Driving systems with chaotic signals

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Negativity of the *conditional Lyapunov exponents* is a necessary and in many cases a sufficient condition for the occurrence of synchronization between a chaotic drive and the response subsystem. Treating one of the variables of coupled logistic maps as the drive, our numerical simulations show that the negativity of the conditional Lyapunov exponents does not always guarantee synchronization and, additionally, the domain of initial conditions for the drive variables needs to be specified in which case the synchronization occurs. Additionally, an example showing the pitfalls of the numerical study of synchronization when the conditional Lyapunov exponent is positive is presented.

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I. INTRODUCTION

A composite, nonlinear dynamical system comprising two subsystems that are internally coupled, such that the behavior of the second system is dependent on the first, but not vice versa, is termed *drive-decomposable.* The subsystem providing the chaotic signal is called the *drive,* and the one being driven is called the *response* subsystem. Consider an m-dimensional system describing the discrete-time map given by

$$v_{n+1}=f_n(u,w)\ . \tag{1}$$

Vector v represents the drive variables that are used in the response. The k-dimensional vector u denotes those drive variables that are not part of the response subsystem and can be defined as

$$u_{n+1}=g_n(v,u)\ . \tag{2}$$

We can construct the l-dimensional response vector w as

$$w_{n+1}=h_n(v,w)\ . \tag{3}$$

The overall dimension of the composite system (1)–(3) is equal to $m+k+l$.

Pecora and Carroll [1] have treated a special case of *homogeneous driving* when $k=l$ and $f=h$. In such a situation, the response is the same as in that part of the drive that is not providing the drive signal and, therefore, their functional forms are identical. Pecora and Carroll studied [1,2] continuous-time, differential, nonlinear dynamical systems such as the Lorenz model, the Rossler model, and hysteretic circuits, and found that when the Lyapunov exponents of the response subsystem (termed the *conditional Lyapunov exponents*) are negative, the response variables will synchronize with the drive variables. This happens even when the subsystem has different initial conditions from that of the drive. Negativity of the conditional Lyapunov exponenets (CLE's) is, therefore, stated to be a necessary condition and, in many cases, also a sufficient condition. The adjective *conditional* has been used to signify the Lyapunov exponent's dependence on the drive vector v.

In this report, a system of two coupled logistic maps has been studied to seek the synchronization effects. The coupled system is given by

$$X_{1_{n+1}}=(1-\epsilon)\mu X_{1_n}(1-X_{1_n})+\epsilon X_{2_n}\ , \tag{4}$$

$$X_{2_{n+1}}=(1-\epsilon)\mu X_{2_n}(1-X_{2_n})+\epsilon X_{1_n}\ . \tag{5}$$

Here, we have considered a case of homogeneous driving where $m=k=l=1$. Out of the two (X_1,X_2) system variables, one (X_2) has been treated as the drive, and the response system corresponding to the remaining variable (X_1) is constructed. In the strict sense, the coupled subsystems (4) and (5) are not drive-decomposable. However, in the situation when ϵ is small, the CLE's do predict correct synchronization behavior. Treating X_2 as the drive, the response subsystem can then be written as

$$X_{3_{n+1}}=(1-\epsilon)\mu X_{3_n}(1-X_{3_n})+\epsilon X_{2_n}\ . \tag{6}$$

It can be easily seen that Eq. (6) is a replica of Eq. (4).

II. RESULTS AND DISCUSSION

The coupled system represented by (4) and (5) can exhibit unique stable, periodic, or chaotic behavior as the coupling strength ϵ is varied. To ascertain that the drive is chaotic, the parameter space was scanned to obtain a region wherein the largest Lyapunov exponent becomes positive. Figure 1 shows a plot of conventional Lyapunov exponents versus coupling parameter ϵ wherein the two exponents have been computed using the algorithm proposed by Wolf *et al.* [3]. Initial conditions of $X_{1_0}=0.4$ and $X_{2_0}=0.2$ have been used for the simulation. For $\mu=3.7$, and $0.045<\epsilon<0.056$ or $\epsilon>0.467$, the coupled system exhibits periodic behavior. For the subsequent simulations, parameter values of $\mu=3.7$ and $\epsilon=0.09$, for which the positive Lyapunov exponent condition is realized, have been used.

Treating X_2 as the drive variable and X_3 as the response variable, Eqs. (4)–(6) were iterated simultaneously. The initial conditions chosen are $X_{1_0}=0.4$,

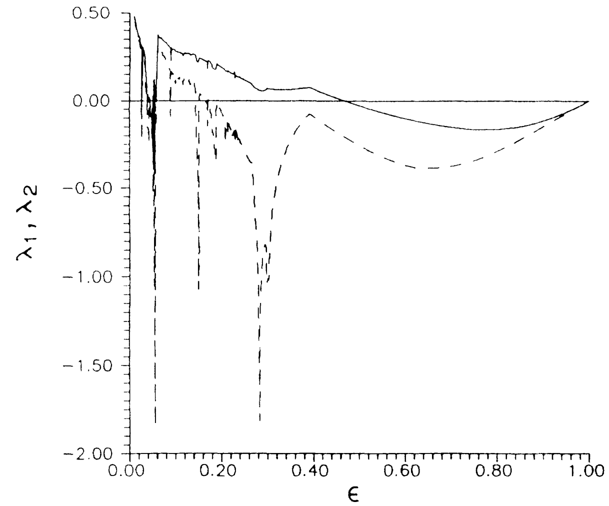

FIG. 1. Plot of Lyapunov exponents vs the coupling parameter ϵ for $\mu=3.7$. Solid and dashed lines represent conventional Lyapunov exponents λ_1,λ_2, respectively.

$X_{2_0}=0.2$, and $X_{3_0}=0.7$. The synchronization between X_{1_n} and X_{3_n} was monitored only after 10 000 iterations. If, for the last 100 iterations, the mean of the absolute value of the difference $(X_{1_n}-X_{3_n})$ equals zero, then we assume that the synchronization has taken place. Figure 2 shows the plot of the mean of abs$(X_{1_n}-X_{3_n})$ versus the coupling parameter ϵ. Additionally, the conditional Lyapunov exponent of the response subsystem represented by (6) has also been plotted. For $\varepsilon>0.21$, the synchronization and negativity of the conditional Lyapunov exponent coincide. A region $(0.067<\epsilon<0.103)$ where the conditional Lyapunov exponent is negative but X_{1_n} and X_{3_n} do not synchronize can also be seen.

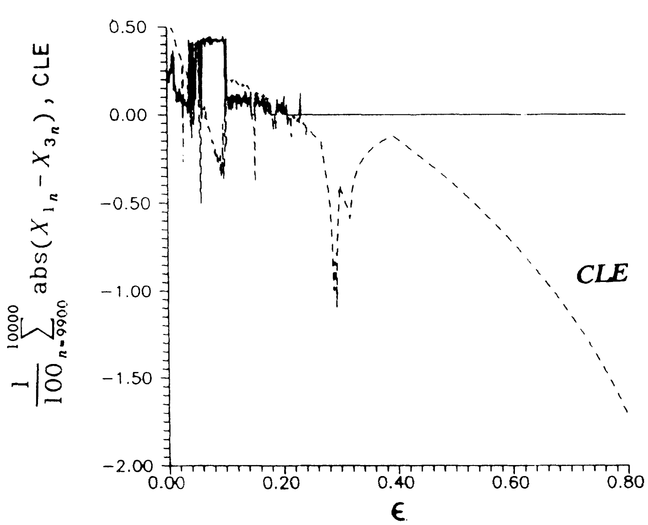

FIG. 2. Plot depicting the synchronization behavior and the conditional Lyapunov exponent (CLE) as a function of coupling parameter ϵ. Solid curve, abs$(X_{1_n}-X_{3_n})$; dashed curve, CLE.

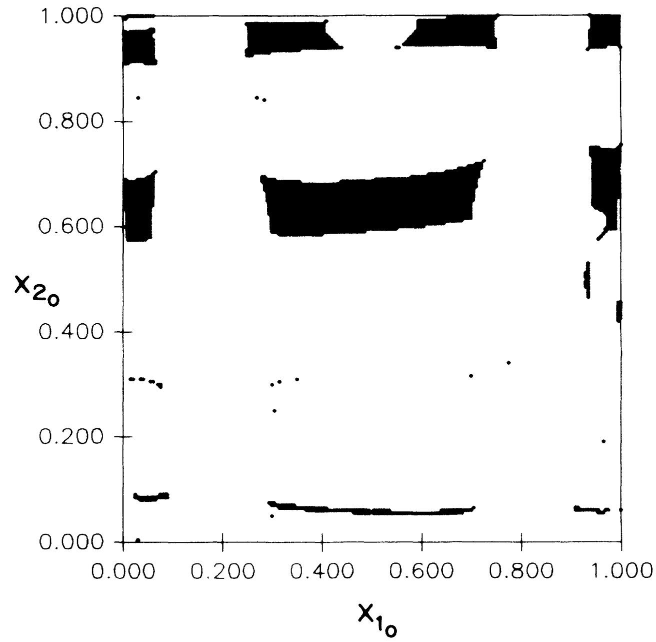

FIG. 3. Initial conditions X_{1_0},X_{2_0} for which synchronization occurs when CLE is negative ($\mu=3.7$, $X_{3_0}=0.7$).

In order to investigate the influence of drive initial conditions on synchronization, these were systematically varied. For $\epsilon=0.09$, and $X_{3_0}=0.7$, the initial conditions X_{1_0} and X_{2_0} were scanned at the interval of 0.005, and the dynamics of (4), (5), and (6) was examined. The results are shown in Fig. 3, where the dots represent those initial conditions for which the synchronization has occurred. It was also found that some of these initial conditions result in the periodic drive.

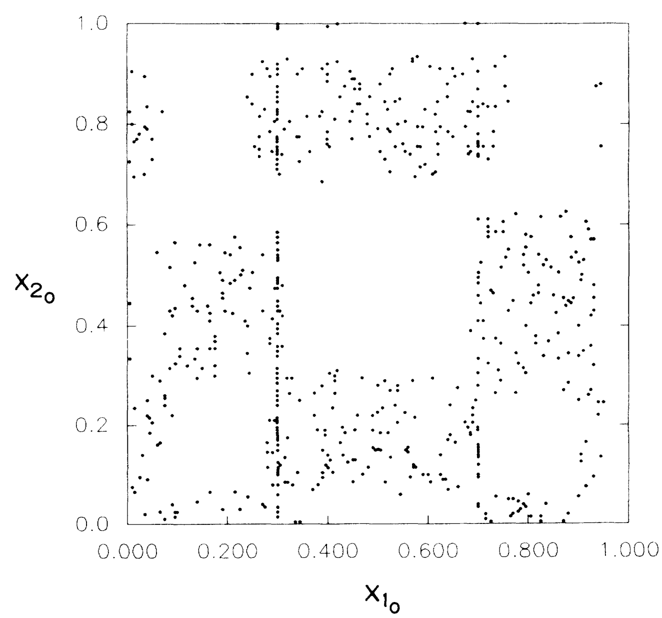

FIG. 4. Initial conditions X_{1_0},X_{2_0}, which evolve to synchronization when CLE is positive.

It has been observed that for some set of initial conditions, X_{1_n} and X_{3_n} synchronize, although the conditional Lyapunov exponent with respect to the response subsystem remains positive. Figure 4 shows the plot of X_{1_0} vs X_{2_0}, wherein the dots correspond to those initial conditions where such a phenomenon is observed. One can easily recognize this to be the pitfall of numerical computations. The flaw arises because of the noninvertible nature of the logistic map and the finite precision of computers. The noninvertibility leads to each value of the map having two preimages. It is thus possible to begin with different initial conditions and get mapped to the same point. The finite precision of the computers in use also ensures such behavior. Once the two points have the same value, they will follow the same trajectory thereafter, regardless of the stability situation. A typical example exemplifying an occurrence of such an event is presented below. Consider a situation where $X_{1_0}=0.3$, $X_{2_0}=0.19$, $X_{3_0}=0.7$, $\epsilon=0.09$, and $\mu=3.7$. In the next iteration, X_1 and X_3 are mapped to the same values as $X_1=X_3=0.72417$ and $X_2=0.54518$.

The values of X_1 and X_3 remain equal thereafter. However, simply changing the last digit in X_3 by 1 will cause the points to diverge, in accordance with the positive conditional Lyapunov exponent. This occurs at several X_2 initial conditions with $X_{1_0}=0.3$ and $X_{3_0}=0.7$, as shown by the many points lying on the vertical line at $X_{1_0}=0.3$. Similar apparent synchronization can occur corresponding to different initial conditions, as seen in Fig. 4.

Pecora and Carroll [1] based the occurrence of synchronization essentially on the negativity of the Lyapunov exponents of the response subsystem only. Considering the composite system under study, this means the computation of the Lyapunov exponent corresponding to the single equation (6). We computed the Lyapunov exponents with respect to the individual drive variables, in a manner similiar to conditional Lyapunov exponents, to check whether they have any effect on the synchronization behavior. Since the drive vectors are represented by (4) and (5), the individual Lyapunov exponents are evaluated according to

$$\lambda_{c_1}=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\ln\left|\frac{\partial f_1}{\partial X_1}\right| , \tag{7}$$

$$\lambda_{c_2}=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\ln\left|\frac{\partial f_2}{\partial X_2}\right| , \tag{8}$$

where f_1 and f_2 correspond to (4) and (5), respectively. An interesting observation about the behavior of λ_{c_1} and λ_{c_2} is that they (along with CLE for the response subsystem) attain negative values when the synchronization takes place, whereas, for no synchronization, λ_{c_1} and λ_{c_2} remain positive.

III. CONCLUSION

It has been stated [1] that the negativity of the conditional Lyapunov exponents is a necessary and, in many cases, a sufficient condition for the occurrence of synchronization between drive and response subsystems. Our numerical simulations show that the negativity of the conditional Lyapunov exponent does not always guarantee synchronization. There also exists a set of initial conditions for the drive variables, within which the synchronization will take place. An example illustrating the pitfalls of the numerical computations in the case of synchronization when the conditional Lyapunov exponent becomes positive has also been presented.

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