Quadric solitons and breathers of n-dimensional nonlinear wave equations

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For the n-dimensional nonlinear wave equations $\partial_\alpha\partial_\alpha\varphi=dF_i(\varphi)/d\varphi$, $i=1,2,\ldots$, a kind of general soliton solution is obtained. It contains some interesting specific solutions, such as N multiple solitons and the quadric solitons. The discussion of their properties shows that there exist some motional sphere solitons with a definite radius and some quadric breathers which have various shapes.

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The multidimensional nonlinear wave equations are a kind of common equation. They may describe many important physical phenomena. Concerning the solutions of the equations, many interesting results have been obtained in several previous articles [1–4].

In the present paper, we study further the equations by using the method of function transformation. Let any solution φ of the equations be a function of another function ξ. The equations are changed into a system of equations for φ and ξ. The general solution of the equations of ξ leads to a general soliton solution of the equations of φ. They include some well-known results [5–7]. In addition, we find many interesting soliton solutions such as N multiple quadric solitons and quadric breathers.

Let us consider the n-dimensional nonlinear wave equation

$$\partial_\alpha\partial_\alpha\varphi=\frac{dF_i(\varphi)}{d(\varphi)} , \tag{1}$$

where $\partial_\alpha=\partial/\partial x_\alpha$, $x_0=it$, and $F_i(\varphi)$ for $i=1,2,\ldots$ denote some functions of φ. Here and throughout the paper we adopt a summation convention for repeated indices: a greek index runs from 0 to $n-1$ and any other index runs from 1 to $n-1$, unless it is particularly stated otherwise.

Setting $\varphi=\varphi(\xi)$ to be a function of another function ξ, we easily calculate the following:

$$\begin{aligned}\partial_\alpha\varphi&=\partial_\alpha\xi\frac{d\varphi}{d\xi} ,\\ \partial_\alpha\partial_\alpha\varphi&=\partial_\alpha\xi\partial_\alpha\xi\frac{d^2\varphi}{d\xi^2}+\partial_\alpha\partial_\alpha\xi\frac{d\varphi}{d\xi} .\end{aligned} \tag{2}$$

Substituting (2) into (1) yields

$$\partial_\alpha\xi\partial_\alpha\xi\frac{d^2\varphi}{d\xi^2}+\partial_\alpha\partial_\alpha\xi\frac{d\varphi}{d\xi}=\frac{dF_i(\varphi)}{d\varphi} . \tag{3}$$

Given (3), we can show that if ξ satisfies the system of equations

$$\partial_\alpha\partial_\alpha\xi=0 , \quad \partial_\alpha\xi\partial_\alpha\xi=1 , \tag{4}$$

then (3) becomes an ordinary differential equation

$$\frac{d^2\varphi}{d\xi^2}=\frac{dF_i(\varphi)}{d\varphi} . \tag{5}$$

By simply integrating this equation, we have a solution in the form

$$\xi-\xi_0=\pm\int\frac{d\varphi_i}{\sqrt{2F_i(\varphi)+c}} , \quad \xi_0,c=\text{const} , \tag{6}$$

which contains some well-known plane soliton solutions. Obviously, (4) may have multidimensional solutions. Applying these solutions to (6) will give many interesting soliton solutions to Eq. (1).

We now solve Eqs. (4). We know that the general solution, the complete solution, and the singular solution include all of the solutions for a equation. For Eqs. (4), the complete solution is a simple hyperplane solution.

$$\xi=a_\alpha x_\alpha+\xi_0 , \quad a_\alpha a_\alpha=1 . \tag{7}$$

There does not exist a singular solution of the equation explicitly. Therefore we only take interest in the general solution. Consider a kind of general solution of (4) in the form

$$\xi=f(\zeta_j)+d_\alpha x_\alpha , \quad \zeta_j=b_{j\alpha}x_\alpha+\epsilon_j , \quad d_\alpha,b_{j\alpha},\epsilon_j=\text{const} , \tag{8}$$

where $f(\zeta_j)$ denotes an arbitrary function of ζ_j. Combining (4) with (8), the direct calculation gives

$$\begin{aligned}\partial_\alpha\partial_\alpha\xi&=b_{j\alpha}b_{k\alpha}\frac{\partial^2 f}{\partial\zeta_j\partial\zeta_k}=0 ,\\ \partial_\alpha\xi\partial_\alpha\xi&=b_{j\alpha}b_{k\alpha}\frac{\partial f}{\partial\zeta_j}\frac{\partial f}{\partial\zeta_k}+2d_\alpha b_{j\alpha}\frac{\partial f}{\partial\zeta_j}+d_\alpha d_\alpha=1 .\end{aligned} \tag{9}$$

The arbitrariness of the function $f(\zeta_j)$ leads to the conditions

$$b_{j\alpha}b_{k\alpha}=0 , \quad b_{j\alpha}d_\alpha=0 , \quad d_\alpha d_\alpha=1 . \tag{10}$$

Taking $f(\zeta_j)$ to be a function of ζ_j for $j=1,2,\ldots,N$, (10) implies $\frac{1}{2}(N+1)(N+2)$ equations with $(N+1)n$ constants $b_{j\alpha},d_\alpha$. Hence the integral number N must satisfy the inequality

$$\tfrac{1}{2}(N+1)(N+2)\le(N+1)n . \tag{11}$$

Equation (11) shows that

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$$N=1 \text{ for } n=1\,,\quad N\le 2 \text{ for } n=2\,,$$
$$N\le 4 \text{ for } n=3\,,\quad N\le 6 \text{ for } n=4\,, \tag{12}$$

and so on. For the $N=1, n=2$ case, (10) implies

$$b_{10}^2 b_{11}^2=0\,,\quad b_{10}d_0+b_{11}d_1=0\,,\quad d_0^2+d_1^2=1\,.$$

It gives a solution

$$b_{10}=b_{11}=0\,,\quad d_0=\sqrt{1-d_1^2}\,.$$

Inserting these into (8) leads to $\xi=d_\alpha x_\alpha+\text{const}$, which is the complete solution (7). For the $n\ge 3$ cases, (8) denotes a general solution with arbitrary function $f(\zeta_j)$. It makes (6) a kind of general solution of the n-dimensional nonlinear wave equations. They contain some interesting specific solutions, such as the plane solitons, the N multiple plane solitons, the quadric solitions, the N multiple quadric solitions, the quadric breathers, and so on.

Taking all of $b_{j\alpha}$ in (10) to be equal to zero, (8) becomes the plane solution (7), and (6) gives some plane solitons. The plane solitons are simple but stable since they are equivalent to some solutions in one-dimensional space.

The N multiple solitons of n-dimensional wave equations are known from previous results [5–7]. We will easily show that they are included by the general solution (8) and (6). Selecting

$$f(\zeta_i)=\ln\sum_{j=1}^{N}e^{\zeta_j}=\ln\sum_{j=1}^{N}e^{b_{j\alpha}x_\alpha+\epsilon_j}\,,$$

then (8) becomes

$$\xi=f(\zeta_j)+d_\alpha x_\alpha=\ln\left[e^{d_\alpha x_\alpha}\sum_{j=1}^{N}e^{b_{j\alpha}x_\alpha+\epsilon_j}\right]=\ln\sum_{j=1}^{N}e^{a_{j\alpha}x_\alpha+\epsilon_j}\,, \tag{13}$$

where $a_{j\alpha}=d_\alpha+b_{j\alpha}$ and N satisfies (11). The conditions (10) are simplified into

$$a_{j\alpha}a_{k\alpha}=1\,. \tag{14}$$

Taking N to be $1,2,\ldots,N$, respectively, (13) gives N multiple wave solutions. This assertion has been proved.

Any concrete soliton is a four-dimensional soliton. Its shape is probably one of the quadrics. Now we solve for the quadric solitons. Setting (8) in the form

$$\xi=\zeta_j\zeta_j+d_\alpha x_\alpha=b_{j\alpha}b_{j\beta}x_\alpha x_\beta+(2\epsilon_j b_{j\alpha}+d_\alpha)x_\alpha+\epsilon_j\epsilon_j\,,\quad j=1,2,\ldots,N \tag{15}$$

it describes a general quadratic surface at any definite time. Then we make a coordinates transformation

$$x_k=c_{k\gamma}X_\gamma\,,\quad \sum_{k=1}^{n-1}c_{k\gamma}^2=1 \tag{16}$$

such that

$$\xi=b_{ji}b_{jk}x_ix_k+(2b_{ji}b_{j0}x_0+2\epsilon_j b_{ji}+d_i)x_i+b_{j0}b_{j0}x_0^2+(2\epsilon_j b_{j0}+d_0)x_0+\epsilon_j\epsilon_j$$
$$=b_{ji}b_{jk}c_{is}c_{k\gamma}X_sX_\gamma+(2b_{ji}b_{j0}x_0+2\epsilon_j b_{ji}+d_i)c_{is}X_s+b_{j0}b_{j0}x_0^2+(2\epsilon_j b_{j0}+d_0)x_0+\epsilon_j\epsilon_j\,,\quad j=1,2,\ldots,N \tag{17}$$

$$b_{ji}b_{jk}c_{is}c_{k\gamma}=\delta_{s\gamma}\,,\quad j=1,2,\ldots,N\,. \tag{18}$$

For the four-dimensional case, (17) and (18) denote a sphere with radius

$$R=\left[\left[b_{ji}b_{j0}x_0+\epsilon_j b_{ji}+\frac{d_i}{2}\right]\times\left[b_{k\gamma}b_{k0}x_0+\epsilon_k b_{k\gamma}+\frac{d_\gamma}{z}\right]c_{is}c_{\gamma s}-b_{j0}b_{j0}x_0^2-(2\epsilon_j d_{j0}+d_0)x_0-\epsilon_j\epsilon_j\right]^{1/2}\,, \tag{19}$$

when $R^2>0$. Let the constants satisfy the equations

$$b_{ji}b_{j0}b_{k\gamma}b_{k0}c_{is}c_{\gamma s}=b_{j0}b_{j0}\,,$$
$$2b_{ji}b_{j0}\left[\epsilon_k b_{k\gamma}+\frac{d_\gamma}{2}\right]c_{is}c_{\gamma s}=2\epsilon_j d_{j0}+d_0\,,\quad j,k=1,2,\ldots,N\,. \tag{20}$$

Inserting (20) into (19) leads to

$$R=\left[\left[\epsilon_j b_{ji}+\frac{d_i}{2}\right]\left[\epsilon_k b_{k\gamma}+\frac{d_\gamma}{2}\right]c_{is}c_{\gamma s}-\epsilon_j\epsilon_j\right]^{1/2}=\text{const}\,,\quad j,k=1,2,\ldots,N\,. \tag{21}$$

The center of the sphere obeys the equation

$$X_s=-\left[b_{ji}b_{j0}x_0+\epsilon_j b_{ji}+\frac{d_i}{2}\right]c_{is}\,,\quad j=1,2,\ldots,N,\quad S=1,2,3\,. \tag{22}$$

It denotes a space curve. Thus (17) and (6) give a sphere soliton solution which has definite radius and motional center. It seems to be a motional particle. On the other hand, if the transformation (16) makes (18) the following

$$b_{ji}b_{jk}c_{is}c_{k\gamma}=0 \quad\text{for } s\ne\gamma\,,$$
$$b_{ji}b_{jk}c_{i1}c_{k1}\ne b_{ji}b_{jk}c_{is}c_{ks} \quad\text{for } s=2,3\,, \tag{23}$$

then (17) and (6) will lead to some ellipsoid solitons and hyperboloid solitons.

Further, by appropriately selecting $c_{k\gamma}$, we let the transformation (16) make (15) into the equation

$$\xi=ax_1^2+bx_2^2+cx_3^2+2fx_2x_3+2gx_1x_3+2hx_1x_2+2p(t)x_1+2q(t)x_2+2r(t)x_3+d(t)$$
$$=AX_1^2+BX_2^2+CX_3^2+\frac{\Delta(t)}{D}\,, \tag{24}$$

where A, B, and C are the characteristic values of the equation $U^3-IU^2+JU-D=0$; Δ, D, I, and J are the invariants of transformations

$$\Delta(t)=\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{vmatrix}, \quad D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \tag{25}$$

$$I=a+b+c\ , \quad J=ab+bc+ca-f^2-g^2-h^2\ . \tag{26}$$

When $A=B=C$ and $\Delta(t)/D<0$, (24) denotes a sphere whose center is inmovable $X_s=0$ and radius is variable:

$$R=\sqrt{A\Delta(t)/D}\ , \tag{27}$$

which is a sphere wave.

We now take a specific solution of (8) in the form

$$\begin{aligned}\xi&=\ln\sum_{j=1}^{N} e^{\zeta_j^2}+d_\alpha x_\alpha \\ &=\ln\left[e^{d_\alpha x_\alpha}\sum_{j=1}^{N} e^{(b_{j\alpha}x_\alpha+\epsilon_j)^2}\right] \\ &=\ln\sum_{j=1}^{N}\exp[(b_{j\alpha}x_\alpha+\epsilon_j)^2+d_\alpha x_\alpha]\ .\end{aligned} \tag{28}$$

According to the discussions on (13) and (15), it is clear that substituting (28) into (6) will yield the N multiple quadric soliton solutions. We take interest in the quadric breathers [8]. In order to obtain them, we discuss the $n=4, N=4$ case. In this case, setting (8) as

$$\begin{aligned}\xi&=\ln[\xi_0(e^{\zeta_1^2+\zeta_3^2}-e^{\zeta_2^2+\zeta_4^2})]+d_\alpha x_\alpha \\ &=\ln[\xi_0(e^{\zeta_1^2+\zeta_3^2+d_\alpha x_\alpha}-e^{\zeta_2^2+\zeta_4^2+d_\alpha x_\alpha})]\ , \quad \xi_0=\text{const}\ ,\end{aligned} \tag{29}$$

we may make a coordinate transformation whose form is (16) such that

$$\begin{aligned}\eta_j&=\zeta_j^2+\zeta_{j+2}^2+d_\alpha x_\alpha=(b_{j\alpha}x_\alpha+\epsilon_j)^2+(b_{j+2,\alpha}x_\alpha+\epsilon_{j+2})^2+d_\alpha x_\alpha \\ &=a_jx_1^2+b_jx_2^2+c_jx_3^2+2f_jx_2x_3+2g_jx_1x_3+2h_jx_1x_2+2p_j(t)x_1+2q_j(t)x_2+2r_j(t)x_3+d_j(t) \\ &=A_jX_1^2+B_jX_2^2+C_jX_3^2+\frac{\Delta_j(t)}{D_j}\ , \quad j=1,2 \text{ with no sum being taken on } j\ ,\end{aligned} \tag{30}$$

where A_j, B_j, C_j, $\Delta_j(t)$, and D_j have similar definitions, respectively, to A, B, C, $\Delta(t)$, and D in (24). Let the constants satisfy

$$A_1=A_2\ , \quad B_1=B_2\ , \quad C_1=C_2\ , \quad \frac{\Delta_1(t)}{D_1}=-\frac{\Delta_2(t)}{D_2}\ . \tag{31}$$

Inserting (30) and (31) into (29) yields

$$\begin{aligned}\xi&=\ln[\xi_0e^{A_1X_1^2+B_1X_2^2+C_1X_3^2}(e^{\Delta_1(t)/D_1}-e^{-\Delta_1(t)/D_1})]\ , \\ &=\ln\{2\xi_0e^{A_1X_1^2+B_1X_2^2+C_1X_3^2}\sinh[\Delta_1(t)/D_1]\}\ .\end{aligned} \tag{32}$$

Let ξ_0 and D_1 be some imaginary numbers so that

$$\xi_0=i\xi_0'\ , \quad D_1=iD_1'\ .$$

Given (32) we have

$$\xi=\ln\left[2\xi_0'e^{A_1x_1^2+B_1X_2^2+C_1X_3^2}\sin\left[\frac{\Delta_1(t)}{D_1'}\right]\right]\ . \tag{33}$$

Equations (33) and (6) will give some quadric breathers such as (a) the sphere breathers, when $A_1=B_1=C_1$; (b) the ellipsoid breathers, when A_1,B_1,C_1 have the same sign; (c) the hyperboloid breathers, when A_1,B_1,C_1 have different signs.

The quadric breathers have many obvious and interesting properties. For example, the sphere breathers behave like some breathing abdomens. They will describe various actual physical phenomena.

We consider some particular examples at the end of the paper. By applying $F_i(\varphi)=\frac{1}{2}a\varphi^2-\frac{1}{4}b\varphi^4$ to (6), we obtain a general soliton solution of the φ^4 field equation as

$$\text{arcsech}(\sqrt{b/2a}\,\varphi)=\xi-\xi_0\ , \quad \xi_0=\text{const}\ . \tag{34}$$

In the $(2+1)$-dimensional case, we have a simple solution of (10)

$$\begin{aligned}&b_{10}=i\ , \quad b_{11}=1\ , \quad b_{1i}=b_{jk}=0\ , \quad i,j>1 \\ &b_{12}=1\ , \quad d_{10}=d_{11}=d_{1i}=0\ , \quad i>2\ .\end{aligned} \tag{35}$$

Inserting (35) into (8) and further into (34), we have an explicit form of the general soliton solution

$$\text{arcsech}(\sqrt{b/2a}\,\varphi)=\xi_1=f(x-t)+y\ , \tag{36}$$

which moves along the **x** direction. Here, any definite value ξ_1 determines a value of φ and a general plane curve $y=\xi_1-f(x-t)$. Therefore (36) includes arbitrary plane-curve solitons with various shapes.

For the $(3+1)$-dimensional case, a similar result is given:

$$\text{arcsech}(\sqrt{b/2a}\,\varphi)=\xi_2=f(x-t)+\frac{1}{\sqrt{2}}(y+z)\ . \tag{37}$$

Equation (37) manifestly implies some interesting quadric solitons. For example, taking $f(x-t)=[R^2-(x-t)^2]^{1/2}$ yields a circular cylindrical surface soliton with radius R.

Now we come to show that the solitons (36) and (37) are stable when $a<0$. We know that the energy of the φ^4

filed can be written in the form

$$H=\int d^{n-1}x\left[\tfrac{1}{2}(\partial_i\varphi\partial_i\varphi-\partial_0\varphi\partial_0\varphi)+\tfrac{1}{2}a\varphi^2-\tfrac{1}{4}b\varphi^4\right]\ . \tag{38}$$

In the case of stable field (38) becomes

$$\begin{aligned}H_0&=\int^{d-1}x(\tfrac{1}{2}\partial_i\varphi\partial_i\varphi+\tfrac{1}{2}a\varphi^2-\tfrac{1}{4}b\varphi^4)\\&=\int d^{n-1}x\,\mathcal{H}[\varphi,\partial_i\varphi]\ ,\end{aligned} \tag{39}$$

and the φ^4 field equation gives

$$\partial_i\partial_i\varphi=a\varphi-b\varphi^3 \tag{40}$$

since $\partial_0\varphi=0$. The application of (39) and (40) leads to the variation and second variation

$$\begin{aligned}\delta H_0&=\int d^{n-1}x\left[\frac{\partial\mathcal{H}}{\partial\varphi}-\partial_i\frac{\partial\mathcal{H}}{\partial(\partial_i\varphi)}\right]\delta\varphi\\&=\int d^{n-1}x(a\varphi-b\varphi^3-\partial_i\partial_i\varphi)\delta\varphi=0\ ,\\\delta^2H_0&=\int d^{n-1}x\left[\frac{\partial^2\mathcal{H}}{\partial\varphi^2}\delta\varphi^2+\frac{\partial^2\mathcal{H}}{\partial(\partial_i\varphi)^2}(\delta\partial_i\varphi)^2\right]\\&=\int d^{n-1}x[(a-3b\varphi^2)\delta\varphi^2+(\delta\partial_i\varphi\delta\partial_i\varphi)]\ .\end{aligned} \tag{41}$$

According to the principle of least energy [9], the conditions

$$\delta H_0=0\ ,\quad \delta^2H_0>0 \tag{42}$$

make the energy a minimum and the corresponding solitons the stable ones. Given (41) and (42), we have the condition of stabilization

$$a-3b\varphi^2>0\ , \tag{43}$$

since $\delta\varphi^2>0$ and $\delta(\partial_i\varphi)\delta(\partial_i\varphi)>0$. The solitons (36) and (37) have the property

$$0\le\sqrt{b/2a}\ \varphi\le1\ . \tag{44}$$

Inserting (44) into (43) yields

$$(1-6)a=-5a>0\ ,$$

that is,

$$a<0\ . \tag{45}$$

Thus we obtain the conclusion which, for the φ^4 field equation with $F_i=\frac{1}{2}a\varphi^2-\frac{1}{4}b\varphi^4$, $a<0$, has general soliton solutions that are stable.

For the sine-Gordon equation with $F_i=-\cos\varphi$ and other wave equations, we may give their general soliton solutions and some interesting properties by using the above-mentioned method.

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