Diffusion in models of modulated area-preserving maps

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We investigate the diffusion in the action variable when the frequency of an integrable isochronous map is modulated. Purely stochastic, hyperbolic, or periodic deterministic modulations are considered. The diffusion coefficient in the invariant for the unperturbed map is exactly determined and shown to be nonzero, except in the last case, when the modulation is smooth and nonzero, due to the presence of topological barriers.

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The diffusion in chaotic regions of phase space still remains an open problem, whose implications are physically quite relevant [1–7]. Indeed, the lifetime of many physical systems, such as a spinning planet, a beam in a circular accelerator, or the toroidal surfaces in a confined plasma, seems to depend critically on the existence of a diffusion process of the orbits in phase space.

It is usually believed that for large perturbations with respect to the integrable case a smooth map exhibits diffusion and that the quasilinear approximation holds [1,2]. This is equivalent to assuming that the angular variables randomize, driving the diffusion of the action. At the moment the only exact result is available for the continuous sawtooth map as proved in [7]: some few results, in a probabilistic setting, also hold in the discontinuous case [8]. Some numerical experiments have shown that in an intermediate regime the diffusion coefficient can exceed the quasilinear value and this new regime has been denominated superlinear [9].

When the perturbation is small and satisfies some regularity conditions, the Kol'mogorov-Arnol'd-Moser (KAM) curves confine the orbits and no large-scale diffusion can exist. However, if the map is coupled with a hyperbolic degree of freedom a slow (Arnol'd) diffusion can appear, even though no rigorous results are yet available for a generic four-dimensional map [10–12].

In this Brief Report we propose a simple model obtained by modulating an integrable area-preserving map with constant frequency with a stochastic, hyperbolic, or periodic modulation. In the first case, describing the effect of an external noise on the system, a process with independent, identically distributed variables whose distribution is absolutely continuous with respect to the Lebesgue measure and a Bernoulli process are considered. In the second case, describing the coupling with a hyperbolic degree of freedom, the deterministic map $T(\alpha)=2\alpha$ mod 1 modulates the frequency. In almost all the cases, the diffusion is proved to exist and the diffusion coefficient is rigorously computed [13,14]. The asymptotic behavior for small and large modulation amplitudes ϵ is found to behave as ϵ^2 as $\epsilon \to 0$ and to reach the quasilinear value as $\epsilon \to \infty$, except for the Bernoulli modulation where asymptotic oscillations prevent the convergence. Still, in this case and for particular values of the parameters, one can also prove the existence of ballistic motions. The intermediate region crucially depends on the particular process, and superlinear regimes are observed. When the modulation is periodic and smooth (analytic), then the existence of 2-tori on the associated volume-preserving map is proved, preventing any diffusion, if the frequencies satisfy a diophantine condition. If the modulation is not sufficiently smooth then the diffusion exists and the coefficient is the quasilinear one if ϵ is an integer.

We consider an integrable map with modulated frequency

$$\begin{aligned} \theta_{n+1} &= \theta_n + \omega + \epsilon f(\alpha_n) \bmod 2\pi , \\ j_{n+1} &= j_n + V(\theta_n) . \end{aligned} \tag{1}$$

We recall that a perturbation of the θ motion was first proposed in [15] in order to compute the diffusion coefficient for the standard map. The following particular cases for the modulation α_n are considered:

(1) Stochastic modulation: α_n are independent identically distributed random variables and $f(\alpha)=\alpha$, the dis-

tribution $d\mu(x)$ of the process being (a) absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$; (b) atomic $d\mu(x)=\sum_{k=1,m}p_k\delta(x-q_k)$, where p_k are positive weights with unit sums and q_k are in [0,1], namely, a Bernoulli process. The α_n are the iterates of a dynamical system $T, \alpha_{n+1}=T(\alpha_n)$ that we choose as the following.

(2) Hyperbolic modulation: $f(\alpha)=\alpha$, $\forall\alpha\in[-\frac{1}{2},\frac{1}{2}[$ and $T(\alpha)=2\alpha \bmod [0,1[-\frac{1}{2}$.

(3) Periodic modulation: $T(\alpha)=\alpha+\Omega \bmod [0,2\pi[$, where the frequencies are strongly nonresonant, namely, ω,Ω satisfy a diophantine condition. Here also we consider two distinct possibilities: (a) the modulation $f(\alpha)=\alpha-\pi, \forall\alpha\in[0,2\pi[, f(\alpha)=f(\alpha+2\pi), \forall\alpha\in\mathbb{R}$, with ϵ integer; (b) the smooth modulation $f(\alpha)=f(\alpha+2\pi)$, $\alpha\in\mathbb{R}$, where f is analytic in a strip $|\mathrm{Im}\alpha|\le\Delta$.

We consider the diffusion of the invariant J for the unperturbed map ($\epsilon=0$), which is given by

$$J=j-\sum_{k=-\infty}^{+\infty}\frac{V_k}{e^{ik\omega}-1}e^{ik\theta} . \tag{2}$$

The potential $V(\theta)$ is assumed to be with zero mean and analytic in a strip $|\mathrm{Im}\theta|\le\Delta_V$ so that its Fourier coefficients V_k decay as $e^{-\Delta_V|k|}$. We determine the diffusion coefficient, which is defined as usual

$$D(J_0,\epsilon)=\lim_{n\to\infty}\frac{1}{2n}\langle (J_{n+1}-J_0)^2\rangle , \tag{3}$$

where the average is taken over the initial angle θ_0 and the distribution of the process in the stochastic case, and over the initial angle θ_0 and the initial value α_0 in the deterministic case. For the models (1a) the diffusion coefficient is positive, independent of J_0, and has the following behavior:

$$D\sim\epsilon^2 \text{ for } \epsilon\to 0, \quad \lim_{\epsilon\to\infty}D=D_{\mathrm{ql}}\equiv\tfrac{1}{2}\sum_{k=-\infty}^{\infty}V_k^2 , \tag{4}$$

where D_{ql} is the quasilinear coefficient. In particular, it is given by

$$D=\sum_{k=-\infty}^{+\infty}\frac{|V_k|^2}{|e^{ik\omega}-1|^2}\mathrm{Re}\left[1-\chi(\epsilon k)-e^{ik\omega}\frac{[1-\chi(\epsilon k)]^2}{1-e^{ik\omega}\chi(\epsilon k)}\right], \tag{5}$$

where $\chi(x)$ is the characteristic function of the probability distribution.

For a Bernoulli process (1b) with states $s_1,\ldots,s_m$ and weights $p_1,\ldots,p_m$, defining $Q_k=\sum_{j=1}^{m}p_je^{i\epsilon s_jk}$ we have

$$D=D_{\mathrm{ql}}+\mathrm{Re}\sum_{k=-\infty}^{+\infty}|V_k|^2\frac{Q_ke^{ik\omega}}{1-Q_ke^{ik\omega}} , \tag{6}$$

provided that a sort of diophantine condition $|\sum_{j'<j}p_jp_{j'}\sin^2k\epsilon 2^{-1}(s_j-s_{j'})|^{-1}\le\gamma|k|^{\mu}$ for some $\gamma,\mu>0$ is satisfied: in this case D is still independent of J_0 and positive. Moreover, it behaves like the first expression in (4) whenever $\epsilon\to 0$. Instead, when $\epsilon\to+\infty$, the diffusion coefficient does not tend to any limit, as is easy to deduce from Eq. (6); this means that we cannot expect the random-phase approximation be true in the limit of large perturbations, as is usually believed to happen for standardlike maps. If conversely $\epsilon s_j+\omega$ are rationally dependent, a ballistic motion is observed (J_n grows linearly with n) and the diffusion coefficient is infinite.

For the hyperbolic modulation [case (2)] the result reads

$$D=D_{\mathrm{ql}}+\sum_{k=-\infty}^{+\infty}|V_k|^2\sum_{r=0}^{\infty}\cos[k\omega(r+1)]\frac{\sin\left[\dfrac{k\epsilon}{2}\left[1-\dfrac{1}{2^{r+1}}\right]\right]}{\dfrac{k\epsilon}{2}\left[1-\dfrac{1}{2^{r+1}}\right]}\prod_{j=1}^{r+1}\cos\left[\frac{\epsilon}{2}k\left[1-\frac{1}{2^j}\right]\right] , \tag{7}$$

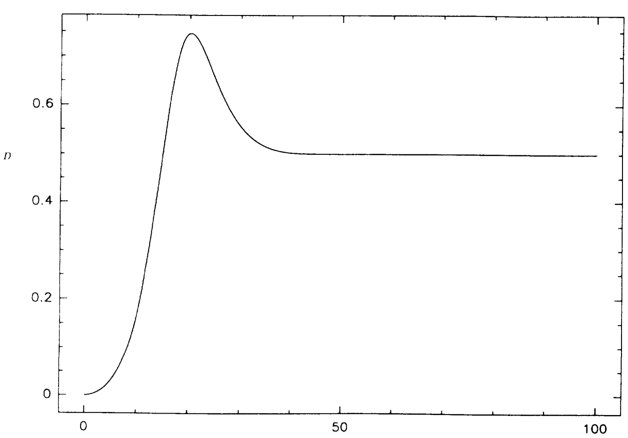

FIG. 1. Diffusion coefficient D vs perturbation parameter ϵ for the case of $V(\theta)=\sqrt{2}\cos(\theta)$. The quasilinear estimate is $\frac{1}{2}$. Gaussian case with mean $a=0.09$ and standard deviation $\sigma=0.08$.

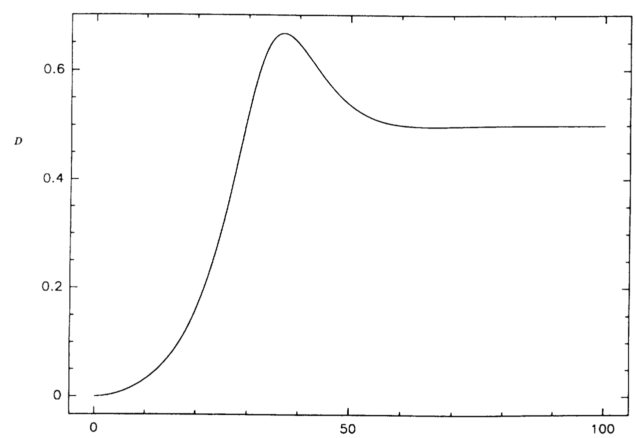

FIG. 2. Diffusion coefficient D vs perturbation parameter ϵ for the case of $V(\theta)=\sqrt{2}\cos(\theta)$. Gaussian case with mean $a=-0.09$ and standard deviation $\sigma=0.05$.

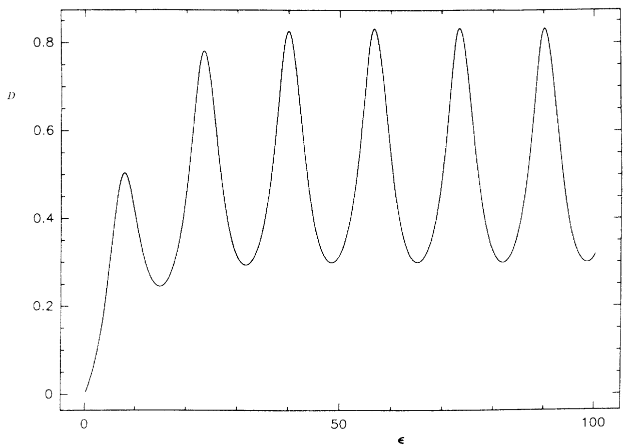

FIG. 3. Diffusion coefficient D vs perturbation parameter ϵ for the case of $V(\theta)=\sqrt{2}\cos(\theta)$. Two-state Bernoulli perturbation with $p_1=0.25$, $p_2=0.75$ and $s_1=0.375$, $s_2=-0.125$.

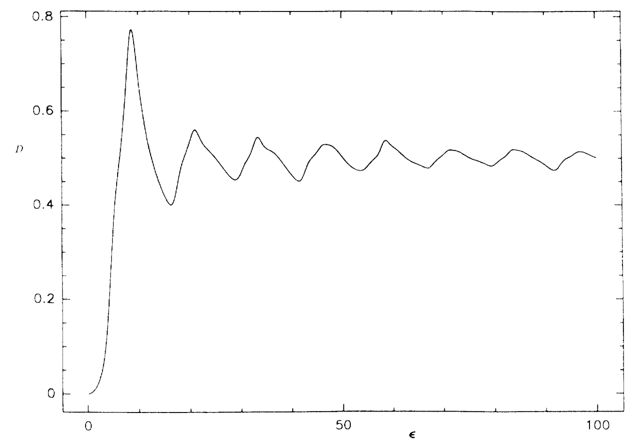

FIG. 4. Diffusion coefficient D vs perturbation parameter ϵ for the case of $V(\theta)=\sqrt{2}\cos(\theta)$. Case of the hyperbolic modulation (2).

and the positivity of D can only be checked numerically; the asymptotic behaviors are still of the type (4). For a periodic modulation the diffusion is present when in case (3a) and $D=D_{\mathrm{ql}}$.

When the modulation is smooth and the frequencies are nonresonant [case (3b)], two-dimensional tori are proved to exist and the diffusion is absent.

Finally, in case (3a), but with noninteger $\epsilon \in \mathbb{R}$, no answer is available. Preliminary numerical computations seem to show that diffusion is present with D close to the quasilinear value.

The intermediate types are quite interesting in all the previous cases in which the diffusion is present. Indeed, a superlinear regime is observed and the approach to the quasilinear value can occur in many different ways. In Figs. 1 and 2 we give an example of such a different behavior for a Gaussian process simply by changing the mean value and variance. It is important to notice that the qualitative behavior of the diffusion coefficient as a function of ϵ does not change if we consider other choices of the probability distribution in place of the Gaussian one. Figure 3 shows the persistency of the oscillating trend of the diffusion coefficient for a two-state Bernoulli perturbation. In Fig. 4 the behavior for the hyperbolic modulation (2) is also shown. The fact that for the Gaussian processes the diffusion coefficient does not oscillate is consistent with a recent work by Ichikawa and coworkers [16], where the addition of a nonlinear frequency dependence to the standard map makes the diffusion coefficient monotonic with respect to the parameter ϵ.

To conclude, the model we have presented describes the diffusion of a generic Hamiltonian map, in the anisochronous case, close to an invariant curve of frequency ω, if the variation of the frequency with the action is small in a large region of phase space, so that to replace it with the isochronous map is (not yet rigorously) justified. The statistics of the system has not been obtained but certainly deserves to be analyzed; relations with physical situations such as the electrostatic model for a plasma should also be explored.

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