)\ , \tag{3.34}$$

where the $d_j(\sigma,k,s)$ coefficients obey the recurrence formula

$$d_j(\sigma+1,k,s)=(2s-2\sigma+1)\{2d_j(\sigma,k-1,s)+d_j(\sigma,k,s)\} \tag{3.35}$$

with the associated conditions

$$d_k(0,k,s)=1\ ,$$

$$d_k(\sigma+1,k,s)=(2s-2\sigma+1)d_k(\sigma,k,s)\quad\text{for } k=j$$

and

$$d_j(k-j,k,s+k-j-1)=2(2s+1)d_j(k-j-1,k-1,s+k-j-1)\quad\text{for } 0\le j\le k-1\ .$$

At this level, let us emphasize that the $d_j(\sigma,k,s)$ coefficients depend neither on the order N of the perturbation nor on the particular problem under consideration. Their determination can be performed, once and for all, by means of the recurrence formula (3.35). Moreover, taking advantage of the underlying connection between the former perturbed factorization of the same eigenequation (3.1) and this last one, the following closed-form expression can be derived (see Appendix B):

$$d_j(\sigma,t,s)=(\tfrac{1}{2})^{\sigma-t+j}\frac{(2s+1)!(s-\sigma)!}{s!(2s-2\sigma+1)!}\begin{bmatrix}\sigma\\t-j\end{bmatrix}. \tag{3.36}$$

Finally, we obtain, via the expression (3.34) of the $C_{S_N-\sigma}^{(N)}(k)$ coefficients, the analytical expression of the perturbed ladder function in terms of the data coefficients $b_s^{(N)}(j)$:

$$K^{(N)}(x,m;\mu)=\sum_{s=0}^{S_N}x^{2s+1}\sum_{t=0}^{S_N-s}C_s^{(N)}(t)\begin{bmatrix}\mu-m\\t\end{bmatrix}, \tag{3.37}$$

where

$$C_s^{(N)}(t)=-\sum_{u=s+1}^{S_N+1}\left[\frac{1}{4b}\right]^{u-s}\sum_{j=0}^{t}\frac{2^{t-j}(2u)!s!}{u!(2s+1)!}\times\begin{bmatrix}u-s-1\\t-j\end{bmatrix}b_u^{(N)}(j)\ .$$

2. Determination of the perturbed eigenvalue

The perturbed factorization function $L^{(N)}(m;\mu)$ is solution of the finite-difference equation (3.11) where $A_0(m)=0$ and $\alpha_0=1$ (see Table I). After introducing the expansion (3.13) of $Z_0^{(N)}(m)$ and noting that $W_0^{(N)}(m)=0$ [see Eq. (3.27)], we obtain

$$L^{(N)}(m;\mu)=\sum_{t=0}^{S_N}\begin{bmatrix}\mu-m\\ t\end{bmatrix}C_0^{(N)}(t)+\sum_{t=0}^{S_N+1}2\begin{bmatrix}\mu-m\\ t\end{bmatrix}C_0^{(N)}(t-1)+\mathcal{L}\ , \tag{3.38}$$

where $\mathcal{L}$ is an arbitrary summation constant.

The associated condition to be fulfilled is $L^{(N)}(m=\mu)=k_0^{(N)}=C_0^{(N)}(0)$ [see Eq. (2.19) and Table I], and it is easily checked that $\mathcal{L}=0$.

Finally, substituting m with j and μ with m in the expression (3.38) of $L^{(N)}(m;\mu)$ and setting $m-j=v$, we obtain

$$\Lambda_v^{(N)}=\sum_{u=0}^{S_N+1}\lambda_u^{(N)}\begin{bmatrix}v\\ u\end{bmatrix}\ , \tag{3.39}$$

where $\lambda_u^{(N)}=C_0^{(N)}(u)+2C_0^{(N)}(u-1)$. Noting that, formally, $\lambda_u^{(N)}=2bC_{-1}^{(N)}(u)$ and using Eq. (3.37), we get

$$\lambda_u^{(N)}=-\sum_{j=0}^{u}\sum_{t=u-j}^{S_N+1-j}\left[\frac{1}{4b}\right]^t 2^{u-j}\frac{(2t)!}{(t)!}\times\begin{bmatrix}t\\ u-j\end{bmatrix}b_t^{(N)}(j)\ . \tag{3.40}$$

Rearranging the terms, the following alternative expression of the perturbed eigenvalue is obtained:

$$\Lambda_v^{(N)}=\sum_{j=0}^{S_N+1}\sum_{t=1}^{S_N+1-j}b_t^{(N)}(j)\mathcal{J}_t(j)\ , \tag{3.41}$$

where

$$\mathcal{J}_t(j)=-\frac{(2t-1)!!}{(2b)^t}\sum_{k=0}^{t}2^k\begin{bmatrix}t\\ k\end{bmatrix}\begin{bmatrix}v\\ k+j\end{bmatrix}\ , \tag{3.42}$$

and

$$(2t-1)!!=1\times 2\times 3\times\cdots\times(2t-3)(2t-1)=\frac{(2t)!}{t!2^t}$$

is a double factorial.

Let us note that, as a by-product of the method, a closed-form expression of the diagonal integrals $\langle x^{2t}\rangle$ between the unperturbed harmonic-oscillator eigenfunctions $\Psi_v^{(0)}(x)$ [see Eq. (3.4)] is obtained as a particular case of expression (3.42). Indeed, at the first order ($N=1$) of the perturbation, we have $W_s^{(1)}(m)=0$ [see Eq. (3.27)]; the data coefficients $b_t^{(1)}(j)$ reduce to the expansion coefficients $b_t^{(1)}(0)=b_t^{(1)}$ of the perturbation $V^{(1)}(x)$ and, consequently, the expression (3.41) of the perturbed eigenvalue reduces to

$$\Lambda_v^{(1)}=\sum_{t=1}^{S_1+1}b_t^{(1)}\mathcal{J}_t(0)\ .$$

When comparing this expression of $\Lambda_v^{(1)}$ with its alternative expression within the classical Rayleigh-Schrödinger framework where the coefficient of $b_t^{(1)}=b_t^{(1)}(0)$ is merely the integral $\langle x^{2t}\rangle$, it follows that $\langle x^{2t}\rangle=-\mathcal{J}_t(0)$. From the expression (3.42) of the $\mathcal{J}_t(j)$, one finds again the already known expression [11]

$$\langle x^{2t}\rangle=\frac{(2t-1)!!}{(2b)^t}\sum_{u=0}^{t}2^u\begin{bmatrix}t\\ u\end{bmatrix}\begin{bmatrix}v\\ u\end{bmatrix}\ . \tag{3.43}$$

Particularly, we have

$$\begin{aligned}
\langle x^2\rangle&=\frac{1}{2b}\left\{2\begin{bmatrix}v\\ 1\end{bmatrix}+1\right\}\ ,\\
\langle x^4\rangle&=\frac{3}{4b^2}\left\{4\begin{bmatrix}v\\ 2\end{bmatrix}+4\begin{bmatrix}v\\ 1\end{bmatrix}+1\right\}\ ,\\
\langle x^6\rangle&=\frac{15}{8b^3}\left\{8\begin{bmatrix}v\\ 3\end{bmatrix}+12\begin{bmatrix}v\\ 2\end{bmatrix}+6\begin{bmatrix}v\\ 1\end{bmatrix}+1\right\}\ ,\\
\langle x^8\rangle&=\frac{105}{16b^4}\left\{16\begin{bmatrix}v\\ 4\end{bmatrix}+32\begin{bmatrix}v\\ 3\end{bmatrix}+24\begin{bmatrix}v\\ 2\end{bmatrix}+8\begin{bmatrix}v\\ 1\end{bmatrix}+1\right\}\ .
\end{aligned} \tag{3.44}$$

In the same way, we get

$$\begin{aligned}
\mathcal{J}_1(1)&=-\frac{1}{2b}\left\{2\begin{bmatrix}v\\ 2\end{bmatrix}+\begin{bmatrix}v\\ 1\end{bmatrix}\right\}\ ,\\
\mathcal{J}_2(1)&=-\frac{3}{4b^2}\left\{4\begin{bmatrix}v\\ 3\end{bmatrix}+4\begin{bmatrix}v\\ 2\end{bmatrix}+\begin{bmatrix}v\\ 1\end{bmatrix}\right\}\ ,\\
\mathcal{J}_3(1)&=-\frac{15}{8b^3}\left\{8\begin{bmatrix}v\\ 4\end{bmatrix}+12\begin{bmatrix}v\\ 3\end{bmatrix}+6\begin{bmatrix}v\\ 2\end{bmatrix}+\begin{bmatrix}v\\ 1\end{bmatrix}\right\}\ ,\\
\mathcal{J}_1(2)&=-\frac{1}{2b}\left\{2\begin{bmatrix}v\\ 3\end{bmatrix}+\begin{bmatrix}v\\ 2\end{bmatrix}\right\}\ ,\\
\mathcal{J}_2(2)&=-\frac{3}{4b^3}\left\{4\begin{bmatrix}v\\ 4\end{bmatrix}+4\begin{bmatrix}v\\ 3\end{bmatrix}+\begin{bmatrix}v\\ 2\end{bmatrix}\right\}\ ,\\
\mathcal{J}_1(3)&=-\frac{1}{2b}\left\{2\begin{bmatrix}v\\ 4\end{bmatrix}+\begin{bmatrix}v\\ 3\end{bmatrix}\right\}\ .
\end{aligned} \tag{3.45}$$

Finally, using either expression (3.40) or expression (3.41), at each order N of the perturbation, the determination of the perturbed eigenvalue reduces to the determination of the data coefficients $b_s^{(N)}(j)$, which depend on the particular problem under consideration.

The anharmonic-oscillator energies are usually computed in a series of $(v+\frac{1}{2})^u$. Particularly, for the case of a quartic anharmonic perturbation, the perturbed eigenvalues of order N are known [12] to be polynomials in $(v+\frac{1}{2})$ of degree $N+1$ and with partity $(-1)^{N+1}$. It is then worthwhile to work out an alternative perturbed factorization of the anharmonic-oscillator eigenequation leading to such an expression of the perturbed eigenvalues.

D. x^{2s} expansion of the perturbation associated with a $(v+\frac{1}{2})^u$ expansion of the perturbed eigenvalue

The perturbation $V^{(N)}(x)$ as well as the perturbed ladder function $K^{(N)}(x,m)$ and the potential-like function $\mathcal{W}^{(N)}(x,m)$ are still given by Eqs. (3.25), (3.26), and (3.27), respectively.

1. Determination of the perturbed ladder function

The required $Z_s^{(N)}(m)$ function is a solution of the finite-difference equation (3.10) with $A_s(m)=\beta_s=0$, $B_s(m)=2b$, and $\alpha_s=2s+1$ (see Table I).

Since we require the final expression of the perturbed eigenvalue in a series of powers of $(v+\frac{1}{2})$, the perturbed factorization function $L^{(N)}(m;\mu)$ has to be expanded in a series of powers of $(m-\mu-\frac{1}{2})$ and we set

$$Z_s^{(N)}(m)=\sum_{t=1}^{S_N-s} C_s^{(N)}(t)(m-\mu-\tfrac{1}{2})^t . \tag{3.46}$$

In order to solve the finite-difference equation (3.10), we require an analytical expression of $\Delta^{-1}Z_s^{(N)}(m)$. Let us introduce the Bernoulli polynomials,

$$\varphi_k(m)=\frac{1}{k!}\sum_{u=0}^{k}\begin{bmatrix}k\\u\end{bmatrix}\mathcal{B}_u m^{k-u} ,$$

and make use of the following relations [10], which are valid within an arbitrary summation constant:

$$\Delta^{-1}m^t=t!\varphi_{t+1}(m)-\frac{1}{t+1}\mathcal{B}_{t+1} = \frac{1}{t+1}\sum_{u=1}^{t+1}\begin{bmatrix}t+1\\u\end{bmatrix}\mathcal{B}_{t+1-u}m^u . \tag{3.47}$$

The $\mathcal{B}_u$ are Bernoulli numbers (see Appendix C). We get

$$\Delta^{-1}Z_s^{(N)}(m)=\sum_{t=1}^{S_N-s} C_s^{(N)}(t)\left[t!\varphi_{t+1}(m-\mu-\tfrac{1}{2})-\frac{1}{t+1}\mathcal{B}_{t+1}\right] = \sum_{u=1}^{S_N-s+1}(m-\mu-\tfrac{1}{2})^u\sum_{t=u-1}^{S_N-s}\frac{C_s^{(N)}(t)}{t+1}\begin{bmatrix}t+1\\u\end{bmatrix}\mathcal{B}_{t+1-u} . \tag{3.48}$$

After choosing the arbitrary summation constant in the expression (3.10) of $Z_{s-1}^{(N)}(m)$ so that $Z_{s-1}^{(N)}(m)$ keeps the same form (3.46) as $Z_s^{(N)}(m)$, we obtain

$$Z_{s-1}^{(N)}(m)=-\frac{1}{2b}\{W_s^{(N)}(m)-w_s^{(N)}(0) +2\alpha_s(m-\mu-\tfrac{1}{2})[k_s^{(N)}-Z_s^{(N)}(\mu)] +\alpha_s Z_s^{(N)}(m)+2\alpha_s\Delta^{-1}Z_s^{(N)}(m)\} , \tag{3.49}$$

where $\Delta^{-1}Z_s^{(N)}(m)$ is defined by Eq. (3.48) and the $W_s^{(N)}(m)$ function can be written [see Eqs. (3.27), (3.26), and (3.46)]

$$W_s^{(N)}(m)=\sum_{t=0}^{S_N-s} w_s^{(N)}(t)(m-\mu-\tfrac{1}{2})^t . \tag{3.50}$$

Then, after introducing into Eq. (3.49) the expansions of the functions $Z_s^{(N)}(m)$, $W_s^{(N)}(m)$, and $\Delta^{-1}Z_s^{(N)}(m)$ in a series of $(m-\mu-\frac{1}{2})^u$ and comparing the resulting expression of $Z_{s-1}^{(N)}(m)$ with its standard expansion (3.46), we get

$$C_{s-1}^{(N)}(1)=-\frac{1}{2b}\left[w_s^{(N)}(1)+2\alpha_s[k_s^{(N)}-Z_s^{(N)}(\mu)] +\alpha_s C_s^{(N)}(1)+2\alpha_s\sum_{t=1}^{S_N-s}C_s^{(N)}(t)\mathcal{B}_t\right] \tag{3.51}$$

and, for $2\le u\le S_N-s+1$,

$$C_{s-1}^{(N)}(u)=-\frac{1}{2b}\left[w_s^{(N)}(u)+\alpha_s C_s^{(N)}(u) +2\alpha_s\sum_{t=u-1}^{S_N-s}\frac{C_s^{(N)}(t)}{t+1}\begin{bmatrix}t+1\\u\end{bmatrix}\mathcal{B}_{t+1-u}\right] . \tag{3.52}$$

Setting $k_s^{(N)}-Z_s^{(N)}(\mu)=C_s^{(N)}(0)$, it is easily checked that this recurrence formula (3.52) also holds for $u=1$. Rearranging the terms, it can be written again ($1\le u \le S_N-s+1$):

$$C_{s-1}^{(N)}(u)=-\frac{1}{2b}\left[w_s^{(N)}(u)+\frac{2\alpha_s}{u}C_s^{(N)}(u-1)+2\alpha_s\sum_{t=u+1}^{S_N-s}\frac{C_s^{(N)}(t)}{t+1}\begin{bmatrix}t+1\\u\end{bmatrix}\mathcal{B}_{t+1-u}\right] . \tag{3.53}$$

Starting from $s=S_N+1$ down to $s=1$ and from $u=S_N-s+1$ down to $u=1$, this relation allows a recursive determination of the $C_s^{(N)}(u)$ in terms of the $C_s^{(N)}(0)=k_s^{(N)}-Z_s^{(N)}(\mu)$.

One has now to obtain analytical expressions of the $C_s^{(N)}(0)=k_s^{(N)}-Z_s^{(N)}(\mu)$ in terms of the expansion coefficients $b_s^{(N)}$ of the given perturbation $V^{(N)}(x)$. On one hand, we have the expression (3.32) giving $k_{s-1}^{(N)}$ in terms of $b_s^{(N)}$ and $k_s^{(N)}$. On the other hand, using the expression (3.49) of $Z_{s-1}^{(N)}(m)$, we get

$$Z_{s-1}^{(N)}(m=\mu)=-\frac{1}{2b}\{W_s^{(N)}(\mu)-w_s^{(N)}(0)-\alpha_s k_s^{(N)}+2\alpha_s[Z_s^{(N)}(\mu)+\Delta^{-1}Z_s^{(N)}(m=\mu)]\}. \tag{3.54}$$

Using the first part of Eq. (3.47) together with its counterpart expression $m^t=t!\Delta\varphi_{t+1}(m)$, we can write

$$m^t+\Delta^{-1}m^t=t!\{\varphi_{t+1}(m)+\Delta\varphi_{t+1}(m)\}-\frac{1}{t+1}\mathcal{B}_{t+1}=t!\varphi_{t+1}(m+1)-\frac{1}{t+1}\mathcal{B}_{t+1} \tag{3.55}$$

and we get

$$Z_s^{(N)}(\mu)+\Delta^{-1}Z_s^{(N)}(m=\mu)=\sum_{t=1}^{S_N-s}C_s^{(N)}(t)\left[t!\varphi_{t+1}(\tfrac{1}{2})-\frac{1}{t+1}\mathcal{B}_{t+1}\right]. \tag{3.56}$$

Then, using this expression together with Eqs. (3.32) and (3.54), and reminding the reader that [13] $k!\varphi_k(\frac{1}{2})=-\{1-\frac{1}{2}^{k-1}\}\mathcal{B}_k$, we obtain the required expression

$$C_{s-1}^{(N)}(0)=-\frac{1}{2b}\left[b_s^{(N)}+w_s^{(N)}(0)+2\alpha_s\sum_{t=1}^{S_N-s}C_s^{(N)}(t)\times\frac{2}{t+1}\{1-(\tfrac{1}{2})^{t+1}\}\mathcal{B}_{t+1}\right]. \tag{3.57}$$

Finally, after setting $\alpha_s=2s+1$ (see Table I), relations (3.53) and (3.57) reduce to the single recurrence formula

$$C_{s-1}^{(N)}(k)=-\frac{1}{2b}\left[b_s^{(N)}(k)+\frac{4s+2}{k}C_s^{(N)}(k-1)+(4s+2)\sum_{t=k+1,2}^{S_N-s}a_{kt}C_s^{(N)}(t)\right], \tag{3.58}$$

where

$$b_s^{(N)}(0)=b_s^{(N)}+w_s^{(N)}(0),\quad b_s^{(N)}(k)=w_s^{(N)}(k)\quad \text{for } 1\le k\le S_N-s \tag{3.59}$$

$$a_{0t}=\frac{2}{t+1}\{1-(\tfrac{1}{2})^{t+1}\}\mathcal{B}_{t+1},$$

$$a_{kt}=\frac{1}{t+1}\binom{t+1}{k}\mathcal{B}_{t+1-k}.$$

Note that since, except $\mathcal{B}_1=-\frac{1}{2}$, the Bernoulli numbers $\mathcal{B}_k$ with odd subscripts k are all zero, the t summation involved in Eq. (3.58) works by steps of two units. It will be seen hereafter that this property of the Bernoulli numbers leads to the expected property [12] that the perturbed eigenvalues are polynomials in $(v+\frac{1}{2})$ of definite parity.

Using this recurrence formula (3.58) successively from $s=S_N+1$, down to $s=1$, it can be inferred that one can write (see Appendix A)

$$C_{S_N-\sigma}^{(N)}(k)=\sum_{j=0}^{\sigma}\sum_{u=j}^{\sigma+j-k}\left[-\frac{1}{2b}\right]^{\sigma+1-u}d_j(\sigma-u,k,S_N-u)\times b_{S_N+1-u}^{(N)}(j), \tag{3.60}$$

where $d_j(0,j,S_N-j)=1$ and the $d_j(\sigma,k,s)$ satisfy the following recurrence formula:

$$d_j(\sigma+1,k,s)=2(2s-2\sigma+1)\left[\frac{1}{k}d_j(\sigma,k-1,s)+\sum_{t=k+1,2}^{\sigma+j}a_{kt}d_j(\sigma,t,s)\right]. \tag{3.61}$$

This recurrence formula depends neither on the order N of the perturbation nor on the data specific to the problem under consideration. Setting

$$d_j(\sigma,k,s)=\frac{(2s+1)!(s-\sigma)!j!}{s!(2s-2\sigma+1)!k!}\Theta_j(\sigma,k), \tag{3.62}$$

it is found that the $\Theta_j(\sigma,k)$ satisfy the recurrence formula

$$\Theta_j(\sigma+1,k)=\sum_{u=0}^{u_M}\mathcal{A}_{ku}\Theta_j(\sigma,k+2u-1), \tag{3.63}$$

where u_M is the integer part of $(\sigma+j-k+1)/2$, $\Theta_j(0,j)=1$, $\Theta_j(0,k)=0$ for $k<j$, and

$$\mathcal{A}_{0u}=\frac{2\mathcal{B}_{2u}}{(2u)!}\{1-(\tfrac{1}{2})^{2u}\},\quad \mathcal{A}_{ku}=\frac{\mathcal{B}_{2u}}{(2u)!}\quad \text{for any } k\neq 0. \tag{3.64}$$

Finally, the analytical expression of the perturbed ladder function in terms of the data coefficients $b_u^{(N)}(j)$ is

$$K^{(N)}(x,m;\mu)=\sum_{s=0}^{S_N}x^{2s+1}\sum_{t=0}^{S_N-s}C_s^{(N)}(t)(m-\mu-\tfrac{1}{2})^t, \tag{3.65}$$

where

$$C_s^{(N)}(t)=\tfrac{1}{2}\sum_{u=1}^{S_N+1}\left[-\frac{1}{b}\right]^{u-s}\frac{(2u-1)!!}{t!(2s+1)!!}\times\sum_{j=0}^{S_N-s-1}j!\Theta_j(u-s-1,t)b_u^{(N)}(j).$$

The $\Theta_j(\sigma,k)$ are easily obtainable, once and for all, by means of the recurrence formula (3.63). Some values have been reported in Appendix C.

2. Determination of the perturbed eigenvalue

The perturbed factorization function $L^{(N)}(m)$ is solution of the finite-difference equation (3.11) with $W_0^{(N)}(m)=0$ [see Eq. (3.27)]. After introducing the m-basis functions $(m-\mu-\frac{1}{2})^u$, we get

$$L^{(N)}(m;\mu)=-2\alpha_0(m-\mu-\tfrac{1}{2})\{k_0^{(N)}-Z_0^{(N)}(\mu)\} -\alpha_0 Z_0^{(N)}(m)-2\alpha_0\Delta^{-1}Z_0^{(N)}(m)+\mathcal{L}\ , \quad (3.66)$$

where $\Delta^{-1}Z_0^{(N)}(m)$ is defined by Eq. (3.47) and $\mathcal{L}$ is an arbitrary summation constant.

The associated condition to be fulfilled is [see Eq. (2.19)]

$$L^{(N)}(m=\mu;\mu)=\alpha_0 k_0^{(N)}$$

and, as a consequence, the arbitrary summation constant is found to be

$$\mathcal{L}=2\alpha_0\{Z_0^{(N)}(\mu)+\Delta^{-1}Z_0^{(N)}(m=\mu)\} =-2\alpha_0\sum_{t=1}^{S_N} a_{0t}C_0^{(N)}(t)\ .$$

Then, using the expansions of $Z_0^{(N)}(m)$ and $\Delta^{-1}Z_0^{(N)}(m)$ in a series of $(m-\mu-\frac{1}{2})^u$, substituting m for μ and $-(v+\frac{1}{2})$ for $(m-\mu-\frac{1}{2})$ into the expression (3.66) of $L^{(N)}(m;\mu)$, and keeping in mind that $\alpha_0=1$, we obtain the following expression of the perturbed eigenvalue:

$$\Lambda_v^{(N)}=\sum_{u=0}^{S_N+1}\lambda_u^{(N)}(v+\tfrac{1}{2})^u\ , \quad (3.67)$$

where

$$\lambda_u^{(N)}=(-1)^{u+1}\left[\frac{2}{u}C_0(u-1)+2\sum_{t=u+1,2}^{S_N}a_{ut}C_0^{(N)}(t)\right]\ .$$

Noting that one can write, formally, $\lambda_u^{(N)}=2b(-1)^u C_{-1}(u)$, we get

$$\lambda_u^{(N)}=(-1)^{u+1}\sum_{j=0}^{S_N+1}\sum_{t=u-j}^{S_N+1-j}\left[-\frac{1}{2b}\right]^t d_j(t,u,t-1) \times b_t^{(N)}(j)\ . \quad (3.68)$$

Rearranging the terms and introducing the expression (3.62) of the $d_j(\sigma,k,s)$, it is easily found that the general expression (3.41) of the perturbed eigenvalue $\Lambda_v^{(N)}$ still holds where

$$\mathcal{J}_t(j)=(-1)^{j+1}\frac{(2t-1)!!}{b^t} \times\sum_{k=0,2}^{t+j}\frac{j!\Theta_j(t,t+j-k)}{(t+j-k)!}(v+\tfrac{1}{2})^{t+j-k}\ . \quad (3.69)$$

Following from the property that $\Theta_j(t,k)$ vanishes unless k is of the same parity as $(t+j)$, the k summation works by steps of two units.

As a by-product of the method, we obtain a closed-term expression of the diagonal integrals $\langle x^{2t}\rangle$ between the unperturbed harmonic-oscillator eigenfunctions $\Psi_v^{(0)}(x)$ in a series of $(v+\frac{1}{2})^u$. We have $\langle x^{2t}\rangle=-\mathcal{J}_t(0)$, i.e.,

$$\langle x^{2t}\rangle=\frac{(2t-1)!!}{b^t}\sum_{k=0,2}^{t}\frac{\Theta_0(t,t-k)}{(t-k)!}(v+\tfrac{1}{2})^{t-k}\ . \quad (3.70)$$

For instance, we get

$$\begin{aligned}
\langle x^2\rangle&=\frac{1}{b}(v+\tfrac{1}{2})\ ,\\
\langle x^4\rangle&=\frac{3}{2b^2}\{(v+\tfrac{1}{2})^2+\tfrac{1}{4}\}\ ,\\
\langle x^6\rangle&=\frac{5}{2b^3}\{(v+\tfrac{1}{2})^3+\tfrac{5}{4}(v+\tfrac{1}{2})\}\ ,\\
\langle x^8\rangle&=\frac{35}{8b^4}\{(v+\tfrac{1}{2})^4+\tfrac{7}{2}(v+\tfrac{1}{2})^2+\tfrac{9}{16}\}\ .
\end{aligned} \quad (3.71)$$

In the same way, we get

$$\begin{aligned}
\mathcal{J}_1(1)&=\frac{1}{2b}\{(v+\tfrac{1}{2})^2+\tfrac{1}{4}\}\ ,\\
\mathcal{J}_2(1)&=\frac{1}{2b^2}\{(v+\tfrac{1}{2})^3+\tfrac{5}{4}(v+\tfrac{1}{2})\}\ ,\\
\mathcal{J}_3(1)&=\frac{5}{8b^3}\{(v+\tfrac{1}{2})^4+\tfrac{7}{2}(v+\tfrac{1}{2})^2+\tfrac{9}{16}\}\ ,\\
\mathcal{J}_1(2)&=-\frac{1}{3b}\{(v+\tfrac{1}{2})^3+\tfrac{1}{2}(v+\tfrac{1}{2})\}\ ,\\
\mathcal{J}_2(2)&=-\frac{1}{4b^2}\{(v+\tfrac{1}{2})^4+2(v+\tfrac{1}{2})^2+\tfrac{3}{16}\}\ ,\\
\mathcal{J}_1(3)&=\frac{1}{4b}\{(v+\tfrac{1}{2})^4+(v+\tfrac{1}{2})^2-\tfrac{1}{16}\}\ .
\end{aligned} \quad (3.72)$$

Note that while the expressions (3.71) of the $\langle x^{2t}\rangle$ can be found again by using the former expressions (3.43) together with the expansion of the binomial functions $\binom{v}{u}$ in a series of $(v+\frac{1}{2})^k$, this is not the case for the expressions of the pseudointegrals $\mathcal{J}_t(j)$ which are specific to the perturbed factorization case under consideration.

Summarizing the results, at each other N of the perturbation, the determination of an analytical expression of the perturbed eigenvalue $\Lambda_v^{(N)}$ and ladder function $K^{(N)}(x,m;\mu)$, associated with the perturbation $V^{(N)}(x)$, merely amounts to the computation of the data coefficients $b_t^{(N)}(j)$. For the three perturbed factorizations considered in the present paper, the data coefficients $b_t^{(N)}(j)$ are defined by Eq. (3.19) and involve the expansion coefficients of the perturbation $V^{(N)}(x)$ in a series of $y_t(x)$ together with the expansion coefficients $w_t^{(N)}(j)$ of the potential-like function

$$\mathcal{W}^{(N)}(x,m)=\sum_{v=1}^{N-1}K^{(v)}(x,m;\mu)K^{(N-v)}(x,m;\mu)$$

in a series of $y_t(x)\binom{\mu-m}{j}$, or $y_t(x)(m-\mu-\frac{1}{2})^j$, according to the factorization case under consideration. When dealing with extensive perturbations and/or high orders of the perturbation, the following expression of the $w_t^{(N)}(j)$ in terms of the $b_u^{(v)}(k)$ of the preceding orders of the perturbation is well adapted for microcomputer programming and can be used (see Appendix D):

$$w_t^{(N)}(j)=\sum_{v=1}^{N-1}\sum_{s=1}^{S_{N-v}+1}\sum_{r=1}^{S_v+1}\sum_{l=0}^{S_{N-v}+1-s}\sum_{m=0}^{S_v+1-r} b_s^{(N-v)}(l)b_r^{(v)}(m)\mathcal{X}_i(s,r,t,l,m,j)\ , \tag{3.73}$$

where, in each of the above three cases, a closed-form expression of the "coupling coefficient" $\mathcal{X}_i(s,r,t,l,m,j)$ is available [see Eqs. (D4), (D7), and (D10), respectively].

Let us emphasize that the values of these $\mathcal{X}_i(s,r,t,l,m,j)$ depend neither on the order of the perturbation nor on the particular problem under consideration: Tables and/or subroutines giving these numbers can be made available once and for all and can serve for the analytical solution of any anharmonic eigenequation.

Let us now consider some illustrative and test applications of the method.

IV. ILLUSTRATIVE APPLICATIONS

Since the main purpose of the present paper is to present the method rather than to give new results or extensive tables, we limit ourselves to some short and comparative test examples of the above three perturbed factorization types.

A. General determination of the perturbed eigenvalues and ladder functions

Let us consider the solution of eigenequation (3.1) up to the third order ($N=3$) of the perturbation. In order to avoid writing down too many cumbersome expressions, let us set $b=1$ [14] and assume that the perturbation corresponds to the choice $S_1=1$ and therefore, $S_2=3$ and $S_3=5$ [see Eq. (3.9)].

1. Expansion of the perturbation in a series of Hermite polynomials

The perturbations are

$$\begin{aligned}&V^{(1)}(x)=g_1y_1+g_2y_2\ ,\\&V^{(2)}(x)=h_1y_1+h_2y_2+h_3y_3+h_4y_4\ ,\\&V^{(3)}(x)=p_1y_1+p_2y_2+p_3y_3+p_4y_4+p_5y_5+p_6y_6\ ,\end{aligned}\tag{4.1}$$

where $y_s=y_s(x)=\chi_s\mathcal{H}_{2s}(x)$.

a. First order ($N=1$) of the perturbation ($S_1=1$). The perturbed eigenvalue is [see Eq. (3.24)]

$$\Lambda_v^{(1)}=-2g_1\begin{bmatrix}v\\1\end{bmatrix}-4g_2\begin{bmatrix}v\\2\end{bmatrix}\ . \tag{4.2}$$

The associated perturbed ladder function is [see Eq. (3.21)]

$$K^{(1)}(x,m;\mu)=-\{g_1Y_0+g_2Y_1\}-2\begin{bmatrix}\mu-m\\1\end{bmatrix}g_2Y_0\ , \tag{4.3}$$

where $Y_s=Y_s(x)=\chi_{s+1}\mathcal{H}_{2s+1}(x)$.

b. Second order ($N=2$) of the perturbation ($S_2=3$). One has first to calculate the data coefficients $b_t^{(2)}(j)$. The potential-like function is $\mathcal{W}^{(2)}=(K^{(1)})^2$ [see Eq. (3.8)] and, since we have

$$\begin{bmatrix}\mu-m\\1\end{bmatrix}\begin{bmatrix}\mu-m\\1\end{bmatrix}=\begin{bmatrix}\mu-m\\1\end{bmatrix}+2\begin{bmatrix}\mu-m\\2\end{bmatrix}\ ,$$

we get

$$\begin{aligned}\mathcal{W}^{(2)}(x,m;\mu)=&\,g_1^2Y_0Y_0+2g_1g_2Y_0Y_1+g_2^2Y_1Y_1\\&+\begin{bmatrix}\mu-m\\1\end{bmatrix}\{(4g_1g_2+4g_2^2)Y_0Y_0\\&\qquad+4g_2^2Y_0Y_1\}\\&+\begin{bmatrix}\mu-m\\2\end{bmatrix}8g_2^2Y_0Y_0\ ,\end{aligned}\tag{4.4}$$

where [see Eq. (3.7)]

$$\begin{aligned}&Y_0Y_0=\tfrac{1}{2}y_0+\tfrac{1}{2}y_1,\quad Y_0Y_1=\tfrac{1}{2}y_1+\tfrac{1}{2}y_2\ ,\\&Y_1Y_1=\tfrac{1}{3}y_0+y_1+\tfrac{3}{2}y_2+\tfrac{5}{6}y_3\ .\end{aligned}$$

Keeping in mind that $w_t^{(N)}(j)$ is the coefficient of $y_t\binom{\mu-m}{j}$ in this expansion of $\mathcal{W}^{(2)}$ and that $b_t^{(N)}(0)=h_t+w_t^{(N)}(0)$ while, for $j\neq 0$, $b_t^{(N)}(j)=w_t^{(N)}(j)$, we get the following nonvanishing $b_t^{(2)}(j)$:

$$\begin{aligned}&b_0^{(2)}(0)=\tfrac{1}{2}g_1^2+\tfrac{1}{3}g_2^2,\quad b_1^{(2)}(0)=h_1+\tfrac{1}{2}g_1^2+g_1g_2+g_2^2\ ,\\&b_2^{(2)}(0)=h_2+g_1g_2+\tfrac{3}{2}g_2^2,\quad b_3^{(2)}(0)=h_3+\tfrac{5}{6}g_2^2\ ,\\&b_4^{(2)}(0)=h_4,\quad b_0^{(2)}(1)=2g_1g_2+2g_2^2\ ,\\&b_1^{(2)}(1)=2g_1g_2+4g_2^2,\quad b_2^{(2)}(1)=2g_2^2\ ,\\&b_0^{(2)}(2)=4g_2^2,\quad b_1^{(2)}(2)=4g_2^2\ .\end{aligned}\tag{4.5}$$

Using Eq. (3.24), the second-order perturbed eigenvalue is found to be

$$\begin{aligned}\Lambda_v^{(2)}=-\Bigg[&\tfrac{1}{2}g_1^2+\tfrac{1}{3}g_2^2+(2h_1+g_1^2+4g_1g_2+4g_2^2)\begin{bmatrix}v\\1\end{bmatrix}\\&+(4h_2+8g_1g_2+18g_2^2)\begin{bmatrix}v\\2\end{bmatrix}\\&+(8h_3+\tfrac{68}{3}g_2^2)\begin{bmatrix}v\\3\end{bmatrix}+16h_4\begin{bmatrix}v\\4\end{bmatrix}\Bigg]\ .\end{aligned}\tag{4.6}$$

The associated perturbed ladder function $K^{(2)}(x,m;\mu)$ is [see Eq. (3.21) and use expressions (4.5)]

$$K^{(2)}(x,m;\mu)=-\{(h_1+\tfrac{1}{2}g_1^2+g_1g_2++g_1^2)Y_0+(h_2+g_1g_2+\tfrac{3}{2}g_2^2)Y_1+(h_3+\tfrac{5}{6}g_2^2)Y_2+h_4Y_3\}$$
$$-\begin{bmatrix}\mu-m\\1\end{bmatrix}\{(2h_2+4g_1g_2+7g_2^2)Y_0+(2h_3+\tfrac{11}{3}g_2^2)Y_1+2h_4Y_2\}$$
$$-\begin{bmatrix}\mu-m\\2\end{bmatrix}\{(4h_3+\tfrac{34}{3}g_2^2)Y_0+4h_4Y_1\}-\begin{bmatrix}\mu-m\\3\end{bmatrix}8h_4Y_0\ . \quad (4.7)$$

c. Third order ($N=3$) of the perturbation ($S_3=5$). The potential-like function is $\mathcal{W}^{(3)}=2K^{(1)}K^{(2)}$, where $K^{(1)}$ and $K^{(2)}$ are given by Eqs. (4.3) and (4.7), respectively. Since we have [10]

$$\begin{bmatrix}\mu-m\\1\end{bmatrix}\begin{bmatrix}\mu-m\\t\end{bmatrix}=(t+1)\begin{bmatrix}\mu-m\\t+1\end{bmatrix}+t\begin{bmatrix}\mu-m\\t\end{bmatrix} \quad (4.8)$$

and [see Eq. (3.7)]

$$\begin{aligned}
&Y_0Y_0=\tfrac{1}{2}y_0+\tfrac{1}{2}y_1,\quad Y_0Y_1=\tfrac{1}{2}y_1+\tfrac{1}{2}y_2\ ,\\
&Y_0Y_2=\tfrac{1}{2}y_2+\tfrac{1}{2}y_3,\quad Y_0Y_3=\tfrac{1}{2}y_3+\tfrac{1}{2}y_4\ ,\\
&Y_1Y_1=\tfrac{1}{3}y_0+y_1+\tfrac{3}{2}y_2+\tfrac{5}{6}y_3\ ,\\
&Y_1Y_2=\tfrac{2}{3}y_1+2y_2+\tfrac{5}{2}y_3+\tfrac{7}{6}y_4\ ,\\
&Y_1Y_3=y_2+3y_3+\tfrac{7}{2}y_4+\tfrac{3}{2}y_5\ ,
\end{aligned} \quad (4.9)$$

we obtain the following nonvanishing third-order data coefficients:

$$\begin{aligned}
b_0^{(3)}(0)&=(h_1+\tfrac{1}{2}g_1^2+g_1g_2+g_2^2)g_1\\
&\quad+\tfrac{2}{3}(h_2+g_1g_2+\tfrac{3}{2}g_2^2)g_2\ ,\\
b_1^{(3)}(0)&=p_1+(h_1+h_2+\tfrac{1}{2}g_1^2+2g_1g_2+\tfrac{5}{2}g_2^2)g_1\\
&\quad+(h_1+2h_2+\tfrac{4}{3}h_3+\tfrac{1}{2}g_1^2+3g_1g_2+\tfrac{46}{9}g_2^2)g_2\ ,\\
b_2^{(3)}(0)&=p_2+(h_2+h_3+g_1g_2+\tfrac{7}{3}g_2^2)g_1\\
&\quad+(h_1+3h_2+4h_3+2h_4+\tfrac{1}{2}g_1^2+4g_1g_2+\tfrac{53}{6}g_2^2)g_2\ ,\\
b_3^{(3)}(0)&=p_3+(h_3+h_4+\tfrac{5}{6}g_2^2)g_1\\
&\quad+(\tfrac{5}{3}h_2+5h_3+6h_4+\tfrac{5}{3}g_1g_2+\tfrac{20}{3}g_2^2)g_2\ ,\\
b_4^{(3)}(0)&=p_4+h_4g_1+(\tfrac{7}{3}h_3+7h_4+\tfrac{35}{18}g_2^2)g_2\ ,\\
b_5^{(3)}(0)&=p_5+3h_4g_2,\quad b_6^{(3)}(0)=p_6\ ,\\
b_0^{(3)}(1)&=(2h_2+4g_1g_2+7g_2^2)g_1\\
&\quad+(2h_1+4h_2+\tfrac{4}{3}h_3+g_1^2+10g_1g_2+\tfrac{166}{9}g_2^2)g_2\ ,\\
b_1^{(3)}(1)&=(2h_2+2h_3+4g_1g_2+\tfrac{32}{3}g_2^2)g_1\\
&\quad+(2h_1+8h_2+8h_3+\tfrac{8}{3}h_4+g_1^2\\
&\qquad+16g_1g_2+\tfrac{122}{3}g_2^2)g_2\ ,\\
b_2^{(3)}(1)&=(2h_3+2h_4+\tfrac{11}{3}g_2^2)g_1\\
&\quad+(4h_2+10h_3+12h_4+6g_1g_2+30g_2^2)g_2\ ,\\
b_3^{(3)}(1)&=2h_4g_1+(\tfrac{16}{3}h_3+16h_4+\tfrac{70}{9}g_2^2)g_2\ ,\\
b_4^{(3)}(1)&=\tfrac{20}{3}h_4g_2\ ,\\
b_0^{(3)}(2)&=(4h_3+\tfrac{34}{3}g_2^2)g_1\\
&\quad+(8h_2+16h_3+\tfrac{8}{3}h_4+16g_1g_2+\tfrac{220}{3}g_2^2)g_2\ ,\\
b_1^{(3)}(2)&=(4h_3+4h_4+\tfrac{34}{3}g_2^2)g_1\\
&\quad+(8h_2+28h_3+24h_4+16g_1g_2+\tfrac{298}{3}g_2^2)g_2\ ,\\
b_2^{(3)}(2)&=4h_4g_1+(12h_3+36h_4+\tfrac{78}{3}g_2^2)g_2\ ,\\
b_4^{(3)}(2)&=\tfrac{20}{3}h_4g_2\ ,\\
b_0^{(3)}(3)&=8h_4g_1+(24h_3+48h_4+68g_2^2)g_2\ ,\\
b_1^{(3)}(3)&=8h_4g_1+(24h_3+80h_4+68g_2^2)g_2\ ,\\
b_2^{(3)}(3)&=32h_4g_2,\quad b_0^{(3)}(4)=b_1^{(3)}(4)=128h_4g_2\ .
\end{aligned}$$

Using these expressions, the third-order perturbed eigenvalue is found to be [see Eq. (3.24)]

$$\begin{aligned}
\Lambda_v^{(3)}=-\Bigg\{&(h_1+\tfrac{1}{2}g_1^2+g_1g_2+g_2^2)g_1+\tfrac{2}{3}(h_2+g_1g_2+\tfrac{3}{2}g_2^2)g_2\\
&+\begin{bmatrix}v\\1\end{bmatrix}\{2p_1+(2h_1+4h_2+g_1^2+8g_1g_2+12g_2^2)g_1+(4h_1+8h_2+4h_3+2g_1^2+16g_1g_2+\tfrac{86}{3}g_2^2)g_2\}\\
&+\begin{bmatrix}v\\2\end{bmatrix}\{4p_2+(8h_2+12h_3+12g_1g_2+42g_2^2)g_1+(8h_1+36h_2+48h_3+16h_4+4g_1^2+64g_1g_2+190g_2^2)g_2\}\\
&+\begin{bmatrix}v\\3\end{bmatrix}\{8p_3+(24h_3+32h_4+44g_2^2)g_1+(\tfrac{136}{3}h_2+160h_3+192h_4+\tfrac{208}{3}g_1g_2+440g_2^2)g_2\}\\
&+\begin{bmatrix}v\\4\end{bmatrix}\{16p_4+64h_4g_1+(176h_3+672h_4+\tfrac{1000}{3}g_2^2)g_2\}+\begin{bmatrix}v\\5\end{bmatrix}\{32p_5+\tfrac{1840}{3}h_4g_2\}+\begin{bmatrix}v\\6\end{bmatrix}64p_6\Bigg\}\ .
\end{aligned} \quad (4.10)$$

The associated perturbed ladder function $K^{(3)}(x,m;\mu)$ is given by Eq. (3.21) and, for the sake of brevity, has not been reproduced.

The computation can be pursued up to any higher-order N of the perturbation without special difficulty. As a matter of fact, when the order N of the perturbation increases the general expression of $\Lambda_v^{(N)}$, involving all the expansion coefficients g_i, h_i, p_i, etc. of the perturbation becomes somewhat cumbersome and, of course, when dealing with a given problem, one is better off giving the nonvanishing expansion coefficients their actual values from the beginning of the computation. Nevertheless, for comparative purposes, let us also work out general expressions when the perturbation is expanded in a series of x^{2s}.

2. *Expansion of the perturbation in a series of x^{2s}*

Let us now assume that the perturbation is

$$V^{(1)}(x)=g_1x^2+g_2x^4 ,$$

$$V^{(2)}(x)=h_1x^2+h_2x^4+h_3x^6+h_4x^8 .$$

a. First order ($N=1$) of the perturbation ($S_l=1$). The perturbed eigenvalue is [see Eq. (3.41)]

$$\Lambda_v^{(1)}=g_1\mathcal{I}_1(0)+g_2\mathcal{I}_2(0) ,$$

where the $\mathcal{I}_t(0)=-\langle x^{2t}\rangle$ are given by Eq. (3.44). We get

$$\Lambda_v^{(1)}=-\left\{(g_1+3g_2)\begin{bmatrix}v\\1\end{bmatrix}+3g_2\begin{bmatrix}v\\2\end{bmatrix}+\tfrac{1}{2}g_1+\tfrac{3}{4}g_2\right\} . \tag{4.11}$$

The associated perturbed ladder function is given by Eq. (3.37) where the only nonvanishing data coefficients are $b_1^{(1)}(0)=g_1$ and $b_2^{(1)}(0)=g_2$. We get

$$K^{(1)}(x,m;m)=-\left\{(\tfrac{1}{2}g_1+\tfrac{3}{4}g_2)x+\tfrac{1}{2}g_2x^3+\tfrac{3}{2}g_2x\begin{bmatrix}\mu-m\\1\end{bmatrix}\right\} . \tag{4.12}$$

b. Second order ($N=2$) of the perturbation ($S_2=3$). One has first to calculate the data coefficients $b_t^{(2)}(0)=h_t+w_t^{(2)}(0)$ and $b_t^{(2)}(j)=w_t^{(2)}(j)$ where $w_t^{(2)}(j)$ is the coefficient of $x^{2t}\binom{\mu-m}{j}$ in the expansion of the potential-like function $\mathcal{W}^{(2)}=(K^{(1)})^2$.

Using the expression (4.12) of $K^{(1)}$ and relation (4.8), we get

$$\begin{aligned}(K^{(1)})^2=&(\tfrac{1}{4}g_1^2+\tfrac{3}{4}g_1g_2+\tfrac{9}{16}g_2^2)x^2+(\tfrac{1}{2}g_1g_2+\tfrac{3}{4}g_2^2)x^4\\&+\tfrac{1}{4}g_2^2x^6+\{(\tfrac{3}{2}g_1g_2+\tfrac{9}{2}g_2^2)x^2+\tfrac{3}{2}g_2^2x^4\}\\&\times\begin{bmatrix}\mu-m\\1\end{bmatrix}+\tfrac{9}{2}g_2^2x^2\begin{bmatrix}\mu-m\\2\end{bmatrix}\end{aligned}$$

and we obtain the following nonvanishing $b_t^{(2)}(j)$:

$$\begin{aligned}&b_1^{(2)}(0)=h_1+\tfrac{1}{4}g_1^2+\tfrac{3}{4}g_1g_2+\tfrac{9}{16}g_2^2 ,\\&b_2^{(2)}(0)=h_2+\tfrac{1}{2}g_1g_2+\tfrac{3}{4}g_2^2 ,\\&b_3^{(2)}(0)=h_3+\tfrac{1}{4}g_2^2,\quad b_1^{(2)}(1)=\tfrac{9}{2}g_2^2+\tfrac{3}{2}g_1g_2 ,\\&b_2^{(2)}(1)=\tfrac{3}{2}g_2^2,\quad b_1^{(2)}(2)=\tfrac{9}{2}g_2^2 .\end{aligned} \tag{4.13}$$

Consequently, using the expression (3.41) of the perturbed eigenvalue together with the expressions (3.44) and (3.45) of the $\mathcal{I}_t(j)$, and rearranging the terms, we obtain

$$\begin{aligned}\Lambda_v^{(2)}=-\Bigg\{&210h_4\begin{bmatrix}v\\4\end{bmatrix}+(420h_4+15h_3+\tfrac{51}{4}g_2^2)\begin{bmatrix}v\\3\end{bmatrix}+(\tfrac{315}{4}h_4+\tfrac{45}{2}h_3+3h_2+3g_1g_2+\tfrac{153}{8}g_2^2)\begin{bmatrix}v\\2\end{bmatrix}\\&+(105h_4+\tfrac{45}{4}h_3+3h_2+h_1+\tfrac{1}{4}g_1^2+3g_1g_2+9g_2^2)\begin{bmatrix}v\\1\end{bmatrix}\\&+\tfrac{105}{8}h_4+\tfrac{15}{8}h_3+\tfrac{3}{4}h_2+\tfrac{1}{2}h_1+\tfrac{1}{8}g_1^2+\tfrac{3}{4}g_1g_2+\tfrac{21}{16}g_2^2\Bigg\} .\end{aligned} \tag{4.14}$$

The associated perturbed ladder function $K^{(2)}(x,m;\mu)$ is given by Eq. (3.37) where the $b_u^{(2)}(i)$ have to be substituted with their expressions (4.13).

The computation can be pursued by the determination of the analytical expression of $\mathcal{W}^{(3)}=2K^{(1)}K^{(2)}$ leading to an analytical expression of $\Lambda_v^{(3)}$, now involving the x^{2s} expansion coefficients of $V(x)$. This expression is somewhat more cumbersome than its counterpart (4.10) involving the $\mathcal{H}_{2s}$-expansion coefficients and, for the sake of brevity, is not reproduced.

Let us now apply the procedure to the determination of the x^4-perturbed harmonic-oscillator energies.

B. The x^4-perturbed harmonic-oscillator energies and ladder functions

Let us consider the eigenequation ($g>0$)

$$\left\{\frac{d^2}{dx^2}-x^2-2gx^4+2E\right\}\Psi(x)=0 . \tag{4.15}$$

This is an eigenequation (3.1) with $b^2=1$ and $S_1=1$.

1. *Application of the first perturbed factorization scheme*

Since $x^4=\frac{3}{4}+\frac{3}{4}\mathcal{H}_2(x)+\frac{1}{16}\mathcal{H}_4(x)$ [see Eq. (B1)], the perturbation is $V(x)=V^{(1)}(x)=-g(3y_1+\frac{3}{2}y_2)$. The perturbed energies are $E^{(1)}=\frac{1}{2}\Lambda^{(1)}+\frac{3}{4}g$ and, for $N>1$, $E^{(N)}=\frac{1}{2}\Lambda^{(N)}$.

a. First order (N=1) of the perturbation ($S_1=1$). The perturbed eigenvalue is $\Lambda_v^{(1)}=6g\binom{v}{1}+6g\binom{v}{2}$ [see Eq. (3.24) with $b_1^{(1)}(0)=-3g$ and $b_2^{(1)}(0)=-\frac{3}{2}g$], and one gets

$$E_v^{(1)}=3g\left\{\begin{bmatrix}v\\2\end{bmatrix}+\begin{bmatrix}v\\1\end{bmatrix}+\frac{1}{4}\right\}. \qquad (4.16)$$

The associated ladder function is [see Eq. (3.21)]

$$K^{(1)}(x,m;\mu)=-g\left\{(3Y_0+\tfrac{3}{2}Y_1)+\begin{bmatrix}\mu-m\\1\end{bmatrix}3Y_0\right\}.$$

b. Second order (N=2) of the perturbation ($S_2=3$). The potential-like function is $\mathcal{W}^{(2)}=(K^{(1)})^2$, i.e.,

$$\mathcal{W}^{(2)}=g^2\left\{9Y_0Y_0+9Y_0Y_1+\tfrac{9}{4}Y_1Y_1+(27Y_0Y_0+9Y_0Y_1)\begin{bmatrix}\mu-m\\1\end{bmatrix}+18Y_0Y_0\begin{bmatrix}\mu-m\\2\end{bmatrix}\right\},$$

to microcomputer programming.

Keeping in mind that the $w_t^{(2)}(j)$ are the coefficients of $y_t\binom{\mu-m}{j}$ in this expression, one gets the following expressions of the second-order data coefficients:

$$b_0^{(2)}(0)=\tfrac{21}{4}g^2,\quad b_1^{(2)}(0)=\tfrac{45}{4}g^2,\quad b_2^{(2)}(0)=\tfrac{63}{8}g^2,$$
$$b_3^{(2)}(0)=\tfrac{15}{8}g^2,\quad b_0^{(2)}(2)=b_1^{(2)}(2)=9g^2.$$

Consequently, the perturbed energy $E_v^{(2)}=\frac{1}{2}\Lambda_v^{(2)}$ and associated ladder function are

$$E_v^{(2)}=-g^2\left\{\frac{51}{2}\begin{bmatrix}v\\3\end{bmatrix}+\frac{153}{4}\begin{bmatrix}v\\2\end{bmatrix}+18\begin{bmatrix}v\\1\end{bmatrix}+\frac{21}{8}\right\},$$

$$K^{(2)}(x,m;\mu)=-g^2\left\{\tfrac{45}{4}Y_0+\tfrac{63}{8}Y_1+\tfrac{45}{24}Y_2+\{\tfrac{135}{4}Y_0+\tfrac{33}{4}Y_1\}\begin{bmatrix}\mu-m\\1\end{bmatrix}+\tfrac{51}{2}Y_0\begin{bmatrix}\mu-m\\2\end{bmatrix}\right\}. \qquad (4.17)$$

c. Third order (N=3) of the perturbation ($S_3=5$). The potential-like function is $\mathcal{W}^{(3)}=2K^{(1)}K^{(2)}$, i.e.,

$$\mathcal{W}^{(3)}=g^3\left\{\tfrac{135}{2}Y_0Y_0+81Y_0Y_1+\tfrac{45}{4}Y_0Y_2+\tfrac{189}{8}Y_1Y_1+\tfrac{45}{8}Y_1Y_2+[\tfrac{945}{2}Y_0Y_0+\tfrac{495}{2}Y_0Y_1+\tfrac{45}{4}Y_0Y_2+\tfrac{99}{4}Y_1Y_1]\begin{bmatrix}\mu-m\\1\end{bmatrix}+[864Y_0Y_0+\tfrac{351}{2}Y_0Y_1]\begin{bmatrix}\mu-m\\2\end{bmatrix}+459Y_0Y_0\begin{bmatrix}\mu-m\\3\end{bmatrix}\right\},$$

where the Y_iY_j are given by Eq. (4.9).

One gets the following expressions of the third-order data coefficients:

$$b_0(0)=\tfrac{333}{8},\quad b_1(0)=\tfrac{813}{8},\quad b_2(0)=\tfrac{1485}{16},\quad b_3(0)=\tfrac{315}{8},$$
$$b_4(0)=\tfrac{105}{16},\quad b_0(1)=\tfrac{489}{2},\quad b_1(1)=\tfrac{1539}{4},\quad b_2(1)=\tfrac{333}{2},$$
$$b_3(1)=\tfrac{105}{4},\quad b_0(2)=432,\quad b_1(2)=\tfrac{2079}{4},$$
$$b_2(2)=\tfrac{351}{4},\quad b_0(3)=\tfrac{459}{2},$$

where, for the sake of brevity, we have set $g=1$ and used the shortened notation $b_t^{(3)}(j)=b_t(j)$.

The perturbed energy $E_v^{(3)}=\frac{1}{2}\Lambda_v^{(3)}$ is found to be

$$E_v^{(3)}=g^3\left\{\tfrac{1125}{2}\begin{bmatrix}v\\4\end{bmatrix}+1125\begin{bmatrix}v\\3\end{bmatrix}+\tfrac{6291}{8}\begin{bmatrix}v\\2\end{bmatrix}+\tfrac{1791}{8}\begin{bmatrix}v\\1\end{bmatrix}+\tfrac{333}{16}\right\} \qquad (4.18)$$

and the associated ladder function $K^{(3)}(x,m;\mu)$ is given by Eq. (3.21).

The computation can be pursued by working out the expansion of $\mathcal{W}^{(4)}=2K^{(1)}K^{(3)}+(K^{(2)})^2$ in a series of $y_t\binom{\mu-m}{j}$. Nevertheless, as the order N of the perturbation increases, one has rather to compute the data coefficients via the general expression (3.73) of the $w_t^{(N)}(j)$ and resort to microcomputer programming.

Of course, the expressions of $E_v^{(1)}$, $E_v^{(2)}$, and $E_v^{(3)}$ are directly obtainable by setting $g_1=-3g$, $g_2=-\frac{3}{2}g$, and $h_i=p_i=0$ in the general expressions (4.2), (4.6), and (4.10) of $\Lambda_v^{(1)}$, $\Lambda_v^{(2)}$, and $\Lambda_v^{(3)}$.

2. Application of the second perturbed factorization scheme

a. First order (N=1) of the perturbation ($S_1=1$). Since the perturbation reduces to $V^{(1)}(x)=-2gx^4$, the associated eigenvalue is $\Lambda_v^{(1)}=-2g\mathcal{J}_2(0)$ and, using the expression (3.44) of $\mathcal{J}_2(0)=-\langle x^4\rangle$, we find again the expression (4.16) of $E_v^{(1)}=\frac{1}{2}\Lambda_v^{(1)}$. The associated ladder function is [see Eq. (3.37) with $b_2^{(1)}(0)=-2g$]

$$K^{(1)}(x,m;\mu)=g\left\{\tfrac{3}{2}x+x^3+3x\begin{bmatrix}\mu-m\\1\end{bmatrix}\right\}. \qquad (4.19)$$

b. Second order (N=2) of the perturbation ($S_2=3$). One gets

$$(K^{(1)})^2=g^2\left\{\tfrac{9}{4}x^2+3x^4+x^6+(18x^2+6x^4)\begin{bmatrix}\mu-m\\1\end{bmatrix}+18x^2\begin{bmatrix}\mu-m\\2\end{bmatrix}\right\} \qquad (4.20)$$

and the perturbed eigenvalue is found to be

$$\Lambda_v^{(2)}=g^2\{\tfrac{9}{4}\mathcal{J}_1(0)+3\mathcal{J}_2(0)+\mathcal{J}_3(0)+18\mathcal{J}_1(1)+6\mathcal{J}_2(1)+18\mathcal{J}_1(2)\} . \tag{4.21}$$

Using the expressions (3.44) and (3.45) of the $\mathcal{J}_t(j)$, we find again the expression (4.17) of $E_v^{(2)}$.

The associated ladder function is

$$K^{(2)}(x,m;\mu)=-g^2\left\{\tfrac{21}{4}x+\tfrac{11}{4}x^3+\tfrac{1}{2}x^5+(\tfrac{51}{2}x+\tfrac{11}{2}x^3)\begin{bmatrix}\mu-m\\1\end{bmatrix}+\tfrac{51}{2}x\begin{bmatrix}\mu-m\\2\end{bmatrix}\right\} . \tag{4.22}$$

c. Third order (N=3) of the perturbation ($S_3=5$). One gets

$$2K^{(1)}K^{(2)}=-g^3\left\{\tfrac{63}{4}x^2+\tfrac{75}{4}x^4+7x^6+x^8+(261x^2+117x^4+14x^6)\begin{bmatrix}\mu-m\\1\end{bmatrix}+(\tfrac{1377}{2}x^2+117x^4)\begin{bmatrix}\mu-m\\2\end{bmatrix}+459x^2\begin{bmatrix}\mu-m\\3\end{bmatrix}\right\} \tag{4.23}$$

and the perturbed eigenvalue is found to be

$$\Lambda_v^{(3)}=-g^3\{\tfrac{63}{4}\mathcal{J}_1(0)+\tfrac{75}{4}\mathcal{J}_2(0)+7\mathcal{J}_3(0)+\mathcal{J}_4(0)+261\mathcal{J}_1(1)+117\mathcal{J}_2(1)+14\mathcal{J}_3(1)+\tfrac{1377}{2}\mathcal{J}_1(2)+117\mathcal{J}_2(2)+459\mathcal{J}_1(3)\} . \tag{4.24}$$

Using the expressions (3.44) and (3.45) of the $\mathcal{J}_t(j)$, we find again the expression (4.18) of $E_v^{(3)}$.

The associated perturbed ladder function $K^{(3)}(x,m;\mu)$ is given by Eq. (3.37) and the computation can be pursued to higher order of the perturbation.

Let us now consider the determination of alternative expressions of the $E_v^{(N)}$ in a series of $(v+\frac{1}{2})^u$.

3. *Application of the third perturbed factorization scheme*

a. First order (N=1) of the perturbation ($S_1=1$). The perturbed eigenvalue is $\Lambda_v^{(1)}=-2g\mathcal{J}_2(0)$, and using the expression (3.71) of $\mathcal{J}_2(0)=-\langle x^4\rangle$, we get the following expression of $E_v^{(1)}=\frac{1}{2}\Lambda_v^{(1)}$:

$$E_v^{(1)}=g\{\tfrac{3}{2}(v+\tfrac{1}{2})^2+\tfrac{3}{8}\} . \tag{4.25}$$

The associated perturbed ladder function is [set $b_2^{(1)}(0)=-2g$ in Eq. (3.65)]

$$K^{(1)}(x,m;\mu)=g\{x^3-3x(m-\mu-\tfrac{1}{2})\} . \tag{4.26}$$

b. Second order (N=2) of the perturbation ($S_2=3$). One gets

$$(K^{(1)})^2=g^2\{x^6-6x^4(m-\mu-\tfrac{1}{2})+9x^2(m-\mu-\tfrac{1}{2})^2\} \tag{4.27}$$

and, consequently, the perturbed eigenvalue is

$$\Lambda_v^{(2)}=g^2\{\mathcal{J}_3(0)-6\mathcal{J}_2(1)+9\mathcal{J}_1(2)\} . \tag{4.28}$$

Using the expressions (3.71) and (3.72) of the $\mathcal{J}_t(j)$, we obtain

$$E_v^{(2)}=-g^2\{\tfrac{17}{4}(v+\tfrac{1}{2})^3+\tfrac{67}{16}(v+\tfrac{1}{2})\} . \tag{4.29}$$

Setting $b_3^{(2)}(0)=g^2$, $b_2^{(2)}(1)=-6g^2$, and $b_1^{(2)}(2)=9g^2$ [and $b_u^{(N)}(j)=0$, otherwise] in the expression (3.65) of $C_s^{(2)}(t)$, we get

$$C_s^{(2)}(t)=g^2\frac{(-1)^{s+1}}{t!(2s+1)!!}\{\tfrac{15}{2}\Theta_0(2-s,t)+9\Theta_1(1-s,t)+9\Theta_2(-s,t)\}$$

and, after introducing the values of the $\Theta_j(\sigma,k)$ (see Appendix C), the associated perturbed ladder function is found to be

$$K^{(2)}(x,m;\mu)=g^2\{-\tfrac{1}{2}x^5+\tfrac{11}{2}(m-\mu-\tfrac{1}{2})x^3-[\tfrac{33}{16}+\tfrac{51}{4}(m-\mu-\tfrac{1}{2})^2]x\} . \tag{4.30}$$

c. Third order (N=3) of the perturbation ($S_3=5$). Using expressions (4.26) and (4.30) of $K^{(1)}$ and $K^{(2)}$, we get

$$\mathcal{W}^{(3)}=-\{[x^8+\tfrac{33}{8}x^4]+[14x^6+\tfrac{99}{8}x^2](m-\mu-\tfrac{1}{2})-\tfrac{117}{2}x^4(m-\mu-\tfrac{1}{2})^2+\tfrac{153}{2}x^2(m-\mu-\tfrac{1}{2})^3\} , \tag{4.31}$$

and the perturbed eigenvalue is found to be

$$\Lambda_v^{(3)}=g^3\{-\mathcal{J}_4(0)-\tfrac{33}{8}\mathcal{J}_2(0)+14\mathcal{J}_3(1)+\tfrac{99}{8}\mathcal{J}_1(1)-\tfrac{117}{2}\mathcal{J}_2(2)+\tfrac{153}{2}\mathcal{J}_1(3)\} . \tag{4.32}$$

Using the expressions (3.71) and (3.72) of the $\mathcal{J}_t(j)$, we obtain

$$E_v^{(3)}=g^3\{\tfrac{375}{16}(v+\tfrac{1}{2})^4+\tfrac{1707}{32}(v+\tfrac{1}{2})^2+\tfrac{1539}{256}\} . \tag{4.33}$$

The above expressions of the x^4-perturbed harmonic-oscillator energies $E_v^{(1)}$, $E_v^{(2)}$, and $E_v^{(3)}$ are in accordance with already known results [12]. The expected property that the quartic anharmonic energies are polynomial in $(v+\frac{1}{2})$ of definite parity is found again as a direct consequence of the vanishing conditions of the $\Theta_j(\sigma,k)$ numbers: the pseudointegrals $\mathcal{J}_t(j)$, as well as the integrals $\langle x^{2t}\rangle=-\mathcal{J}_t(0)$, are polynomials in $(v+\frac{1}{2})$ of the same parity as $(t+j)$.

C. Eigenenergies for the $x^2+\lambda x^2/(1+gx^2)$ interaction

Let us consider the solution of the eigenequation

$$\left\{\frac{d^2}{dx^2}-x^2-\frac{\lambda x^2}{(1+gx^2)}+\mathcal{E}\right\}\Psi(x)=0 . \tag{4.34}$$

Since, on one hand, we have at our disposal general expressions of the anharmonic-oscillator eigenvalues (up to the third order of the perturbation) and, on the other hand, it has been shown [15] that the eigenequation (4.34) can be viewed as a particular case of the perturbed harmonic-oscillator eigenequation (3.1) with a perturbation which is expandable as a convergent series of Hermite polynomials, we can obtain analytical expressions of the eigenenergies $\mathcal{E}_v^{(N)}$ with a minimum effort.

The perturbation to be considered in the anharmonic-oscillator eigenequation (3.1), with the eigenvalue $\Lambda=\mathcal{E}-b(2m+1)$, is

$$V(x)=-(1-b^2)x^2-\frac{\lambda}{g}+\frac{\lambda}{g(1+gx^2)}\,, \tag{4.35}$$

where the scaling real parameter b is used to improve the zeroth-order harmonic Hamiltonian.

As it has been shown [15], the choice $b^2=1+\lambda/(1+g/2)$ is well adapted to most low-lying states while, if one is interested solely in one specific vibrational level v, it is more convenient to choose $b^2=1+\lambda/[1+g(v+\frac{1}{2})]$.

The perturbation (4.35) can be expanded in the following convergent series of Hermite polynomials [15,8]:

$$\begin{aligned}V(x)=&-\frac{\lambda}{g}(1-\mathcal{D}_0)-\frac{1-b^2}{2b}\\&-\left\{\frac{1-b^2}{4b}-\frac{\lambda}{g}\mathcal{D}_2\right\}\mathcal{H}_2(b^{1/2}x)\\&+\frac{\lambda}{g}\sum_{k=2}\mathcal{D}_{2k}\mathcal{H}_{2k}(b^{1/2}x)\,,\end{aligned} \tag{4.36}$$

where

$$\mathcal{D}_{2k}=\sum_{u=0}^{k}(-1)^u\{u!(2k-2u)!2^{2u}\}^{-1}I_{k-u}\,,$$

$$I_u=\left[-\frac{b}{g}\right]^u(I_0-1)-\sum_{s=1}^{u-1}\frac{(2s-1)!!}{2^s}\left[-\frac{b}{g}\right]^{u-s}\,,$$

$$I_0=\left[\frac{\pi b}{g}\right]^{1/2}\exp\left[\frac{b}{g}\right]\operatorname{erfc}\left\{\left[\frac{b}{g}\right]^{1/2}\right\}.$$

$\operatorname{erfc}(u)=1-\operatorname{erf}(u)$ is the complementary error function. Tables, series, and asymptotic expansions of the $\operatorname{erfc}(u)$ functions can be found, for instance, in Ref. [13]. Formulas allowing the computation of $I_0(u)$ for several ranges of u are available [15].

Now, we note that when setting $x=b^{-1/2}X$, the eigenequation (3.1) to be considered becomes

$$\left\{\frac{d^2}{dX^2}-X^2+2m+1+\frac{1}{b}V(b^{-1/2}X)+\frac{1}{b}[\mathcal{E}-b(2m+1)]\right\}\Psi(X)=0\,. \tag{4.37}$$

The general results of the preceding section, which correspond to an unperturbed harmonic-oscillator potential $U^{(0)}(X,m)=-X^2+2m+1$ and basis functions $y_s(X)=\chi_s\mathcal{H}_{2s}(X)$, can be used. We have $\mathcal{E}_v^{(0)}=2b(v+\frac{1}{2})$ and the perturbed factorization procedure provides the required perturbed eigenenergies

$$\begin{aligned}&\mathcal{E}_v^{(1)}=b\Lambda_v^{(1)}+\frac{\lambda}{g}(I_0-1)-\frac{1-b^2}{2b}\,,\\&\mathcal{E}_v^{(N)}=b\Lambda_v^{(N)}\ \text{ for } N>1\end{aligned} \tag{4.38}$$

when using the following expressions for the expansion coefficients of the perturbation:

$$\begin{aligned}&b_1(0)=-\frac{2}{b}\left\{\frac{1-b^2}{4b}-\frac{\lambda}{g}\mathcal{D}_2\right\}\\&b_k(0)=\frac{(2k)!\lambda}{k!bg}\mathcal{D}_{2k}\,.\end{aligned} \tag{4.39}$$

Using the expression (4.2) of $\Lambda_v^{(1)}$ with $g_1=b_1(0)$ and $g_2=b_2(0)$, we get the following approximate expression of the total energy in terms of λ, g, b, and v:

$$\begin{aligned}\mathcal{E}_v\simeq&\frac{1}{b}-\frac{\lambda}{g}(I_0-1)+\left[\frac{1}{b}-\frac{4\lambda}{g}\mathcal{D}_2\right]\begin{bmatrix}v\\1\end{bmatrix}\\&-\frac{48\lambda}{g}\mathcal{D}_4\begin{bmatrix}v\\2\end{bmatrix}\,,\end{aligned} \tag{4.40}$$

where

$$4\mathcal{D}_2=-\frac{2b}{g}(I_0-1)-I_0\,,$$

$$48\mathcal{D}_4=\left[\frac{2b^2}{g^2}+\frac{6b}{g}+\frac{3}{2}\right](I_0-1)+\frac{b}{g}\,.$$

This simple and compact expression gives again, as particular cases, the analytical expressions already obtained [15] for the ground and first excited states ($v=0$, 1, and 2): setting $b=[1+\lambda/(1+g/2)]^{1/2}$, it reproduces the trend of the exact values of the energies $\mathcal{E}_{v=0}$, $\mathcal{E}_{v=1}$, and $\mathcal{E}_{v=2}$, for a rather large range of values of λ and g [15].

Using both the expressions (4.2) and (4.6) of $\Lambda_v^{(1)}$ and $\Lambda_v^{(2)}$, we obtain the more elaborate expression of the total energy,

$$\mathcal{E}_v=e_0+e_1\begin{bmatrix}v\\1\end{bmatrix}+e_2\begin{bmatrix}v\\2\end{bmatrix}+e_3\begin{bmatrix}v\\3\end{bmatrix}+e_4\begin{bmatrix}v\\4\end{bmatrix}\,, \tag{4.41}$$

where

$$\begin{aligned}e_0=&\frac{1}{b}\left\{1-\frac{(1-b^2)^2}{8b^2}\right\}-\frac{\lambda}{g}\left\{I_0-1+\frac{1-b^2}{b}\mathcal{D}_2\right\}\\&-\frac{\lambda^2}{g^2b}\{2\mathcal{D}_2^2+48\mathcal{D}_4^2\}\,,\end{aligned}$$

$$e_1=\frac{1}{b}-\frac{4\lambda}{g}\mathcal{D}_2+\frac{4}{b}\left[\frac{1-b^2}{4b}-\frac{\lambda}{g}(\mathcal{D}_2+12\mathcal{D}_4)\right]^2\,,$$

$$e_2=\frac{48\lambda}{g}\mathcal{D}_4\left[\frac{1}{b^2}-\frac{2\lambda}{gb}(2\mathcal{D}_2+27\mathcal{D}_4)\right]\,,$$

$$e_3=-\frac{96\lambda}{g}\left\{10\mathcal{D}_6+\frac{34\lambda^2}{bg^2}\mathcal{D}_4^2\right\},\quad e_4=\frac{2\lambda}{3g}8!\mathcal{D}_8\,,$$

$$6!\mathcal{D}_6=-\left[\frac{b^3}{g^3}+\frac{15b^2}{2g^2}\frac{+45b}{4g}+\frac{15}{8}\right](I_0-1)$$
$$-\frac{b^2}{2g^2}-\frac{3b}{g}-\frac{15}{8},$$
$$8!\mathcal{D}_8=-\left[\frac{b^4}{g^4}+\frac{14b^3}{g^3}+\frac{105b^2}{2g^2}+\frac{105b}{2g}+\frac{105}{16}\right](I_0-1)$$
$$+\frac{b^3}{2g^3}+\frac{25b^2}{4g^2}+\frac{141b}{8g}+\frac{105}{16}.$$

This expression, showing the v dependence of the energies, gives a rather good approximation of the exact values.

With the help of a computer program and the use of expressions (4.38) and (4.39), the perturbed factorization procedure can provide the perturbed energies $\mathcal{E}_v^{(N)}$ in terms of the $\binom{v}{u}$, up to any high-order N of the perturbation without special difficulty.

V. CONCLUSION

Three perturbed factorizations of the symmetric perturbed harmonic-oscillator eigenequation have been proposed, allowing an analytical determination of the perturbed eigenvalues by means of a few algebraic manipulations. For the three cases, the same expression (3.41) of the perturbed eigenvalue $\Lambda_v^{(N)}$ in terms of the pseudointegrals $\mathcal{J}_t(j)$ and of the data coefficients $b_t^{(N)}(j)$ holds and serves successively at each order N of the perturbation. For each perturbed factorization case, closed-form expressions of the pseudointegrals $\mathcal{J}_t(j)$ in a series of $\binom{v}{u}$ or $(v+\frac{1}{2})^u$ have been made available. Therefore, for any anharmonic-oscillator eigenequation under consideration, the determination of the eigenvalue $\Lambda_v^{(N)}$ of order N simply amounts to the computation of the data coefficients $b_t^{(N)}(j)$ in terms of the expansion coefficients of the perturbation, either via the closed-form expressions of the perturbed ladder functions or by means of formula (3.73). In comparison with previous results (see paper I or [8]), the computational algorithm has been drastically simplified. Moreover, it requires only algebraic recursive manipulations and, when high orders of the perturbation are required, it is well adapted for microcomputer programming. Let us add that, since the procedure provides the perturbed ladder operator, the perturbed eigenfunctions can also be determined in closed form (for details, see paper II or [8]).

Owing to the present results obtained for perturbed type D and also for perturbed type F (see paper II), one can hopefully conjecture that these new techniques, which have been elaborated for handling efficiently the perturbed-ladder-operator method, can also be applied to the remaining factorization types A, B, C, and E: very likely, for each factorization type, one has first to find adequate x-basis and finite-difference m-basis functions and, then, to make available the associated pseudointegrals $\mathcal{J}_t(j)$ and coupling coefficients $\mathcal{K}_i(s,r,t;l,m,j)$.

Particularly, the present results can be easily extended to the case of the x^{2s}-perturbed rotating harmonic oscillator (perturbed type C), which corresponds to the unperturbed potential

$$U^{(0)}(x,m)=-b^2x^2-\frac{m(m+1)}{x^2},$$

and analytical expressions of the $\mathcal{J}_t(j)$ in a series of $(v+\frac{1}{2})^u(m+\frac{1}{2})^k$ or $\binom{v}{u}(m+\frac{1}{2})^k$ can be derived. The procedure also works nicely for giving an elaborate perturbative solution of the nuclear diatomic Morse-Pekeris vibration-rotation (perturbed type-B) wave equation, which is needed, for instance, for a theoretical determination of the centrifugal contributions to the rotational spectra of diatomic molecules [16]. Results concerning these studies, together with a more extensive investigation of the computational aspect of the method, will be given elsewhere.

APPENDIX A: DETERMINATION OF THE $C_s^{(N)}(t)$ COEFFICIENTS

1. x^{2s} expansion of the perturbation associated with a $\binom{v}{u}$ expansion of the perturbed eigenvalue

Using the recurrence formula (3.48) successively for $s=S_N+1, S_N, \ldots$, and introducing the shortened notation $C_{S_N-\sigma}(k)=C_{S_N-\sigma}^{(N)}(k)$ and $b_s(u)=b_s^{(N)}(u)$, we have

$$C_{S_N}(0)=-\frac{1}{2b}b_{S_N+1}(0),$$
$$C_{S_N-1}(0)=-\frac{1}{2b}b_{S_N}(0)-\left[\frac{1}{2b}\right]^2(2S_N+1)b_{S_N+1}(0),$$
$$C_{S_N-1}(1)=-\frac{1}{2b}b_{S_N}(1)-\left[\frac{1}{2b}\right]^2(4S_N+2)b_{S_N+1}(0),$$
$$C_{S_N-2}(0)=-\frac{1}{2b}b_{S_N-1}(0)-\left[\frac{1}{2b}\right]^2(2S_N-1)b_{S_N}(0)-\left[\frac{1}{2b}\right]^3(2S_N-1)(2S_N+1)b_{S_N+1}(0), \quad \text{(A1)}$$
$$C_{S_N-2}(1)=-\left[\frac{1}{2b}\right]^2(4S_N-2)b_{S_N}(0)-\left[\frac{1}{2b}\right]^3(4S_N-2)(4S_N+2)b_{S_N+1}(0)-\frac{1}{2b}b_{S_N-1}(1)-\left[\frac{1}{2b}\right]^2(2S_N-1)b_{S_N}(1),$$
$$C_{S_N-2}(2)=-\left[\frac{1}{2b}\right]^3(4S_N-2)(4S_N+2)b_{S_N+1}(0)-\left[\frac{1}{2b}\right]^2(4S_N-2)b_{S_N}(1)-\frac{1}{2b}b_{S_N-1}(2),$$

and so on.

More generally, it is easily checked that Eq. (3.34) holds.

2. x^{2s} expansion of the perturbation associated with a $(v+\frac{1}{2})^u$ expansion of the perturbed eigenvalue

Using the recurrence formula (3.25) successively for $s=S_N+1, S_N, \ldots,$ we have

$$
\begin{aligned}
C_{S_N}(0) &= -\frac{1}{2b} b_{S_N+1}(0) , \\
C_{S_N-1}(0) &= -\frac{1}{2b} b_{S_N}(0) , \\
C_{S_N-1}(1) &= -\frac{1}{2b} b_{S_N}(1) \\
&\quad + \left[-\frac{1}{2b}\right]^2 (4S_N+2) b_{S_N+1}(0) , \\
C_{S_N-2}(0) &= -\frac{1}{2b} b_{S_N-1}(0) \\
&\quad + \left[-\frac{1}{2b}\right]^2 a_{01}(4S_N-2) b_{S_N}(1) \\
&\quad + \left[-\frac{1}{2b}\right]^3 a_{01}(4S_N-2)(4S_N+2) b_{S_N+1}(0) , \\
C_{S_N-2}(1) &= -\frac{1}{2b} b_{S_N-1}(1) \\
&\quad + \left[-\frac{1}{2b}\right]^2 (4S_N-2) b_{S_N}(0) , \\
C_{S_N-2}(2) &= -\frac{1}{2b} b_{S_N-1}(2) + \left[-\frac{1}{2b}\right]^2 (2S_N-1) b_{S_N}(1) \\
&\quad + \left[-\frac{1}{2b}\right]^3 (4S_N-2)(2S_N+1) b_{S_N+1}(0) ,
\end{aligned}
\tag{A2}
$$

and so on. The shortened notation $C_{S_N-\sigma}(k) = C^{(N)}_{S_N-\sigma}(k)$ and $b_s(u) = b_s^{(N)}(u)$ has been used.

More generally, it is easily checked that the expression (3.60) for $C^{(N)}_{S_N-2\sigma}(k)$ holds. Then, substituting for $C^{(N)}_{S_N-\sigma}(k)$ from Eq. (3.60) into the recurrence formula (3.58), we get the recurrence formula (3.61) allowing the determination of the $d_j(\sigma,k,s)$.

APPENDIX B: INTERRELATIONS BETWEEN THE FIRST TWO PERTURBED FACTORIZATIONS OF THE ANHARMONIC-OSCILLATOR EIGENEQUATION

Let us make use of the following relations:

$$x^{2s} = \sum_{t=0}^{s} \frac{(2s)!}{2^{2s}(2t)!(s-t)!} \mathcal{H}_{2t}(x) \tag{B1}$$

and

$$\mathcal{H}_{2s+1}(x) = \sum_{k=0}^{s} \frac{(-1)^{s-k}(2s+1)!}{(s-k)!(2k+1)!} (2x)^{2k+1} . \tag{B2}$$

Since we are dealing with the same perturbed eigenequation, at any order N of the perturbation we can write

$$\underline{\Lambda}_v^{(N)} + \sum_{s=1}^{S_N+1} \underline{b}_s^{(N)}(0) \chi_s \mathcal{H}_{2s}(b^{1/2}x) = \Lambda_v^{(N)} + \sum_{s=1}^{S_N+1} b_s^{(N)}(0) x^{2s} , \tag{B3}$$

where the left and right sides are related to the first and second perturbed factorization cases, respectively. After introducing the expansion (B1) of x^{2s} in a series of $\chi_t \mathcal{H}_{2t}(b^{1/2}x)$, making some rearrangements, and equating the coefficients of $\chi_t \mathcal{H}_{2t}(b^{1/2}x)$ in both sides of Eq. (B3), we obtain

$$\underline{b}_s^{(N)}(0) = \sum_{t=s}^{S_N+1} \begin{bmatrix} t \\ s \end{bmatrix} \left[\frac{1}{b}\right]^{t+1} \frac{(2t)!}{t! 2^{2t}} b_t^{(N)}(0) , \quad 1 \le s \le S_N+1 . \tag{B4}$$

Let us now focus our attention on the first order $(N=1)$ of the perturbation and compare the alternative expressions of the associated perturbed ladder function. On one hand, we have [see Eq. (3.37)]

$$K^{(1)}(x,m;\mu) = \sum_{s=0}^{S_1} x^{2s+1} \sum_{t=0}^{S_1-s} C_s^{(1)}(t) \begin{bmatrix} \mu-m \\ t \end{bmatrix} , \tag{B5}$$

where

$$C_s^{(1)}(t) = -\sum_{u=s+t+1}^{S_1+1} \left[\frac{1}{2b}\right]^{u-s} d_0(u-s-1,t,u-1) \times b_u^{(1)}(0) . \tag{B6}$$

On the other hand, we have [see Eq. (3.21)]

$$K^{(1)}(x,m;\mu) = -b^{-1/2} \sum_{s=0}^{S_1} \chi_{s+1} \mathcal{H}_{2s+1}(b^{1/2}x) \times \sum_{t=0}^{S_1-s} \begin{bmatrix} \mu-m \\ t \end{bmatrix} 2^t \underline{b}^{(1)}_{s+t+1}(0) , \tag{B7}$$

where the $\underline{b}^{(1)}_{s+t+1}(0)$ are given in terms of the $b_u^{(1)}(0)$ by Eq. (B3).

Substituting for the Hermite polynomials $\mathcal{H}_{2s+1}(b^{1/2}x)$ from Eq. (B2) into Eq. (B7), making some rearrangements and comparing the result with expression (B5), we get

$$C_s^{(1)}(t) = -\sum_{u=s+t+1}^{S_1+1} \left[\frac{1}{2b}\right]^{u-s} \frac{2^{s+t-u}(2u)!}{u!} b_u^{(1)}(0) \sum_{k=s}^{u-t-1} (-1)^{k-s} \frac{k!}{(k-s)!(2s+1)!} \begin{bmatrix} u \\ k+t+1 \end{bmatrix} . \tag{B8}$$

Consequently, keeping in mind that the $C_s^{(1)}(t)$ coefficients are given by Eq. (B6), we obtain the following closed-form expression of the $d_0(\sigma,k,s)$ coefficients associated with the second case of perturbed type-D factorization:

$$d_0(u-s-1,t,u-1)=\frac{2^{s+t-u}(2u)!}{u!}\sum_{k=s}^{u-t-1}(-1)^{k-s}\frac{k!}{(k-s)!(2s+1)!}\begin{bmatrix} u \\ k+t+1 \end{bmatrix} . \tag{B9}$$

Setting $u-1=S_N$, $S_N-s=\sigma$, this expression can be written again:

$$d_0(\sigma,t,S_N)=(\tfrac{1}{2})^{\sigma-t}\frac{(2S_N+1)!(S_N-\sigma)!}{S_N!(2S_N-2\sigma+1)!} \times \sum_{u=0}^{\sigma-t}(-1)^u\begin{bmatrix} S_N-\sigma+u \\ u \end{bmatrix}\begin{bmatrix} S_N+1 \\ \sigma-t-u \end{bmatrix} , \tag{B10}$$

where

$$\begin{bmatrix} n \\ u \end{bmatrix}=\frac{\Gamma(n+1)}{\Gamma(u+1)\Gamma(n-u+1)}$$

is a generalized [10] binomial coefficient. Using the relation

$$(-1)^u\begin{bmatrix} S_N-\sigma+u \\ u \end{bmatrix}=\begin{bmatrix} -S_N+\sigma-1 \\ u \end{bmatrix}$$

and applying Cauchy's formula

$$\begin{bmatrix} n+m \\ z \end{bmatrix}=\sum_{u=0}^{z}\begin{bmatrix} n \\ u \end{bmatrix}\begin{bmatrix} m \\ z-u \end{bmatrix} ,$$

we obtain the compact expression

$$d_0(\sigma,t,s)=(\tfrac{1}{2})^{\sigma-t}\frac{(2s+1)!(s-\sigma)!}{s!(2s-2\sigma+1)!}\begin{bmatrix} \sigma \\ t \end{bmatrix} . \tag{B11}$$

As expected, this expression satisfies the recurrence formula (3.35). Moreover, we note that when applying this recurrence formula for $j\neq 0$, the j dependence of the $d_j(\sigma,k,s)$ coefficients is generated only via the starting value $d_j(0,j,s)=1$. The following relation holds:

$$d_j(\sigma,k,s)=d_0(\sigma,k-j,s) . \tag{B12}$$

Hence we obtain the general expression (3.36). It is easily checked that this expression satisfies the recurrence relation (3.35) together with its associated conditions.

APPENDIX C: DETERMINATION OF THE $\Theta_j(\sigma,k)$ COEFFICIENTS

Values of the Bernoulli numbers $\mathcal{B}_t$ can be found in tables [10,13] or can be calculated recursively by means of the symbolic equation $(1+\mathcal{B})^t-\mathcal{B}_t=0$. In the expansion of $(1+\mathcal{B})^t$, $\mathcal{B}_k$ is to be put instead of $\mathcal{B}^k$. For instance, we have

$$\begin{aligned} &\mathcal{B}_0=1,\ \mathcal{B}_2=\tfrac{1}{6},\ \mathcal{B}_4=-\tfrac{1}{30},\ \mathcal{B}_6=\tfrac{1}{42},\ \mathcal{B}_8=-\tfrac{1}{30} ,\\ &\mathcal{B}_{10}=\tfrac{5}{66},\ \mathcal{B}_{12}=-\tfrac{691}{2730},\ldots ,\\ &\mathcal{B}_1=-\tfrac{1}{2},\ \mathcal{B}_{2t+1}=0 \text{ for any } t>0 . \end{aligned} \tag{C1}$$

Using the expressions (3.64), we get

$$\begin{aligned} &\mathcal{A}_{00}=0,\ \mathcal{A}_{01}=\tfrac{1}{8},\ \mathcal{A}_{02}=-\tfrac{1}{384},\ldots ,\\ &\mathcal{A}_{k0}=1,\ \mathcal{A}_{k1}=\tfrac{1}{12},\ \mathcal{A}_{k2}=-\tfrac{1}{720} \text{ for any } k . \end{aligned} \tag{C2}$$

Applying the recurrence formula (3.63), we obtain

$$\Theta_j(\sigma,\sigma+j)=1 \text{ for any } j \tag{C3}$$

together with the following values which are required for the computation of the expressions (3.71) and (3.72) of the $\mathcal{I}_t(j)$:

$$\begin{aligned} &\Theta_0(2,0)=\tfrac{1}{8},\ \Theta_0(3,1)=\tfrac{5}{24},\ \Theta_0(4,0)=\tfrac{3}{128} ,\\ &\Theta_0(4,2)=\tfrac{7}{24},\ \Theta_1(1,0)=\tfrac{1}{8},\ \Theta_1(2,1)=\tfrac{5}{24} ,\\ &\Theta_1(3,0)=\tfrac{3}{128},\ \Theta_1(3,2)=\tfrac{7}{24},\ \Theta_2(1,1)=\tfrac{1}{12} ,\\ &\Theta_2(2,0)=\tfrac{1}{128},\ \Theta_2(2,2)=\tfrac{1}{6},\ \Theta_3(1,0)=-\tfrac{1}{384} ,\\ &\Theta_3(1,2)=\tfrac{1}{12} . \end{aligned} \tag{C4}$$

APPENDIX D: DETERMINATION OF THE DATA COEFFICIENTS $b_t^{(N)}(j)$

1. $\chi_s\mathcal{H}_{2s}(b^{1/2}x)$ expansion of the perturbation together with a $\binom{v}{u}$ expansion of the perturbed eigenvalues

Using the expression (3.21) of the perturbed ladder function and keeping in mind that $S_{N-\nu}+S_\nu+1=S_N$, one can write, after some rearrangement,

$$K^{(N-\nu)}K^{(\nu)}=\sum_{t=0}^{S_N}y_t\sum_{k=0}^{S_{N-\nu}}\sum_{l=0}^{S_\nu}\begin{bmatrix} \mu-m \\ k \end{bmatrix}\begin{bmatrix} \mu-m \\ l \end{bmatrix}\sum_{s=0}^{S_{N-\nu}-k}\sum_{r=0}^{S_\nu-l}h(s,r,t)C_s^{(N-\nu)}(k)C_r^{(\nu)}(l) , \tag{D1}$$

where $h(s,r,t)$ is given by Eq. (3.7).

Using the relation [10]

$$\begin{bmatrix} \mu-m \\ k \end{bmatrix}\begin{bmatrix} \mu-m \\ l \end{bmatrix}=\sum_{j=k}^{k+l}\begin{bmatrix} j \\ k \end{bmatrix}\begin{bmatrix} k \\ j-l \end{bmatrix}\begin{bmatrix} \mu-m \\ j \end{bmatrix} \tag{D2}$$

and keeping in mind that $w_t^{(N)}(j)$ is the coefficient of $y_t \begin{bmatrix} \mu-m \\ j \end{bmatrix}$ in the expansion of the potential-like function $\mathcal{W}^{(N)}=\sum_{v=1}^{N-1} K^{(N-v)}K^{(v)}$, we obtain, after some rearrangements ($t=0, S_N$; $j=0, S_N-t$),

$$w_t^{(N)}(j)=\sum_{v=1}^{N-1}\sum_{k=0}^{S_v}\sum_{u=k}^{S_v}\begin{bmatrix} j \\ k \end{bmatrix}\begin{bmatrix} j-k \\ u-k \end{bmatrix}\sum_{s=0}^{S_{N-v}-j+k}\sum_{r=0}^{S_v-u} h(s,r,t)C_s^{(N-v)}(j-k)C_r^{(v)}(u) . \tag{D3}$$

Finally, after introducing the expression (3.20) of the $C_s^{(N)}(t)$ and again making some rearrangements, we obtain the expression (3.73) of $w_t^{(N)}(j)$, where

$$\mathcal{X}_1(s,r,t,k,l,j)=\sum_{u=0}^{s-1}\sum_{v=0}^{r-1}2^{u+v}\begin{bmatrix} j \\ u+k \end{bmatrix}\begin{bmatrix} u+k \\ j-v-l \end{bmatrix}h(s-u-1,r-v-1,t) . \tag{D4}$$

2. x^{2s} expansion of the perturbation associated with a $\binom{v}{u}$ expansion of the perturbed eigenvalue

Using the expression (3.37) of the perturbed ladder function, one can write

$$K^{(N-v)}K^{(v)}=\sum_{t=1}^{S_N}x^{2t}\sum_{s=0}^{t-1}\sum_{k=0}^{S_{N-v}-s}\sum_{l=0}^{S_v-t+s+1}C_s^{(N-v)}(k)C_{t-s-1}^{(v)}(l)\begin{bmatrix} \mu-m \\ k \end{bmatrix}\begin{bmatrix} \mu-m \\ l \end{bmatrix} . \tag{D5}$$

Using Eq. (D2) and keeping in mind that $w_t^{(N)}(j)$ is the coefficient of $x^{2t}\binom{\mu-m}{j}$ in the expansion of the potential-like function $\mathcal{W}^{(N)}=\sum_{v=1}^{N-1}K^{(N-v)}K^{(v)}$, we obtain, after some rearrangements

$$w_t^{(N)}(j)=\sum_{v=1}^{N-1}\sum_{s=0}^{t-1}\sum_{k=0}^{j}\sum_{l=j-k}^{S_v-t+s+1}\begin{bmatrix} j \\ k \end{bmatrix}\begin{bmatrix} k \\ l-j+k \end{bmatrix}C_s^{(N-v)}(k)C_{t-s-1}^{(v)}(l) . \tag{D6}$$

After substituting for the $C_s^{(N-v)}(k)$ and $C_{t-s-1}^{(v)}(l)$ from Eq. (3.37) into Eq. (D6), and again making some rearrangements, we obtain the expression (3.73) of $w_t^{(N)}(j)$, where the "coupling coefficient" is

$$\mathcal{X}_2(s,r,t;l,m,j)=\left[\frac{1}{2b}\right]^{s+r-t+1}\sum_{u=0}^{t-1}\frac{(2s-1)!!(2r-1)!!}{(2u+1)!!(2t-2u-1)!!}\times\sum_{k=0}^{j-l}2^k\begin{bmatrix} s-u-1 \\ k \end{bmatrix}\begin{bmatrix} j \\ k+l \end{bmatrix}\sum_{i=0}^{j-m}2^i\begin{bmatrix} r-t+u+1 \\ i \end{bmatrix}\begin{bmatrix} k \\ j-m-i \end{bmatrix} . \tag{D7}$$

3. x^{2s} expansion of the perturbation associated with a $(v+\frac{1}{2})^u$ expansion of the perturbed eigenvalue

Using the expression (3.65) of the perturbed ladder function, we get, after some rearrangements,

$$K^{(N-v)}K^{(v)}=\sum_{t=1}^{S_N}x^{2t}\sum_{j=0}^{S_N-t}(m-\mu-\tfrac{1}{2})^j\sum_{u=0}^{t-1}\sum_{k=0}^{j}C_u^{(N-v)}(k)C_{t-u-1}^{(v)}(j-k) , \tag{D8}$$

and we have

$$w_t^{(N)}(j)=\sum_{v=1}^{N-1}\sum_{u=0}^{t-1}\sum_{k=0}^{j}C_u^{(N-v)}(k)C_{t-u-1}^{(v)}(j-k) . \tag{D9}$$

Substituting for the $C_s^{(N-v)}(k)$ and $C_{t-s-1}^{(v)}(j-k)$ from Eq. (3.65) into Eq. (D9), we obtain the expression (3.73) of the $w_t^{(N)}(j)$ where the "coupling coefficient" is

$$\mathcal{X}_3(s,r,t;l,m,j)=\left[-\frac{1}{b}\right]^{s+r-t+1}\sum_{u=0}^{t-1}\sum_{k=0}^{j}\frac{(2s-1)!!(2r-1)!!l!m!}{k!(j-k)!(2u+1)!!(2t-2u-1)!!4}\times\Theta_j(s-u-1,k)\Theta_m(r-t-u,j-k) , \tag{D10}$$

and the $\Theta_j(\sigma,k)$ are obtainable by means of the recurrence formula (3.63).

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