Three-dimensional theory of the small-signal high-gain free-electron laser including betatron oscillations

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We have developed a three-dimensional free-electron laser (FEL) theory in the small-signal high-gain regime based upon the Maxwell-Vlasov equations including the effects of the energy spread, the emittance, and the betatron oscillations of the electron beam. The radiation field is expressed in terms of the Green's function of the inhomogeneous wave equation and the distribution function of the electron beam. The distribution function is expanded in terms of a set of orthogonal functions determined by the unperturbed electron distributions. The coupled Maxwell-Vlasov equations are then reduced to a matrix equation, from which a dispersion relation for the eigenvalues is derived. The growth rate for the fundamental mode can be obtained for any initial beam distribution including the hollow-beam, the water-bag, and the Gaussian distribution. Comparisons of our numerical solutions with simulation results and with other analytical approaches show good agreements except for the one-dimensional limit. We present a handy interpolating formula for the FEL gain of a Gaussian beam, as a function of the scaled parameters, that can be used for a quick estimate of the gain. The present theory can be applied to the beam-conditioning case by a few modifications.

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I. INTRODUCTION

Various analytical approaches have been proposed for the calculation of the gain in a free-electron laser (FEL) operating in the high-gain regime before saturation. It is widely known that transverse emittance and betatron oscillation can significantly reduce the gain in this regime, due to a spread in the longitudinal velocity of electrons. One approach to study these effects is based on an integro-differential eigenvalue equation involving the radiation field alone, derived by reducing the coupled Maxwell-Vlasov equations [1]. However, the inclusion of the emittance and the betatron oscillation effects makes it very difficult to solve the equation exactly. Recently, Yu, Krinsky, and Gluckstern [2] have used a variational method to solve the equation approximately for the water-bag distribution of the beam. The principle behind this method is the fact that the error in the eigenvalue depends quadratically on errors in the trial function. However, the success of their analysis depends largely on the electron-beam distribution and the choice of the trial function.

In this article, we present an approach based on an orthogonal expansion of the electron distribution function. This method has been widely used in the study of beam instabilities in particle accelerators [3]. Starting with the Maxwell-Vlasov equations and equations of motion for an electron, we combine them into a single integral equation for the *electron distribution function.* Since the betatron oscillation, the emittance, and the energy spread are all beam parameters, it may be simpler to find the change in the beam distribution due to these effects rather than in the radiation field. The radiation field is expressed explicitly in terms of the Green's function of the inhomogeneous wave equation and the electron distribution function. The perturbed distribution function is then expanded in terms of a set of orthogonal functions determined by the unperturbed distribution function. This expansion converts the integral equation into a matrix equation, from which a dispersion relation for the eigenvalues is derived. This dispersion relation has a form similar to that in plasma physics. The present method has the advantage that the higher-order terms in the expansion can in principle be determined in a systematic fashion. The series expansion converges very quickly, unless the Rayleigh range is much longer than the gain length of the one-dimensional theory (in which case the three-dimensional effects are unimportant). As a matter of fact, one can obtain an accurate eigenvalue by taking only the lowest-order expansion term. In this approximation, the dispersion relation becomes a scalar equation.

Recently, the idea of electron beam "conditioning" has been proposed to reduce the longitudinal velocity spread within the beam by correlating transverse oscillation amplitude and the electron energy, in order to enhance the FEL gain [4]. The present theory can be applied to the beam-conditioning case by a few modifications of the formulation.

This article is organized as follows. In Sec. II, starting from the Hamiltonian, we derive equations of motion for a single electron in the FEL system and construct the Vlasov equation. In Sec. III we calculate the vector potential for the radiation field, and present an explicit expression of the vector potential. In Sec. IV we expand the transverse electron distribution function with respect to the azimuthal angle in the transverse phase space and obtain an integral equation for the radial distribution function of electrons. We solve this integral equation in Sec. V by using the orthogonal-expansion technique. The matrix form of the dispersion relation is derived. In Sec.

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VI we consider the approximation of taking the lowest order in the expansion, and show that the resulting scalar dispersion relation for the hollow-beam distribution of the electrons reduces to the well-known results in both the small- and large-beam size limits when the betatron oscillation is neglected. In Sec. VII we show numerical results of the FEL gain as a function of the four scaling parameters. They are compared with simulation results and analytical results obtained by other approaches. We present a handy interpolating formula for the FEL gain of a Gaussian beam as a function of the scaled energy spread, the betatron frequency, and the transverse emittance, that can be used for a quick estimate of the FEL gain. In Sec. VIII we turn our discussion to the planar-wiggler case. So far, we have assumed that the FEL radiation takes place in the helical wiggler. However, the FEL with a planar wiggler can be treated in parallel with the preceding formulation with a few modifications. The more general results for the asymmetric focusing case are summarized in Appendix D. In Sec. IX we briefly discuss how to apply the present theory to the beam-conditioning case. The paper is concluded in Sec. X.

II. VLASOV EQUATION

To construct the Vlasov equation, one first writes down equations of motion for a single electron. A rigorous way to derive equations of motion is to start with the Hamiltonian. The detail of the derivation is described in Appendix A. We here mostly refer the results from there. We consider the electron beam moving in the z direction through a periodic helical wiggler with wave number k_w and peak wiggler parameter K. We choose z, the distance from the wiggler entrance, as the independent variable. After averaging over the fast wiggling motion, the transverse electron motion can be described by the harmonic betatron oscillation in the spatial transverse vector $\mathbf{x}_\beta$ and its canonical momentum conjugate $\mathbf{p}_\beta$:

$$\frac{d\mathbf{x}_\beta}{dz}=\mathbf{p}_\beta, \quad \frac{d\mathbf{p}_\beta}{dz}=-k_\beta^2\mathbf{x}_\beta \ , \tag{1}$$

where k_β is the betatron wave number. (In the absence of external focusing $k_\beta=Kk_w/\gamma\sqrt{2}$, where γ is the electron energy in units of its rest mass energy, mc^2, and c is the speed of light.) The transverse variables to be used in the Vlasov equation are those $\mathbf{x}_\beta$ and $\mathbf{p}_\beta$. Here, we neglect the nonlinear terms in the betatron focusing. This is a good approximation, since the betatron wavelength in practice is typically longer than the power gain length by one order of magnitude. The total transverse trajectory of the electron, $\mathbf{x}$, includes the helical motion $\mathbf{x}_h$ around the betatron motion:

$$\mathbf{x}=\mathbf{x}_h+\mathbf{x}_\beta \ , \tag{2}$$

where

$$\mathbf{x}_h=-\mathbf{i}_x r_h \sin k_w z+\mathbf{i}_y r_h \cos k_w z \ , \tag{3}$$

and

$$\mathbf{x}_\beta=\mathbf{x}_{\beta_0}\cos k_\beta z+\frac{\mathbf{p}_{\beta_0}}{k_\beta}\sin k_\beta z \ , \tag{4}$$

where

$$\mathbf{x}_{\beta_0}=\mathbf{x}_\beta(z=0) \quad \text{and} \quad \mathbf{p}_{\beta_0}=\mathbf{p}_\beta(z=0) \ . \tag{5}$$

Here, $r_h=Kc/(\gamma k_w v_\parallel)$ is the radius of the helical motion, $v_\parallel$ is the longitudinal velocity of the electron, and $\mathbf{i}_x$ and $\mathbf{i}_y$ are unit vectors in the x and y directions, respectively.

With z as the independent variable, the time t denotes the longitudinal coordinate. For convenience, we define a new longitudinal coordinate τ, as the arrival-time difference of an electron at the position z relative to that of the reference electron. The reference electron arrives at z at time $t_r=z/v_r$, where v_r is the longitudinal velocity of the reference electron. The electron of concern arrives at the position z at time t. The new coordinate τ is defined by

$$\tau=t-t_r=t-\frac{z}{v_r} \ . \tag{6}$$

The quantity τv_r gives the internal longitudinal position of an electron relative to that of the reference electron. An equation of motion of τ is approximately given by

$$\frac{d\tau}{dz}\approx\frac{1}{c}\left[-2\frac{k_w}{k_1}\frac{\gamma-\gamma_r}{\gamma_r}+\frac{1}{2}(\mathbf{p}_\beta^2+k_\beta^2\mathbf{x}_\beta^2)\right] , \tag{7}$$

where γ_r is the resonant energy of the reference electron with zero transverse oscillation amplitude and $k_1=2k_w\gamma_r^2/(1+K^2)$ is the resonant radiation wave number corresponding to the energy γ_r. The energy change is produced by the interaction of the electron's helical motion and the radiation field. An equation of motion of the energy γ is

$$mc^2\frac{d\gamma}{dz}\approx-e\frac{d\mathbf{x}_h}{dz}\frac{\partial \mathbf{A}_r}{\partial t} \ , \tag{8}$$

where e is the electron charge and $\mathbf{A}_r=\mathbf{A}_r(\mathbf{x},z,t)$ is the vector potential for the radiation field.

The Vlasov equation for the electron distribution $f(\mathbf{x}_\beta,\mathbf{p}_\beta,\tau,\gamma;z)$ is as follows:

$$\frac{\partial f}{\partial z}+\frac{d\mathbf{x}_\beta}{dz}\frac{\partial f}{\partial \mathbf{x}_\beta}+\frac{d\mathbf{p}_\beta}{dz}\frac{\partial f}{\partial \mathbf{p}_\beta}+\frac{d\tau}{dz}\frac{\partial f}{\partial \tau}+\frac{d\gamma}{dz}\frac{\partial f}{\partial \gamma}=0 \ . \tag{9}$$

Here, f is normalized such that

$$\int_1^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty f(\mathbf{x}_\beta,\mathbf{p}_\beta,\tau,\gamma;z)\,d^2\mathbf{x}_\beta\, d^2\mathbf{p}_\beta\, d\tau\, d\gamma=N \ , \tag{10}$$

where N is the total number of electrons in the beam. Throughout this article, we apply a rule that the integration signs in the multiple integrals are paired with the differentials from the inside to outside, unless otherwise specified.

We solve Eq. (9) by the perturbation method. The distribution function f can be decomposed into the unperturbed part f_0 and the perturbed part f_1, respectively:

$$f=f_0+f_1 \ . \tag{11}$$

The unperturbed distribution function f_0 satisfies

$$\frac{\partial f_0}{\partial z}+\mathbf{p}_\beta\frac{\partial f_0}{\partial \mathbf{x}_\beta}-k_\beta^2\mathbf{x}_\beta\frac{\partial f_0}{\partial \mathbf{p}_\beta}+\frac{d\tau}{dz}\frac{\partial f_0}{\partial \tau}=0\ , \quad (12)$$

where we have substituted Eq. (1). The perturbed distribution function f_1 is a solution of the linearized Vlasov equation

$$\frac{\partial f_1}{\partial z}+\mathbf{p}_\beta\frac{\partial f_1}{\partial \mathbf{x}_\beta}-k_\beta^2\mathbf{x}_\beta\frac{\partial f_1}{\partial \mathbf{p}_\beta}+\frac{d\tau}{dz}\frac{\partial f_1}{\partial \tau}+\frac{d\gamma}{dz}\frac{\partial f_0}{\partial \gamma}=0\ . \quad (13)$$

In this article, we assume that the focusing in the wiggler is matched to the electron beam so that f_0 is a function of $\mathbf{x}_\beta^2+\mathbf{p}_\beta^2/k_\beta^2$ and γ only (i.e., f_0 is uniform in the longitudinal direction). Furthermore, for the cases considered in this article, the distribution in γ is sharply peaked around an average value. It is then a good approximation to assume that f_0 can be factorized as follows:

$$f_0=f_{0\perp}(\mathbf{x}_\beta^2+\mathbf{p}_\beta^2/k_\beta^2)f_{0\parallel}(\gamma)\ . \quad (14)$$

III. VECTOR POTENTIAL FOR RADIATION FIELD

The vector potential $\mathbf{A}_r(\mathbf{r},t)$ for the radiation field satisfies the inhomogeneous wave equation

$$\nabla^2\mathbf{A}_r-\frac{1}{c^2}\frac{\partial^2\mathbf{A}_r}{\partial t^2}=-\mu_0\mathbf{J}_\perp(\mathbf{r},t)\ , \quad (15)$$

where $\mathbf{J}_\perp(\mathbf{r},t)$ is the transverse current density, μ_0 is the permeability of free space, and $\mathbf{r}$ is the three-dimensional vector $\mathbf{r}=(\mathbf{x},z)$. The solution of Eq. (15) can be written as

$$\mathbf{A}_r(\mathbf{r},t)=\mu_0\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{G}(\mathbf{r},t|\mathbf{r}',t')\mathbf{J}_\perp(\mathbf{r}',t')d^3\mathbf{r}'dt'\ . \quad (16)$$

Here, the Green's function $\mathbf{G}(\mathbf{r},t|\mathbf{r}',t')$ satisfies

$$\nabla^2\mathbf{G}(\mathbf{r},t|\mathbf{r}',t')-\frac{1}{c^2}\frac{\partial^2\mathbf{G}}{\partial t^2}=-\overleftrightarrow{I}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\ , \quad (17)$$

where $\overleftrightarrow{I}$ is the unit dyad (identical to the unit matrix in this case). The solution of $\mathbf{G}$ in free space is well known [5] and is given by

$$\mathbf{G}(\mathbf{r},t|\mathbf{r}',t')=\overleftrightarrow{I}\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{\mathbf{k}^2-(\omega/c)^2}d^3\mathbf{k}\,e^{-i\omega(t-t')}d\omega\ . \quad (18)$$

The transverse current density $\mathbf{J}_\perp(\mathbf{r},t)$ in Eq. (15) is given by

$$\mathbf{J}_\perp(\mathbf{r},t)=\sum_{i=1}^{N}e\frac{d\mathbf{x}_i}{dt}\delta(\mathbf{x}-\mathbf{x}_i)\delta(z-z_i)\ , \quad (19)$$

where $(\mathbf{x}_i(t),z_i(t))$ describes the orbit of the ith electron as a function of time t. Equation (19) can be rewritten using z as the independent variable as

$$\mathbf{J}_\perp(\mathbf{r},t)=\sum_{i=1}^{N}e\frac{d\mathbf{x}_i}{dz}\delta(\mathbf{x}-\mathbf{x}_i)\delta(\tau-\tau_i)\ , \quad (20)$$

where $\tau_i(z)$ is the arrival-time difference of the ith electron at z relative to that of the reference electron. We can express $\mathbf{J}_\perp(\mathbf{r},t)$ in terms of the density distribution of the betatron orbit, $\rho_1(\mathbf{x}_\beta,\tau;z)$, given by

$$\rho_1(\mathbf{x}_\beta,\tau;z)=\int_1^{\infty}\int_{-\infty}^{\infty}f_1(\mathbf{x}_\beta,\mathbf{p}_\beta,\tau,\gamma;z)d^2\mathbf{p}_\beta d\gamma\ . \quad (21)$$

This is done below:

$$\begin{aligned}\mathbf{J}_\perp(\mathbf{r},t)&=\sum_{i=1}^{N}e\frac{d\mathbf{x}_i}{dz}\delta(\mathbf{x}-\mathbf{x}_i)\delta(\tau-\tau_i)\\&=\sum_{i=1}^{N}e\frac{d\mathbf{x}_i}{dz}\delta(\mathbf{x}-\mathbf{x}_h-(\mathbf{x}_i-\mathbf{x}_h))\delta(\tau-\tau_i)\\&=\sum_{i=1}^{N}e\frac{d\mathbf{x}_i}{dz}\delta(\mathbf{x}_\beta-\mathbf{x}_{\beta_i})\delta(\tau-\tau_i)\\&=e\frac{d\mathbf{x}}{dz}\rho_1(\mathbf{x}_\beta,\tau;z)\ . \end{aligned} \quad (22)$$

The vector $\mathbf{x}$ includes both the rapidly oscillating helical orbit $\mathbf{x}_h$ and the slowly varying betatron orbit $\mathbf{x}_\beta$. By retaining only the helical motion $\mathbf{x}_h$ in $d\mathbf{x}/dz$, we have an approximate expression of $\mathbf{J}_\perp(\mathbf{r},t)$:

$$\mathbf{J}_\perp(\mathbf{r},t)\approx e\frac{d\mathbf{x}_h(z)}{dz}\rho_1(\mathbf{x}_\beta,\tau;z)\ . \quad (23)$$

By inserting Eq. (23) into Eq. (16) and changing the volume element from $d^3\mathbf{r}'dt'$ to $d^2\mathbf{x}_\beta'dz'd\tau'$, we obtain

$$\mathbf{A}_r(\mathbf{r},t)=e\mu_0\int_0^z\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{G}(\mathbf{r},t|\mathbf{r}',t')\left[\frac{d\mathbf{x}_h'(z')}{dz'}\rho_1(\mathbf{x}_\beta',\tau';z')\right]d^2\mathbf{x}_\beta'd\tau'\right]dz'\ . \quad (24)$$

For the later use, it is convenient now to seek an alternative expression of $\mathbf{A}_r$ in terms of the Fourier-Laplace transform of $\rho_1(\mathbf{x}_\beta,\tau;z)$ with respect to $\mathbf{x}_\beta$, τ, and z. After lengthy calculation (see Appendix B), we obtain the expression for the vector potential for the radiation field:

$$\mathbf{A}_r(\mathbf{r},t)=e\mu_0\int_{-\infty}^{\infty}\left[\frac{1}{2\pi i}\int_{q_0-i\infty}^{q_0+i\infty}\left[\int_{-\infty}^{\infty}\mathbf{H}_{\omega q}(\mathbf{k}_\perp,z)\rho_{\omega q}(\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}}d^2\mathbf{k}_\perp\right]e^{qz}dq\right]e^{-i\omega\tau}d\omega\ , \quad (25)$$

where $\mathbf{H}_{\omega q}(\mathbf{k}_\perp,z)$ is given by

$$\mathbf{H}_{\omega q}(\mathbf{k}_\perp,z)=-i\frac{\pi}{(2\pi)^4}\frac{K}{\gamma_r}\frac{1}{(k^2-k_\perp^2)^{1/2}}\sum_{p=-\infty}^{\infty}\frac{e^{-ipk_w z}\mathbf{V}_p(\mathbf{k}_\perp)}{q-i[pk_w+(k^2-k_\perp^2)^{1/2}-\omega/v_r]}\,, \tag{26}$$

where $k=\omega/c$, $k_\perp=|\mathbf{k}_\perp|$, and $\rho_{\omega q}(\mathbf{k}_\perp)$ is the Laplace-Fourier transform of $\rho_1(\mathbf{x}_\beta,\tau;z)$, which is related to $f_1(\mathbf{x}_\beta,\mathbf{p}_\beta,\tau,\gamma;z)$ by

$$\rho_{\omega q}(\mathbf{k}_\perp)=\int_{-\infty}^{\infty}\left\{\int_0^{\infty}\left[\int_{-\infty}^{\infty}\left[\int_1^{\infty}\int_{-\infty}^{\infty}f_1(\mathbf{x}_\beta,\mathbf{p}_\beta,\tau,\gamma;z)d^2\mathbf{p}_\beta d\gamma\right]e^{-i\mathbf{k}_\perp\cdot\mathbf{x}_\beta}d^2\mathbf{x}_\beta\right]e^{-qz}dz\right\}e^{i\omega\tau}d\tau\,. \tag{27}$$

The integer p represents the harmonic number of the radiation. The positive value $p>0$ and the negative value $p<0$ corresponds to the forward and the backward radiations in the electron rest frame, respectively. The vector $\mathbf{V}_p(\mathbf{k}_\perp)$ in Eq. (26) is defined by

$$\mathbf{V}_p(\mathbf{k}_\perp)=(-1)^{p-1}e^{-ip\theta_k}\left[\mathbf{i}_x\frac{1}{2}[e^{-i\theta_k}J_{p+1}(k_\perp r_h)+e^{i\theta_k}J_{p-1}(k_\perp r_h)]+\mathbf{i}_y\frac{1}{2i}[e^{-i\theta_k}J_{p+1}(k_\perp r_h)-e^{i\theta_k}J_{p-1}(k_\perp r_h)]\right]\,, \tag{28}$$

where $J_p(x)$ is the Bessel function and $\theta_k=\tan^{-1}(k_y/k_x)$.

Now, we can calculate the energy change by the radiation field, with use of the vector potential $\mathbf{A}_r(\mathbf{r},t)$ given by Eq. (25). After some algebra (see Appendix B), we obtain

$$\frac{d\gamma}{dz}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi i}\int_{q_0-i\infty}^{q_0+i\infty}\left[\int_{-\infty}^{\infty}P_{\omega q}(k_\perp)\rho_{\omega q}(\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\beta}d^2\mathbf{k}_\perp\right]e^{qz}dq\right]e^{-i\omega\tau}d\omega\,. \tag{29}$$

Here

$$P_{\omega q}(k_\perp)=\sum_{p=-\infty}^{\infty}\frac{r_e}{2\pi c}\left[\frac{K}{\gamma_r}\right]^2\frac{(-1)^p\left[J_p^2(k_\perp r_h)\left[\frac{p}{k_\perp r_h}\right]^2+J_p'^2(k_\perp r_h)\right]}{[1-(k_\perp/k)^2]^{1/2}\{q-i[pk_w+(k^2-k_\perp^2)^{1/2}-\omega/v_r]\}}\,, \tag{30}$$

where $r_e=e^2/(4\pi\epsilon_0 mc^2)$ is the classical electron radius, ϵ_0 is the permittivity of free space, and $J_p'(x)$ is the derivative of the Bessel function. The quantity $P_{\omega q}(k_\perp)$ is proportional to the total radiation power emitted from a single electron into the transverse angle $\theta=\sin^{-1}(k_\perp/k)$ with the transverse wave number $k_\perp$ in the frequency range $(\omega,\omega+d\omega)$. Equation (30) contains all the higher-harmonic components of the radiation. We are mostly interested in only the lowest-harmonic term in the forward direction, $p=1$. If we retain only this term in the summation and note that $\theta\ll 1$, Eq. (30) becomes

$$P_{\omega q}(k_\perp)\approx-\frac{r_e}{2\pi c}\left[\frac{K}{\gamma_r}\right]^2\frac{\dfrac{J_1^2(kr_h\theta)}{(kr_h\theta)^2}+J_1'^2(kr_h\theta)}{q+ik_\omega\dfrac{k-k_1}{k_1}+i\dfrac{k}{2}\theta^2}\,. \tag{31}$$

IV. AZIMUTHAL MODE EXPANSION

Now, let us come back to the linearized Vlasov equation (13). If we substitute Eqs. (29) and (30) into Eq. (13) and take its Fourier-Laplace transform, the linearized Vlasov equation becomes

$$\left[q-i\omega\frac{d\tau}{dz}\right]f_{\omega q}+\mathbf{p}_\beta\frac{\partial f_{\omega q}}{\partial\mathbf{x}_\beta}-k_\beta^2\mathbf{x}_\beta\frac{\partial f_{\omega q}}{\partial\mathbf{p}_\beta}=-f_{0\perp}\frac{df_{0\parallel}}{d\gamma}\int P_{\omega q}(k_\perp)\rho_{\omega q}(\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\beta}d^2\mathbf{k}_\perp\,, \tag{32}$$

where $f_{\omega q}$ is the Fourier-Laplace transform of $f_1(\mathbf{x}_\beta,\mathbf{p}_\beta,\tau,\gamma;z)$:

$$f_{\omega q}(\mathbf{x}_\beta,\mathbf{p}_\beta,\gamma)=\int_{-\infty}^{\infty}\left[\int_0^{\infty}f_1(\mathbf{x}_\beta,\mathbf{p}_\beta,\tau,\gamma;z)e^{-qz}dz\right]e^{i\omega\tau}d\tau\,. \tag{33}$$

In Eq. (32) we did not include the Fourier transform of the initial distribution at $z=0$, because we consider only the eigenvalue problem in this paper. (If we retain this term, the problem becomes an initial-value problem.)

Since the betatron motion of the electron is a simple harmonic oscillation, it is natural to introduce polar coordinates in the transverse planes as

$$x_\beta=r_x\cos\phi_x\,,\quad y_\beta=r_y\cos\phi_y\,, \tag{34}$$

$$\frac{p_{\beta x}}{k_\beta}=r_x\sin\phi_x,\quad \frac{p_{\beta y}}{k_\beta}=r_y\sin\phi_y\ . \tag{35}$$

Then, the second and third terms in the left-hand side (LHS) of the Vlasov equation, Eq. (32), are written as

$$\mathbf{p}_\beta\frac{\partial f_{\omega q}}{\partial \mathbf{x}_\beta}-k_\beta^2\mathbf{x}_\beta\frac{\partial f_{\omega q}}{\partial \mathbf{p}_\beta}=-k_\beta\left[\frac{\partial f_{\omega q}}{\partial\phi_x}+\frac{\partial f_{\omega q}}{\partial\phi_y}\right]\ . \tag{36}$$

Now, due to the periodic boundary condition for $f_{\omega q}$ in the azimuthal angles ϕ_x and ϕ_y, $f_{\omega q}$ can be Fourier decomposed with respect to ϕ_x and ϕ_y into an infinite series of modes:

$$f_{\omega q}(\mathbf{x}_\beta,\mathbf{p}_\beta,\gamma)=\sum_{m,n=-\infty}^{\infty}F_{\omega q}^{(m,n)}(r_x,r_y,\gamma)e^{im\phi_x}e^{in\phi_y}\ , \tag{37}$$

where m and n are integers. If we insert the above equation into Eq. (27), the Fourier-Laplace transform of the charge density can be expressed in terms of $F_{\omega q}^{(m,n)}(r_x,r_y,\gamma)$ as follows:

$$\rho_{\omega q}(\mathbf{k}_\perp)=\int_1^\infty\left[\sum_{m,n=-\infty}^{\infty}\int_0^{2\pi}\int_0^\infty\int_0^{2\pi}\int_0^\infty F_{\omega q}^{(m,n)}(r_x,r_y,\gamma)e^{im\phi_x-ik_xr_x\cos\phi_x}e^{in\phi_y-ik_yr_y\cos\phi_y}k_\beta r_x\,dr_x\,d\phi_x\,k_\beta r_y\,dr_y\,d\phi_y\right]d\gamma$$

$$=(2\pi k_\beta)^2\int_1^\infty\left[\sum_{m,n=-\infty}^{\infty}i^{-(|m|+|n|)}\int_0^\infty\int_0^\infty F_{\omega q}^{(m,n)}(r_x,r_y,\gamma)J_{|m|}(k_xr_x)J_{|n|}(k_yr_y)r_x\,dr_x\,r_y\,dr_y\right]d\gamma\ , \tag{38}$$

where we have used the formula [6]

$$\frac{1}{2\pi}\int_0^{2\pi}e^{il\phi-ix\cos\phi}d\phi=i^{-l}J_l(x)\ . \tag{39}$$

Combining Eqs. (32), (37), and (38), we obtain an integral equation for $F_{\omega q}^{(m,n)}$,

$$\left[q-i\omega\frac{d\tau}{dz}-ik_\beta(m+n)\right]F_{\omega q}^{(m,n)}(r_x,r_y,\gamma)$$

$$=-f_{0\perp}(r^2)\frac{df_{0\parallel}(\gamma)}{d\gamma}\int_1^\infty\left[\sum_{m',n'}\int_0^\infty\int_0^\infty K_{\omega q}^{(m,n,m',n')}(r_x,r_y|r_x',r_y')F_{\omega q}^{(m',n')}(r_x',r_y',\gamma')r_x'\,dr_x'\,r_y'\,dr_y'\right]d\gamma'\ , \tag{40}$$

where the kernel $K_{\omega q}^{(m,n,m',n')}$ is given by

$$K_{\omega q}^{(m,n,m',n')}(r_x,r_y|r_x',r_y')$$

$$=i^{|m|+|n|-(|m'|+|n'|)}(2\pi k_\beta)^2\int_{-\infty}^{\infty}P_{\omega q}(k_\perp)[J_{|m|}(k_xr_x)J_{|n|}(k_yr_y)][J_{|m'|}(k_xr_x')J_{|n'|}(k_yr_y')]d^2\mathbf{k}_\perp \tag{41}$$

and $r=(r_x^2+r_y^2)^{1/2}$ is the amplitude of the electron position in four-dimensional transverse phase space.

V. GENERAL SOLUTION

By inspecting Eq. (40), it can be seen that the γ dependence of $F_{\omega q}^{(m,n)}$ is such that

$$F_{\omega q}^{(m,n)}(r_x,r_y,\gamma)\propto\frac{\dfrac{df_{0\parallel}(\gamma)}{d\gamma}}{q-i\omega\dfrac{d\tau}{dz}(r,\gamma)-ik_\beta(m+n)}\ . \tag{42}$$

It is then useful to define a radial function $R_{\omega q}^{(m,n)}$ as the γ integral of $F_{\omega q}^{(m,n)}$ to eliminate the obvious γ dependence:

$$R_{\omega q}^{(m,n)}(r_x,r_y)=\int_1^\infty F_{\omega q}^{(m,n)}(r_x,r_y,\gamma)d\gamma\ . \tag{43}$$

Dividing the Vlasov equation (40) by

$$[q-i\omega(d\tau/dz)-ik_\beta(m+n)]$$

and integrating over γ, we obtain

$$R_{\omega q}^{(m,n)}(r_x,r_y)$$

$$=-f_{0\perp}(r^2)\int_1^\infty\frac{\dfrac{df_{0\parallel}(\gamma)}{d\gamma}d\gamma}{q-i\omega\dfrac{d\tau}{dz}(r,\gamma)-ik_\beta(m+n)}\sum_{m',n'}\int_0^\infty\int_0^\infty K_{\omega q}^{(m,n,m',n')}(r_x,r_y|r_x',r_y')R_{\omega q}^{(m',n')}(r_x',r_y')r_x'\,dr_x'\,r_y'\,dr_y'\ . \tag{44}$$

The integral equation (44) can be solved in a general way as follows [3]. We expand the radial function $R_{\omega q}^{(m,n)}$ using a complete set of orthogonal functions $f_k^{(|m|,|n|)}(r_x,r_y)$ as

$$R_{\omega q}^{(m,n)}(r_x,r_y)=W_\perp(r^2)\sum_{k=0}^{\infty} a_k^{(m,n)} f_k^{(|m|,|n|)}(r_x,r_y) r_x^{|m|} r_y^{|n|} \ . \tag{45}$$

Here, the weight function $W_\perp(r^2)$ is defined by

$$W_\perp(r^2)=Cf_{0\perp}(r^2) \ , \tag{46}$$

where C is a normalization constant to be chosen. The functions $f_k^{(|m|,|n|)}(r_x,r_y)$ are determined so as to satisfy the following orthogonality relationship:

$$\int_0^\infty \int_0^\infty W_\perp(r^2) f_k^{(|m|,|n|)}(r_x,r_y) f_l^{(|m|,|n|)}(r_x,r_y) r_x^{2|m|+1} r_y^{2|n|+1} dr_x dr_y=\delta_{kl} \ . \tag{47}$$

Using $f_k^{(|m|,|n|)}(r_x,r_y)$, we expand the Bessel functions as

$$J_{|m|}(k_x r_x) J_{|n|}(k_y r_y)=\sum_{k=0}^{\infty} C_{|m|,|n|,k}(k_x,k_y) f_k^{(|m|,|n|)}(r_x,r_y) r_x^{|m|} r_y^{|n|} \ , \tag{48}$$

where

$$C_{|m|,|n|,k}(k_x,k_y)=\int_0^\infty \int_0^\infty J_{|m|}(k_x r_x) J_{|n|}(k_y r_y) W_\perp(r^2) f_k^{(|m|,|n|)}(r_x,r_y) r_x^{|m|+1} r_y^{|n|+1} dr_x dr_y \ . \tag{49}$$

For many models of the unperturbed transverse distribution $f_{0\perp}(r^2)$ [or the weight function $W_\perp(r^2)$], the corresponding orthogonal functions $f_k^{(|m|,|n|)}(r_x,r_y)$ can be expressed in terms of the well-known analytical functions. In Appendix C we present explicit expressions of $f_k^{(|m|,|n|)}(r_x,r_y)$ and $C_{|m|,|n|,k}(k_x,k_y)$ for the hollow-beam and the Gaussian-beam models of $f_{0\perp}(r^2)$.

The lowest-order term $C_{|m|,|n|,0}(k_x,k_y)$ has a simpler expression, since the corresponding lowest-order orthogonal function $f_0^{(|m|,|n|)}(r_x,r_y)$ is just a constant. In this case, the integration over the angle $\theta_r=\tan^{-1}(r_y/r_x)$ can be carried out in Eq. (49), with the result,

$$C_{|m|,|n|,0}(k_x,k_y)=f_0^{(|m|,|n|)}\cos^{|m|}\theta_k \sin^{|n|}\theta_k \int_0^\infty \frac{J_{|m|+|n|+1}(k_\perp r)}{k_\perp r} W_\perp(r^2) r^{|m|+|n|+3} dr \ , \tag{50}$$

where $\theta_k=\tan^{-1}(k_y/k_x)$.

Inserting Eqs. (45) and (48) into Eq. (44), multiplying by $f_k^{(|m|,|n|)}(r_x,r_y) r_x^{|m|+1} r_y^{|n|+1}$ and integrating over r_x and r_y, we have a matrix equation for the coefficients $a_k^{(m,n)}$:

$$a_k^{(m,n)}+\sum_{m',n',l,j} \beta_{k,l}^{m,n} M_{m',n',j}^{m,n,l} a_j^{(m',n')}=0 \ , \tag{51}$$

where

$$\beta_{k,l}^{m,n}=\int_1^\infty \int_0^\infty \int_0^\infty \frac{W_\perp(r^2) f_k^{(|m|,|n|)}(r_x,r_y) f_l^{(|m|,|n|)}(r_x,r_y) r_x^{2|m|+1} r_y^{2|n|+1}}{q-i\omega \dfrac{d\tau}{dz}(r,\gamma)-ik_\beta(m+n)} \frac{df_{0\parallel}}{d\gamma} dr_x dr_y d\gamma \tag{52}$$

and the matrix elements are given by

$$M_{m',n',j}^{m,n,l}=i^{|m|+|n|-(|m'|+|n'|)} \frac{(2\pi k_\beta)^2}{C} \int_{-\infty}^\infty \int_{-\infty}^\infty P_{\omega q}(k_\perp) C_{|m|,|n|,l}(k_x,k_y) C_{|m'|,|n'|,j}(k_x,k_y) dk_x dk_y \ . \tag{53}$$

The matrix equation can be symbolically written as

$$(\mathbf{I}+\boldsymbol{\beta}\mathbf{M})\mathbf{a}=\mathbf{0} \ , \tag{54}$$

where $\mathbf{a}$ is the vector of the coefficient $a_k^{(m,n)}$, $\mathbf{I}$ is the unit matrix, and the matrix elements of $\boldsymbol{\beta}$ and $\mathbf{M}$ are given by Eqs. (52) and (53), respectively. The nontrivial solution of Eq. (54) requires that

$$\det(\mathbf{I}+\boldsymbol{\beta}\mathbf{M})=0 \ . \tag{55}$$

This dispersion relation gives eigenvalues q as a function of ω or vice versa.

The matrix $\boldsymbol{\beta}$ represents the Landau damping due to the energy spread and the betatron oscillation via the longitudinal velocity spread. If the variation of the longitudinal velocity due to the betatron oscillation is negligible, the integration over r_x and r_y can be carried out readily, noting the orthogonality relationship, Eq. (47). As the result, the k and l dependence of the matrix $\boldsymbol{\beta}$ becomes just the Kronecker's δ_{kl}. The matrix $\mathbf{M}$ expresses the rest of the interaction between the electron beam and the radiation field.

In principle, all the eigenmodes can be obtained by solving the above dispersion relation. For example, for sufficiently large electron-beam radius, many modes are excited, and the degeneracy problem of the growth rates of the self-similar modes arises [7]. This problem can be analyzed by taking large matrices of **M** and $\boldsymbol{\beta}$. In this article, however, we are concerned about the FEL operating in the high-gain regime where the electron-beam size is relatively small and the full transverse coherence is achieved. In this regime, only a few modes, or even a single dominant mode, are needed to describe the beam-radiation system.

VI. THE LOWEST-ORDER DISPERSION RELATION

It is straightforward to seek zeros of the dispersion relation by computer and the computation requires little CPU time, if the matrix size is not too large. Numerical studies show a quite rapid convergence of solutions as a function of the matrix size. As a matter of fact, we have found that one can obtain an accurate eigenvalue for the functional mode by taking only the lowest-order term $m=n=k=0$ in both the azimuthal and the radial expansions [see Eqs. (37) and (45)]. In this case, an approximate expression for the dispersion relation can be written in a general form as

$$1=4i\frac{k}{k_1}\frac{r_e}{c}\left[\frac{K}{\gamma_r}\right]^2\frac{k_w}{\gamma_r}\int_0^\infty\int_1^\infty\frac{f_{0\parallel}(\gamma)d\gamma}{\left[q+2i\frac{k}{k_1}k_w\frac{\gamma-\gamma_r}{\gamma_r}-i\frac{1}{2}kk_\beta^2r^2\right]^2}2\pi^2k_\beta^2f_{0\perp}(r^2)r^3dr \times\int_0^{\pi/2}\frac{k^2\theta\,d\theta}{q+ik_w\frac{k-k_1}{k_1}+i\frac{k\theta^2}{2}}\left[\int_0^\infty 2\pi^2k_\beta^2f_{0\perp}(r'^2)\frac{J_1(kr'\theta)}{kr'\theta}r'^3dr'\right]^2, \tag{56}$$

where $f_{0\perp}(r^2)$ is normalized such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{0\perp}(\mathbf{x}_\beta^2+\mathbf{p}_\beta^2/k_\beta^2)d^2\mathbf{x}_\beta d^2\mathbf{p}_\beta=\int_0^\infty 2\pi^2k_\beta^2f_{0\perp}(r^2)r^3dr=1\ . \tag{57}$$

Here, we have used the approximated expression of $P_{\omega q}(k_\perp)$, Eq. (31), and have approximated

$$\frac{J_1^2(kr_h\theta)}{(kr_h\theta)^2}+J_1'^2(kr_h\theta)\approx\frac{1}{2}\ , \tag{58}$$

assuming that the radius of the helical orbit r_h is much smaller than the beam size.

In what follows, we write down the above equation in a more specific way for various models of $f_{0\perp}(r^2)$. Longitudinally, we assume a Gaussian distribution with the rms energy spread, σ_γ:

$$f_{0\parallel}(\gamma)=\frac{N}{\hat{\tau}}\frac{1}{\sqrt{2\pi}\sigma_\gamma\gamma_r}e^{-(\gamma-\gamma_r)^2/2\gamma_r^2\sigma_\gamma^2}\ , \tag{59}$$

where $\hat{\tau}$ is the length of the electron beam in time units.

For the *hollow beam*,

$$f_{0\perp}(r^2)=\frac{1}{(\pi R_0^2k_\beta)^2}\delta\left[1-\left[\frac{r}{R_0}\right]^2\right],$$

we have

$$1=2i\frac{k}{k_1}\frac{(2\rho k_w)^3}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-t^2/2}dt}{\left[q+2i\frac{k}{k_1}k_w\sigma_\gamma t-i\frac{1}{2}kk_\beta^2R_0^2\right]^2}\int_0^{\pi/2}\frac{\left[\frac{J_1(kR_0\theta)}{kR_0\theta}\right]^2(kR_0)^2\theta\,d\theta}{q+ik_w\frac{k-k_1}{k_1}+i\frac{k\theta^2}{2}}\ . \tag{60}$$

For the *water-bag beam*,

$$f_{0\perp}(r^2)=\frac{2}{(\pi R_0^2k_\beta)^2}\Theta\left[1-\left[\frac{r}{R_0}\right]^2\right],$$

we have

$$1=2i\frac{k}{k_1}\frac{(2\rho k_w)^3}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^1\frac{x^3e^{-t^2/2}dx\,dt}{\left[q+2i\frac{k}{k_1}k_w\sigma_\gamma t-i\frac{1}{2}kk_\beta^2R_0^2x^2\right]^2}\int_0^{\pi/2}\frac{2\left[\frac{4J_2(kR_0\theta)}{(kR_0\theta)^2}\right]^2(kR_0)^2\theta\,d\theta}{q+ik_w\frac{k-k_1}{k_1}+i\frac{k\theta^2}{2}} . \quad (61)$$

For the *Gaussian beam*,

$$f_{0\perp}(r^2)=\frac{1}{(2\pi\sigma_x^2k_\beta)^2}e^{-\frac{1}{2}(r/\sigma_x)^2} ,$$

we have

$$1=i\frac{k}{2k_1}\frac{(2\rho k_w)^3}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{-x^2/2}x^3e^{-t^2/2}dx\,dt}{\left[q+2i\frac{k}{k_1}k_w\sigma_\gamma t-i\frac{1}{2}kk_\beta^2\sigma_x^2x^2\right]^2}\int_0^{\pi/2}\frac{e^{-(k\sigma_x\theta)^2}(k\sigma_x)^2\theta\,d\theta}{q+ik_w\frac{k-k_1}{k_1}+i\frac{k\theta^2}{2}} . \quad (62)$$

In the above equation, ρ is the Pierce parameter [8] defined by

$$(2\rho k_w)^3=2\pi r_e n_0\left[\frac{K}{\gamma_r}\right]^2\frac{k_w}{\gamma_r} , \quad (63)$$

n_0 is the peak volume density on axis, σ_x is the rms transverse beam size, $\Theta(x)$ is the step function $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$.

Let us investigate the dispersion relation (60) for the hollow-beam model to obtain a physical picture of how the gain is determined. The integral over t characterizes the Landau damping due to the energy spread. In the hollow-beam model, the electron beam has a uniform transverse distribution inside a circle with the radius R_0 in the x-y plane. The function in the θ integral, $[J_1(kR_0\theta)/(kR_0\theta)]^2$, is the diffraction pattern of a plane wave by a uniform source of circular shape with the radius R_0. The factor

$$1/[q+ik_w(k-k_1)/k_1+ik\theta^2/2]$$

is related to the angular distribution of the radiated power from a single electron into the transverse angle θ. Equation (60) implies that the amount of overlap between the angular spectrum of radiation from a single electron and the angular diffraction pattern of the radiation wave by the electron beam plays a key role in determining the FEL gain.

In the limit of large beam size, $R_0\to\infty$, when $\sigma_\gamma=0$ and $k_\beta=0$, the dispersion relation (60) for the hollow beam reduces to the well-known cubic equation of the one-dimensional theory. This can be shown as follows. In this limit, Eq. (60) can be approximated by

$$1=2i\frac{k}{k_1}\frac{(2\rho k_w)^3}{q^2}\frac{1}{q+ik_w\frac{k-k_1}{k_1}}\int_0^{\infty}\left[\frac{J_1(x)}{x}\right]^2x\,dx . \quad (64)$$

Using

$$\int_0^{\infty}\left[\frac{J_1(x)}{x}\right]^2x\,dx=\frac{1}{2} , \quad (65)$$

it follows that

$$q^2\left[q+ik_w\frac{k-k_1}{k_1}\right]-i\frac{k}{k_1}(2\rho k_w)^3=0 . \quad (66)$$

Introducing $\mu=iq/k_w$, Eq. (66) becomes the well-known cubic equation of the high-gain regime [8]:

$$\mu^3-\mu^2\frac{k-k_1}{k_1}-\frac{k}{k_1}(2\rho)^3=0 . \quad (67)$$

The maximum growth rate as the solution of the above equation can be expressed using Moore's scaled growth rate

$$\hat{g}=q/[(2\rho k_w)^{3/2}(2k_1R_0^2)^{1/2}]$$

and scaled beam size

$$\hat{a}=(2\rho k_w)^{3/4}(2k_1R_0^2)^{3/4}$$

as [7]

$$\hat{g}=\frac{\sqrt{3}}{2\hat{a}^{2/3}} . \quad (68)$$

In the limit of small beam size, $R_0\to0$, when $\sigma_\gamma=0$ and $k_\beta=0$, the dispersion relation (60) also gives the correct asymptotic growth rate derived by Moore [7]. In this limit, the dispersion relation (60) when $k=k_1$ can be approximated by

$$\frac{\hat{g}^2}{2}-\int_0^{\infty}\frac{x}{x^2-i\hat{g}\hat{a}^2}\left[\frac{J_1(x)}{x}\right]^2dx=0 . \quad (69)$$

By performing the partial integral in Eq. (69) and neglecting the $\ln\hat{g}$ term, we obtain Moore's expression,

$$\frac{\hat{g}^2}{2}\simeq\frac{1}{4}\ln\frac{2}{\hat{a}} \quad\text{or}\quad \hat{g}\simeq\left[\frac{1}{2}\ln\frac{2}{\hat{a}}\right]^{1/2} . \quad (70)$$

In Fig. 1 we plot numerical results of the scaled growth

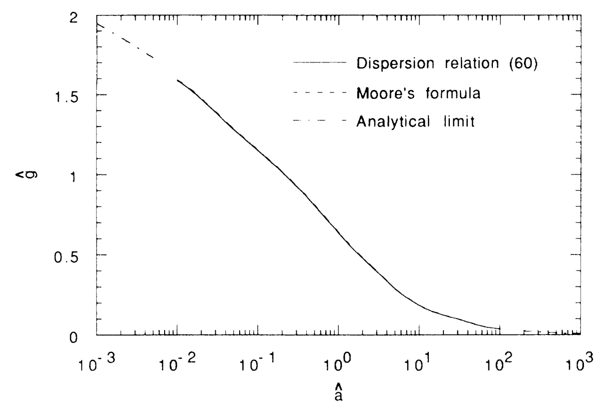

FIG. 1. Scaled growth rate $\hat{g}=q/[(2\rho k_w)^{3/2}(2k_1R_0^2)^{1/2}]$ vs the scaled beam size $\hat{a}=(2\rho k_w)^{3/4}(2k_1R_0^2)^{3/4}$ for the constant-beam-current case. Here, the energy spread $\sigma_\gamma=0$ and the betatron wave number $k_\beta=0$. The solid curve shows the result obtained by the present dispersion relation (60), while the dashed curve denotes the work of Moore. The dot-dashed lines on the right and on the left show the analytical results obtained from Eqs. (68) and (70), for the two extreme cases, respectively.

rate $\hat{g}$ versus the scaled beam size $\hat{a}$ for the constant current case, when $\sigma_\gamma=0$ and $k_\beta=0$ are assumed. The solid curve shows the result obtained by the present dispersion relation (60), while the dashed curve denotes the work of Moore. The dot-dashed lines on the right and on the left show the analytical results obtained from Eqs. (68) and (70), for the two extreme cases, respectively. As anticipated from the above argument, they are in excellent agreement in the entire range of beam size.

The truncated dispersion relations (61) and (62) for the water-bag and the Gaussian models do not converge to the cubic equation of the one-dimensional theory in the limit of large beam size. In fact, it can be shown that the growth rate calculated from Eq. (56) in this limit is given by the one-dimensional theory multiplied by $\frac{2}{3}^{1/3}\approx 87\%$ for the water-bag model or $\frac{1}{2}^{1/3}\approx 79\%$ for the Gaussian model. This is because the truncation of the matrix $(\mathbf{I}+\boldsymbol{\beta}\mathbf{M})$ in Eq. (55) at the lowest-order term no longer provides a good approximate eigenfunction if the beam size is sufficiently large. In this limit, a large degeneracy of the growth rates of the self-similar modes happens, and a single (fundamental) mode does not dominate [7]. (In the hollow-beam model, all the radial functions $R_{\omega q}^{(m,n)}$ degenerate into the δ function, so that the lowest-order radial expansion term, which is also the δ-function, gives the exact eigenfunction.) As the beam size increases, therefore, more expansion terms are needed to achieve the correct one-dimensional result. However, it is found that the critical beam size in which the truncation at the lowest order breaks down is so large that the expression (56) remains a good approximation to the exact dispersion relation (55) for most of the practically interesting parameter ranges. We investigate this problem further in the Sec. VII.

VII. NUMERICAL RESULTS

As Yu, Krinsky, and Gluckstern [2] have pointed out, the growth rate of the fundamental guided mode can be expressed in a scaled form using four dimensionless scaling parameters. One form of such a scaling relation, that which can be derived by inspecting Eq. (56), is

$$\frac{\mathrm{Re}(q)}{k_w\rho}=G\left[\frac{L_R}{L_G^{(1-D)}},\frac{\sigma_\gamma}{\rho},\frac{k_\beta}{k_w\rho},\frac{k-k_1}{k_1\rho}\right] . \tag{71}$$

Here, $L_G^{(1-D)}$ is the power-gain length of the one-dimensional theory given by

$$L_G^{(1-D)}=\frac{1}{2\sqrt{3}\rho k_w} . \tag{72}$$

Also, L_R is the Rayleigh range given by

$$L_R=\frac{2\Sigma_\perp}{\lambda_1}=\frac{k_1\Sigma_\perp}{\pi} , \tag{73}$$

where $\Sigma_\perp$ is the transverse beam area defined by

$$\Sigma_\perp=\frac{I_0}{ecn_0} , \tag{74}$$

where I_0 is the total beam current and n_0 is the peak volume density on axis. The transverse beam area can be calculated from the unperturbed distribution $f_0(\mathbf{x}_\beta,\mathbf{p}_\beta,\gamma)$ as follows. The transverse density distribution $n(\mathbf{x}_\beta)$ is obtained by

$$\begin{aligned} n(\mathbf{x}_\beta)&=\int_1^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty f_0(\mathbf{x}_\beta,\mathbf{p}_\beta,\gamma)d^2\mathbf{p}_\beta d\tau\, d\gamma \\ &=n_0 g(\mathbf{x}_\beta) , \end{aligned} \tag{75}$$

where $g(\mathbf{x}_\beta)$ is normalized so that $g(0)=1$. Then, the transverse beam area $\Sigma_\perp$ is given by

$$\Sigma_\perp=\int_{-\infty}^\infty g(\mathbf{x}_\beta)d^2\mathbf{x}_\beta . \tag{76}$$

The quantity $\Sigma_\perp$ can be also calculated from $f_{0\perp}$ as

$$\begin{aligned} \frac{1}{\Sigma_\perp}&=\int_{-\infty}^\infty f_{0\perp}(\mathbf{x}_\beta=0,\mathbf{p}_\beta)d^2\mathbf{p}_\beta \\ &=2\pi k_\beta^2\int_{-\infty}^\infty f_{0\perp}(r^2)r\, dr . \end{aligned} \tag{77}$$

The scaling relation (71) is convenient when the current density is constant. An alternative form of the scaling relation is convenient when the total beam current is constant:

$$\frac{\mathrm{Re}(q)}{k_wD}=F\left[2k_1\epsilon_x,\frac{\sigma_\gamma}{D},\frac{k_\beta}{k_wD},\frac{k-k_1}{k_1D}\right] , \tag{78}$$

where ϵ_x is the rms transverse emittance of the electron beam which is related to the square of the rms beam size $\langle x^2\rangle$ as

$$\epsilon_x=k_\beta\langle x^2\rangle . \tag{79}$$

The quantity D is the scaling parameter defined by

$$D=\left[\frac{8}{\gamma_r}\frac{K^2}{1+K^2}\frac{I_0}{I_A}\right]^{1/2} , \tag{80}$$

where $I_A=ec/r_e\approx 17.05$ kA is the Alfvén current. The parameter D was originally introduced by Yu, Krinsky, and Gluckstern [2]. However, the value of D defined here is smaller than that defined by Yu, Krinsky, and Gluckstern by a factor of $\sqrt{2}$ [9,10]. The scaling parameter D is related to ρ as

$$\frac{D}{\rho}=\frac{2\sqrt{2}}{3^{1/4}}\left[\frac{L_R}{L_G^{(1-D)}}\right]^{1/2}\approx 2.15\left[\frac{L_R}{L_G^{(1-D)}}\right]^{1/2} . \quad (81)$$

It should be emphasized that D is independent of the model for $f_{0\perp}(r^2)$ and that the scaled growth rate $\mathrm{Re}(q)/(k_w D)$ is identical to Moore's scaled growth rate $\hat{g}$ when the same physical parameters are used:

$$\hat{g}=\frac{\mathrm{Re}(q)}{k_w D} . \quad (82)$$

The dispersion relation for the Gaussian-beam model, Eq. (62), for instance, can be written in the above scaling form as

$$1=\frac{i}{4\sqrt{2\pi}}\frac{\dfrac{k_\beta}{k_w D}}{2k_1\epsilon_x}\times\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{-x^2/2}x^3 e^{-t^2/2}dx\,dt}{\left[\dfrac{q}{k_w D}+2i\dfrac{\sigma_\gamma}{D}t-\dfrac{i}{4}2k_1\epsilon_x\dfrac{k_\beta}{k_w D}x^2\right]^2}\times\int_0^{\infty}\frac{e^{-x^2}x\,dx}{\dfrac{q}{k_w D}+i\dfrac{k-k_1}{k_1 D}+i\dfrac{x^2}{2k_1\epsilon_x}\dfrac{k_\beta}{k_w D}} , \quad (83)$$

where we have replaced k by k_1 except in the detuning term $(k-k_1)/(k_1 D)$ to a good approximation. Since we are interested in the constant total beam current case in this article, we mostly use the scaling relation (78) in what follows.

For the convenience of readers, we summarize the explicit expressions of the square of the rms beam size, $\langle x^2\rangle$, and the transverse beam area $\Sigma_\perp$ for the hollow beam, the water-bag model, and the Gaussian distribution in Table I. The emittance and the Rayleigh range

TABLE I. Expressions of $\langle x^2\rangle$ and $\Sigma_\perp$ for the hollow beam, the water-bag model, and the Gaussian distribution.

Quantity	Hollow beam	Water-bag model	Gaussian distribution
$\langle x^2\rangle$	$\frac{R_0^2}{4}$	$\frac{R_0^2}{6}$	σ_x^2
$\frac{\Sigma_\perp}{\pi}$	R_0^2	$\frac{R_0^2}{2}$	$2\sigma_x^2$

can be calculated from $\langle x^2\rangle$ using Eqs. (79) and (73), respectively.

We have solved the dispersion relation numerically. The results are as follows: First, let us compare the growth rate obtained by the dispersion relation (61) for the water-bag model with the results obtained by Yu, Krinsky, and Gluckstern's variational method for the same water-bag model. The solid curves in Fig. 2 show the scaled growth rate $\mathrm{Re}(q)/(k_w D)$ as a function of $2k_1\epsilon_x$ for several values of $k_\beta/(k_w D)$. Here, the energy spread σ_γ/D is set to 0. The detuning parameter $(k-k_1)/(k_1 D)$ is chosen to yield the maximum growth rate. Generally, the optimal detuning depends on the quantity $k_1\epsilon_x(k_\beta/k_w D)$ and the energy spread σ_γ/D. This detuning is the result of the reduction of the average longitudinal velocity of the electron beam due to the transverse emittance, the betatron focusing force, and the energy spread. The dashed curves show the numerical results from Yu, Krinsky, and Gluckstern's variational method for the water-bag model. Good agreement is found. It is known that the growth rates for the water-bag model obtained by the variational method agree well with the simulations [11].

Now, we consider the case of the Gaussian distribution. In Figs. 3(a), 3(b), and 3(c), we plot $\mathrm{Re}(q)/(k_w D)$ against $2k_1\epsilon_x$ for several values of $k_\beta/(k_w D)$, for $\sigma_\gamma/D=0$, $\sigma_\gamma/D=0.2$, and $\sigma_\gamma/D=0.4$, respectively. These figures cover most of the practical range of FEL parameters. The solid curves show exact solutions of dispersion relation (62), while the dashed curves show approximate values calculated with a pair of empirical expressions of the dispersion relation (62) which agree well with the exact solutions for $\sigma_\gamma/D\lesssim 0.5$ and $k_\beta/(k_w D)\lesssim 10$. The pair of expressions is given by

$$\ln\frac{\mathrm{Re}(q)}{k_w D}=-(0.759+0.238\chi+0.0139\chi^2)\left\{1+\left[2k_1\epsilon_x\frac{k_\beta}{k_w D}\right]^2\Big/\left[0.149+0.0268\ln\frac{k_\beta}{k_w D}\right]+(44.03+3.32\chi+5.45\chi^2)\left[\left[\frac{\sigma_\gamma}{D}\right]^2-0.713\left[\frac{\sigma_\gamma}{D}\right]^4+68.65\left[\frac{\sigma_\gamma}{D}\right]^6\right]\right\}^{1/2}$$

$$\text{for } 2k_1\epsilon_x\frac{k_w D}{k_\beta}\geq 0.05 \text{ and } \frac{k_\beta}{k_w D}\leq 1 \quad (84)$$

and

$$\frac{\mathrm{Re}(q)}{k_w D} = (0.0628 - 0.219\chi - 0.000\,568\chi^2)^{1/2} \exp\left[-\frac{\left[2k_1\epsilon_x \frac{k_\beta}{k_w D}\right]^2}{\left[1.091 + 0.1345\frac{k_\beta}{k_w D}\right]} - (11.92 + 2.202\chi + 0.1414\chi^2)\left[\frac{\sigma_\gamma}{D}\right]^2 \right]$$

$$\text{for } 2k_1\epsilon_x \frac{k_w D}{k_\beta} < 0.05 \text{ or } \frac{k_\beta}{k_w D} > 1 , \quad (85)$$

where

$$\chi = \ln\left[2k_1\epsilon_x \frac{k_w D}{k_\beta}\right] . \quad (86)$$

The parameter $2k_1\epsilon_x(k_w D/k_\beta)$ is a function only of the ratio of the Rayleigh range to the one-dimensional gain length:

$$2k_1\epsilon_x \frac{k_w D}{k_\beta} = \frac{2\sqrt{2}}{3^{3/4}} \frac{\langle x^2 \rangle}{\Sigma_\perp/\pi} \left[\frac{L_R}{L_G^{(1-D)}}\right]^{3/2} . \quad (87)$$

This pair of expressions can be used as a handy formula for a quick estimate of the growth rate.

Comparing Fig. 2 with Fig. 3(a), we notice that the Gaussian distribution shows a considerably smaller growth rate $\mathrm{Re}(q)/(k_w D)$ due to Landau damping than the water-bag model for large $k_\beta/(k_w D)$ when $2k_1\epsilon_x > 0.1$ [compare $k_\beta/(k_w D) = 10$ curves, for example]. This is also the case with the parabolic distribution of $f_{0\perp}(r^2)$ in phase space, which is not shown here; however, it gives similar curves to those of the Gaussian model. This observation implies that the FEL gain for the strong focusing and the large emittance depends sensitively on the details of the transverse distribution. In contrast, we notice that the two figures show more or less identical values of $\mathrm{Re}(q)/(k_w D)$ for the small emittance region $2k_1\epsilon_x < 0.1$. In this region, the variation of the longitudinal velocity inside the beam due to the betatron oscillation is small, and the effect of Landau damping due to the transverse motion becomes negligible. The similar behavior of $\mathrm{Re}(q)/(k_w D)$ in Figs. 2 and 3(a) implies that the FEL gain becomes insensitive to the shape of the transverse distribution of the electron beam for the small beam size, and therefore, it is convenient to calculate the FEL gain using the rms beam size or emittance in the small beam size region. Then, the FEL gain becomes independent of the transverse beam distribution.

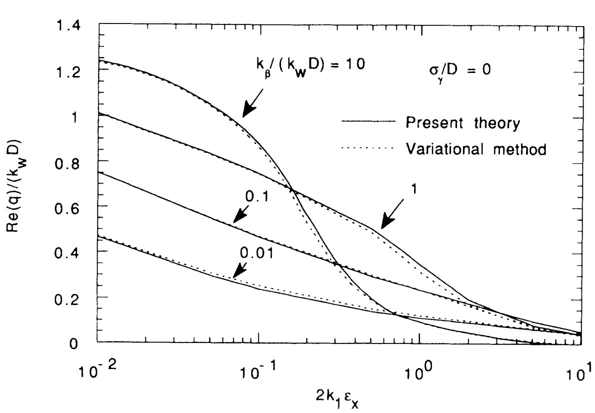

FIG. 2. Scaled growth rate $\mathrm{Re}(q)/(k_w D)$ as a function of $2k_1\epsilon_x$ for several values of the scaled betatron wave number $k_\beta/(k_w D)$ for the water-bag model, where k_1 is the radiation wave number and ϵ_x is the transverse emittance. Here, the scaled energy spread $\sigma_\gamma/D = 0$. The solid curves show solutions of the dispersion relation (61), while the dashed curves show the numerical results obtained by Yu, Krinsky, and Gluckstern's variational method.

We have compared the above results obtained by solving the dispersion relation with those obtained by simulation using the computer code TDA [12]. The nominal FEL parameters used in the simulation are given in Table II. Here, we have chosen the FEL parameters such that the scaled betatron wave number $k_\beta/(k_w D) = 1$, a value large enough to show clearly the effects of Landau damping due to the betatron focusing and the emittance. The detuning parameters used for the simulations are identical to those for the analytical results which yield the maximum growth rates. In Fig. 4(a) we plot the scaled growth rate $\mathrm{Re}(q)/(k_w D)$ as a function of $2k_1\epsilon_x$ for the zero-energy spread for the Gaussian and the water-bag beam distributions. The agreement is excellent. The benchmark for the nonzero-energy spread $\sigma_\gamma/D = 0.2$ is shown in Fig. 4(b) for the Gaussian-beam distribution. The agreement is also excellent.

In Fig. 3 it appears that one can increase the FEL gain by increasing the betatron focusing for a given emittance, until its increase is overwhelmed by the reduction due to Landau damping. However, this increase in the FEL gain is actually originating from the reduced beam size due to the strengthened focusing. This may be more clearly seen if one plots the FEL gain as a function of the beam size instead of the emittance, using a scaling relation of the following form:

$$\frac{\mathrm{Re}(q)}{k_w D} = H\left[\frac{L_R}{L_G^{(1-D)}}, \frac{\sigma_\gamma}{D}, \frac{k_\beta}{k_w D}, \frac{k - k_1}{k_1 D}\right] . \quad (88)$$

[Note the similarity of the above equation with the scaling relation (71) for the constant ρ case.] In Fig. 5 we plot $\mathrm{Re}(q)/k_w D$ as a function of $L_R/L_G^{(1-D)}$ for the Gaussian distribution for $\sigma_\gamma/D = 0$, which is equivalent to Fig. 3(a). When the beam size is small, all the curves for different $k_\beta/(k_w D)$ become identical. As the beam size increases, the curves for large $k_\beta/(k_w D)$ start to

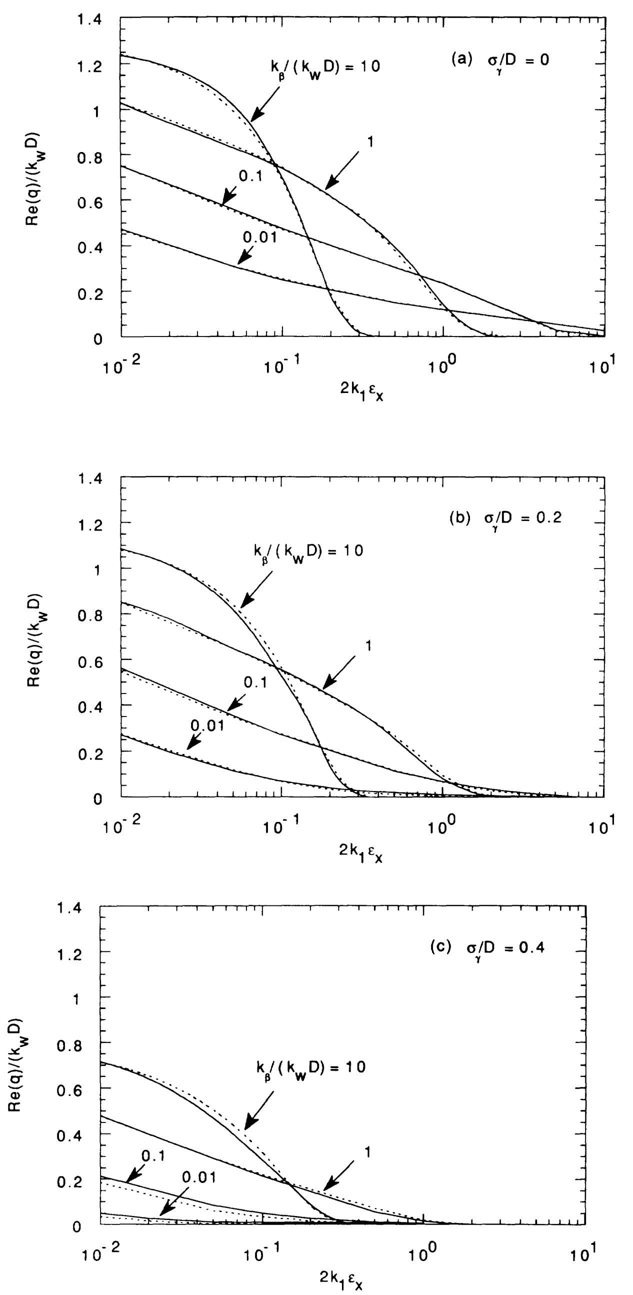

FIG. 3. (a) Scaled growth rate $\mathrm{Re}(q)/(k_w D)$ as a function of $2k_1\epsilon_x$ for several values of the scaled betatron wave number $k_\beta/(k_w D)$ for the Gaussian model, where k_1 is the radiation wave number and ϵ_x is the transverse emittance. Here, the scaled energy spread $\sigma_\gamma/D=0$. The solid curves show solutions of the dispersion relation (62), while the dashed curves show the approximate values calculated by the handy formulas (84) and (85). (b) Scaled growth rate $\mathrm{Re}(q)/(k_w D)$ as a function of $2k_1\epsilon_x$ for several values of $k_\beta/(k_w D)$ for the Gaussian model. Here, $\sigma_\gamma/D=0.2$. The solid curves show solutions of the dispersion relation (62), while the dashed curves show the approximate values calculated by the handy formulas (84) and (85). (c) Scaled growth rate $\mathrm{Re}(q)/(k_w D)$ as a function of $2k_1\epsilon_x$ for several values of $k_\beta/(k_w D)$ for the Gaussian model. Here, $\sigma_\gamma/D=0.4$. The solid curves show solutions of the dispersion relation (62), while the dashed curves show the approximate values calculated by the handy formulas (84) and (85).

TABLE II. Nominal FEL parameters used for the simulations.

Parameter	Value
The Lorentz factor of the reference electron, γ_r	100
The wiggler period λ_w	3 cm
The peak wiggler parameter K	2
The total beam current I_0	53.28 A
The resonant radiation wavelength $\lambda_1=2\pi/k_1$	7.5 μm
The scaling parameter D	0.014 142
The betatron wavelength $\lambda_\beta=2\pi/k_\beta$	2.121 32 m

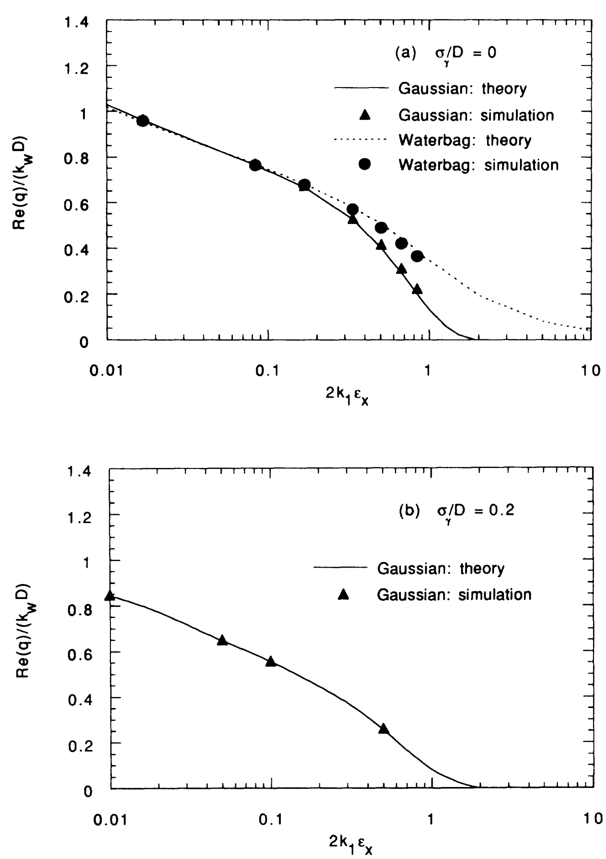

FIG. 4. (a) Comparison of the scaled growth rate $\mathrm{Re}(q)/(k_w D)$ with the simulation results for the Gaussian and the water-bag beam distributions. Here, the scaled betatron wave number $k_\beta/(k_w D)=1$ and the scaled energy spread $\sigma_\gamma/D=0$. The solid and the dashed curves show the solutions of the dispersion relations for the Gaussian and the water-bag beam distributions, respectively, while the triangles and the circles show the simulation results for the Gaussian and the water-bag beam distributions, respectively. (b) Comparison of the scaled growth rate $\mathrm{Re}(q)/(k_w D)$ with the simulation results for the Gaussian-beam distribution. Here, $k_\beta/(k_w D)=1$ and $\sigma_\gamma/D=0.2$. The solid curve shows the solution of the dispersion relation for the Gaussian-beam distribution, while the triangles show the simulation results for the Gaussian-beam distribution.

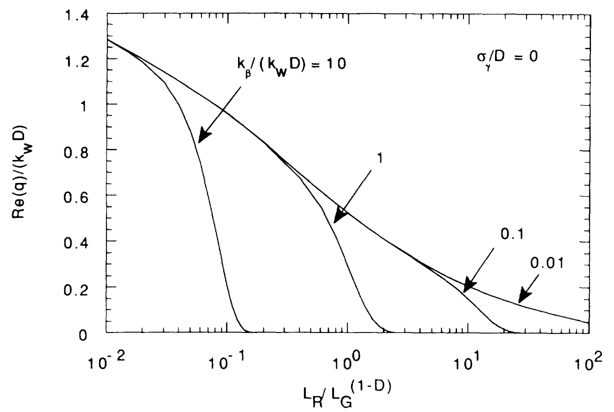

FIG. 5. Scaled growth rate $\mathrm{Re}(q)/(k_w D)$ as a function of the ratio of the Rayleigh range to the one-dimensional gain length, $L_R/L_G^{(1-D)}$, for several values of the scaled betatron wave number $k_\beta/(k_w D)$ for the Gaussian model. Here, the scaled energy spread $\sigma_\gamma/D=0$. This figure is equivalent to Fig. 3(a).

break away. This figure shows that one can have a larger FEL gain for a weaker betatron focusing for a fixed beam size. The scaling relation (88) may be a better expression for understanding of the physical picture, while the other one (78) may be more suitable to the practical application.

Finally, we examine the accuracy of the truncation to the dispersion relation at the lowest order of the azimuthal and the radial expansions. First, let us concentrate on the azimuthal expansion and keep the radial expansion at the lowest order $k=0$. Numerical evaluation of the matrix $\mathbf{M}$ defined in Eq. (53) shows that the off-diagonal elements $M_{n,n,0}^{0,0,0}$ $(n>0)$ are normally smaller than $M_{0,0,0}^{0,0,0}$ by more than one order of magnitude. The accuracy of the truncation at the lowest order $m=n=0$ depends on the square of these off-diagonal elements. Therefore, the inclusion of the higher-order azimuthal modes are unlikely to change the gain of the fundamental mode very much. Indeed, we have found during the computation of data for Figs. 2 and 3 that the change in the gain of the fundamental mode due to the higher-order azimuthal modes $(m,n>0)$ is less than 1%. Next, we consider the radial expansion and keep the azimuthal expansion at the lowest order $m=n=0$. In this case, the numerical calculations of the growth rate for the Gaussian model showed that when $L_R/L_G^{(1-D)}\lesssim 30$, the changes in the growth rate of the fundamental mode due to the inclusion of the first-order radial expansion term $k=1$ is less than a few percent, while for $L_R/L_G^{(1-D)}\sim 140$, it increases to 6.3%. Normally, the change in the growth rate of the fundamental mode by including the $k=1$ expansion term becomes smaller as the beam size becomes smaller. From this result we conclude that the truncation of the dispersion relation at the lowest order provides a good approximated eigenvalue, unless the beam size is so large that the three-dimensional effects such as the diffraction effect become negligible. Therefore, in the practical range of the beam size, the truncated dispersion relation (56) is a useful and valid approximation.

VIII. PLANAR WIGGLER

So far, we have assumed that an electron beam goes through a helical wiggler. However, the same formulation can be applied to the FEL using a planar wiggler with a few changes. We still need to assume that the betatron focusing in the wiggler is matched to the electron beam, either by the alternating field of the wiggler magnet or by suitable external focusing devices. For simplicity, we also assume the betatron focusing is equal in the x and y directions. [Without this assumption, we need to introduce different betatron numbers $k_{\beta x}$ and $k_{\beta y}$ and different emittances ϵ_x and ϵ_y for x and y planes, respectively. Then, the scaling relation, Eq. (78), requires six independent scaling parameters, $2k_1\epsilon_x$, $2k_1\epsilon_y$, σ_y/D, $k_{\beta x}/(k_w D)$, $k_{\beta y}/(k_w D)$, and $(k-k_1)/(k_1 D)$. This asymmetric focusing case can be treated parallel with the preceding symmetric focusing case. The results are summarized in Appendix D.] Now, the main change is that we need a new evaluation of the angular distribution of radiated power spectrum, $P_{\omega q}(k_\perp)$, in Eq. (30). The result is

$$P_{\omega q}(k_x,k_y)=-\sum_{p=-\infty}^{\infty}\frac{r_e}{2\pi c}\left[\frac{K}{\gamma_r}\right]^2\frac{A_p(k_x)}{[1-(k_\perp/k)^2]^{1/2}\left[q-i\left[pk_w+(k^2-k_\perp^2)^{1/2}-\dfrac{w}{v_r}\right]\right]}, \tag{89}$$

where

$$A_p(k_x)=\frac{1}{4}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(-i)^n i^m J_n(k_x r_w)J_m(k_x r_w)$$
$$\times[J_{(p+n-1)/2}(S_w)-J_{(p+n+1)/2}(S_w)][J_{(p+m-1)/2}(S_w)-J_{(p+m+1)/2}(S_w)]\ . \tag{90}$$

Here,

$$S_w=\frac{k}{k_1}\frac{K^2}{4(1+K^2/2)}\ , \tag{91}$$
$$k_1=2\gamma_r^2 k_w/(1+K^2/2)\ ,$$
$$r_w=K/(\gamma k_w)$$

is the radius of the wiggler motion, and K is the peak value of the wiggler parameter on axis. In the limit of small amplitude of the wiggler motion, $r_w \to 0$, $P_{\omega q}$ can be approximated by

$$P_{\omega q}(k_x,k_y)=-\sum_{p=-\infty}^{\infty}\frac{r_e}{2\pi c}\left[\frac{K}{\gamma_r}\right]^2\frac{[JJ]_p^2/4}{[1-(k_\perp/k)^2]^{1/2}\left[q-i\left[pk_w+(k^2-k_\perp^2)^{1/2}-\frac{w}{v_r}\right]\right]}\ , \tag{92}$$

where

$$[JJ]_p=J_{(p-1)/2}(S_w)-J_{(p+1)/2}(S_w)\ . \tag{93}$$

The function $P_{\omega q}$ can be further approximated by retaining only the fundamental harmonic term of the forward radiation, $p=1$. In this approximation, we simply denote $[JJ]_1$ as $[JJ]$ in what follows. Thus, the dispersion relation for the fundamental mode, Eq. (56), should be multiplied by the factor $[JJ]/2$ on the right-hand side (RHS). It becomes

$$1=2i\frac{k}{k_1}\frac{r_e}{c}\left[\frac{K}{\gamma_r}\right]^2\frac{k_w}{\gamma_r}[JJ]^2\int_0^\infty\int_1^\infty\frac{f_{0\parallel}(\gamma)d\gamma}{\left[q+2i\frac{k}{k_1}k_w\frac{\gamma-\gamma_r}{\gamma_r}-i\frac{1}{2}kk_\beta^2r^2\right]^2}2\pi^2k_\beta^2f_{0\perp}(r^2)r^3dr$$
$$\times\int_0^{\pi/2}\frac{k^2\theta\,d\theta}{q+ik_w\frac{k-k_1}{k_1}+i\frac{k\theta^2}{2}}\left[\int_0^\infty 2\pi^2k_\beta^2f_{0\perp}(r')\frac{J_1(kr'\theta)}{kr'\theta}r'^3dr'\right]^2\ . \tag{94}$$

This implies that the same factor $[JJ]^2/2$ should be multiplied to the RHS of Eq. (63) for the Pierce parameter:

$$(2\rho k_w)^3=\pi r_e n_0\left[\frac{K}{\gamma_r}\right]^2\frac{k_w}{\gamma_r}[JJ]^2\ . \tag{95}$$

Accordingly, D is changed to

$$D=\left[\frac{4}{\gamma_r}\frac{K^2}{1+K^2/2}\frac{I_0}{I_A}\right]^{1/2}[JJ]\ . \tag{96}$$

With these changes, the dispersion relations, Eqs. (60)–(62), and the handy formulas of the growth rate, Eqs. (84) and (85), are all valid.

IX. BEAM CONDITIONING

The idea of beam conditioning is an attempt to reduce the longitudinal velocity spread within the beam by correlating the transverse oscillation amplitude and the electron energy, in order to increase the FEL gain [4]. This can be briefly explained as follows. Before entering the FEL, an electron beam goes through a device, called a "conditioner," that consists of a focusing channel and suitably phased rf cavities operating in the TM_{210} mode. This device provides a different energy increment to individual electrons with different transverse oscillation amplitude so that the RHS of Eq. (7) vanishes in the ideal case:

$$\frac{\gamma-\gamma_r}{\gamma_r}=\frac{1}{2\frac{k_w}{k_1}}\frac{1}{2}(\mathbf{p}_\beta^2+k_\beta^2\mathbf{x}_\beta^2)\ . \tag{97}$$

Therefore, if all electrons have the same energy before entering the conditioner, they all move with the same longitudinal velocity after the device. In this case, there is no gain reduction due to the electron-beam emittance. In reality, however, the electron beam is likely to have a nonzero initial energy spread. In this case, only the part of the longitudinal velocity spread due to the electron-beam emittance is cancelled, and we still have the reduction of the gain due to the energy spread.

The beam-conditioned FEL can be analyzed in the present theory with a few modifications. We define a new longitudinal variable α, instead of γ, by

$$\alpha=\gamma-\gamma_r-\frac{\gamma_r}{2\frac{k_w}{k_1}}\frac{1}{2}(\mathbf{p}_\beta^2+k_\beta^2\mathbf{x}_\beta^2)\ . \tag{98}$$

The distribution of α is determined by the initial γ distribution (before the device) and the performance of the beam-conditioning device. Assuming an ideal operation of the beam conditioner, the distribution in α after the device will be the same as the distribution in γ before the device. Note that the energy distribution will be changed after the device.

The equation of motion of τ, Eq. (7), can be expressed with α as

$$\frac{d\tau}{dz}=-\frac{2}{c}\frac{k_w}{k_1}\frac{\alpha}{\gamma_r}\ . \tag{99}$$

The equation of motion of α follows from Eq. (98) that

$$\frac{d\alpha}{dz}=\frac{d\gamma}{dz}-\frac{1}{2\frac{k_w}{k_1}}\frac{1}{2}\frac{d}{dz}(\mathbf{p}_\beta^2+k_\beta^2\mathbf{x}_\beta^2)=\frac{d\gamma}{dz}\ , \tag{100}$$

where we have used the fact that the transverse oscillation amplitude $\mathbf{p}_\beta^2+k_\beta^2\mathbf{x}_\beta^2$, is the constant of motion.

Now, the electron distribution f is regarded as a function of $\mathbf{x}_\beta$, $\mathbf{p}_\beta$, τ, α, and z:

$$f=f(\mathbf{x}_\beta,\mathbf{p}_\beta,\tau,\alpha;z) \ . \tag{101}$$

Accordingly, the linearized Vlasov equation (13) is changed to

$$\frac{\partial f_1}{\partial z}+\mathbf{p}_\beta\frac{\partial f_1}{\partial \mathbf{x}_\beta}-k_\beta^2\mathbf{x}_\beta\frac{\partial f_1}{\partial \mathbf{p}_\beta}+\frac{d\tau}{dz}\frac{\partial f_1}{\partial \tau}+\frac{d\alpha}{dz}\frac{\partial f_0}{\partial \alpha}=0. \tag{102}$$

Our assumption on the factorization of f_0, Eq. (14), should be replaced by

$$f_0=f_{0\perp}(\mathbf{x}_\beta^2+\mathbf{p}_\beta^2/k_\beta^2)f_{0\parallel}(\alpha) \ . \tag{103}$$

The rest of the procedure closely follows the preceding formulation except that most of the γ appearing in association with the electron distribution function must be replaced by α. Finally, we arrive at the general form of the approximate dispersion relation [cf. Eq. (56)]

$$1=4i\frac{k}{k_1}\frac{r_e}{c}\left[\frac{K}{\gamma_r}\right]^2\frac{k_w}{\gamma_r}\int_{-\infty}^{\infty}\frac{f_{0\parallel}(\alpha)d\alpha}{\left[q+2i\frac{k}{k_1}k_w\frac{\alpha}{\gamma_r}\right]^2}\int_0^{\pi/2}\frac{k^2\theta\, d\theta}{q+ik_w\frac{k-k_1}{k_1}+i\frac{k\theta^2}{2}}\left[\int_0^\infty 2\pi^2k_\beta^2 f_{0\perp}(r')\frac{J_1(kr'\theta)}{kr'\theta}r'^3dr'\right]^2 . \tag{104}$$

Note that we have performed the r integration in the first line of Eq. (104), since there is no r dependence in the denominator. Now, the reduction of the gain is solely determined by the α distribution. If we assume a Gaussian distribution for α with the rms spread, σ_α,

$$f_{0\parallel}(\alpha)=\frac{N}{\hat{\tau}}\frac{1}{\sqrt{2\pi}\sigma_\alpha\gamma_r}e^{-\alpha^2/2\gamma_r^2\sigma_\alpha^2} , \tag{105}$$

the dispersion relations for various models of $f_{0\perp}(r^2)$, Eqs. (60)–(62), are still valid. The only changes are to set $\frac{1}{2}kk_\beta^2R_0^2=0$ in Eqs. (60) and (61) and to set $\frac{1}{2}kk_\beta^2\sigma_x^2=0$ in Eq. (62), and to replace σ_γ by σ_α. As in Eq. (104), we can carry out the x integration readily. The handy empirical formulas (84) and (85) can also be used only by deleting the terms proportional to $[2k_1\epsilon_x(k_\beta/k_wD)]^2$ and replacing σ_γ by σ_α.

Since the transverse emittance does not contribute to the gain reduction, the gain is now insensitive to the shape of the transverse beam distribution. For example, the dispersion relation for the hollow-beam model, Eq. (60), can be used for an estimate of the gain, regardless of the actual transverse beam distribution. (This choice is convenient, since this dispersion relation keeps a good approximation even in the limit of large beam size.) The handy empirical formulas (84) and (85) without the terms proportional to $[2k_1\epsilon_x(k_\beta/k_wD)]^2$ serve as good approximations to the dispersion relation for the hollow-beam model.

X. CONCLUSIONS

We have developed the 3D FEL theory in the high-gain regime before saturation based upon the Maxwell-Vlasov equation, including the effects of the energy spread, the transverse emittance, the angular distribution of the radiation from a single particle, the betatron focusing and oscillation of the electron beam, and the diffraction and the guiding of the radiation field. Our numerical results of the FEL gain show good agreement with results obtained by Moore's approach for the hollow beam (see Fig. 1) and Yu, Krinsky, and Gluckstern's approach for the water-bag model (see Fig. 2) of $f_{0\perp}(\mathbf{x}_\beta^2+\mathbf{p}_\beta^2/k_\beta^2)$, respectively. We presented a dispersion relation for the FEL gain of a Gaussian beam, Eq. (62), and its approximate expressions, Eqs. (84) and (85), for a quick estimate of the growth rate with a pocket calculator. Comparisons of numerical solutions of this dispersion relation with the simulation results for the Gaussian beam show excellent agreement. We have shown that the present theory can handle the beam-conditioning case easily by changing the longitudinal coordinate and by implementing a few modifications.

One eminent advantage of the present orthogonal expansion method is that an accurate eigenvalue for the fundamental mode can be obtained by taking only the lowest-order expansion term, unless the beam size is too large. As a result, the matrix form of the dispersion relation can be reduced to just a scalar equation. This is not always the case with any expansion method. If one uses an arbitrary set of the orthogonal functions to expand the electron-distribution function, one normally has to sum a number of the expansion terms, or one has no guarantee that the expansion even converges. In the present expansion method, the orthogonal functions are uniquely determined by the unperturbed electron distribution so as to satisfy the orthogonal relationship (47). We have found that this procedure provides a good approximate eigenfunction even if the expansion is truncated at the first term for a wide range of the unperturbed electron-distribution function. Once a good approximate eigenfunction is prepared, a relatively accurate eigenvalue is obtained because the error in the eigenvalue depends quadratically on errors in the approximate eigenfunction. In contrast with the variational method, however, the accuracy of calculation can be determined by evaluating the higher-order expansion terms, and if necessary, one can improve the accuracy systematically by including these higher-order terms.

The present method can be easily extended to the asymmetric betatron focusing case, which may be more realistic in a storage-ring FEL system with a planar wiggler. The results are briefly summarized in Appendix D. In this case, the gain of the fundamental mode becomes a function of six scaling parameters.

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APPENDIX A: EQUATIONS OF MOTION

The Hamiltonian for a single electron is given by

$$H = mc^2\gamma = c\,[(\mathbf{p}_\perp - e\,\mathbf{A})^2 + p_z^2 + m^2c^2]^{1/2}\ , \tag{A1}$$

where $\mathbf{p}_\perp$ and p_z are the canonical momentum conjugates to the transverse coordinates, $\mathbf{x}=(x,y)$ and z, respectively. The vector potential $\mathbf{A}=\mathbf{A}(\mathbf{x},z,t)$ consists of the vector potential of the wiggler field, $\mathbf{A}_w$, and that for the radiation field, $\mathbf{A}_r$:

$$\mathbf{A} = \mathbf{A}_w + \mathbf{A}_r\ . \tag{A2}$$

For a small transverse displacement from the wiggler axis, $\mathbf{A}_w$ can be well approximated by

$$\mathbf{A}_w = A_w[\mathbf{i}_x(1+\tfrac{1}{2}k_w^2y^2)\cos k_w z + \mathbf{i}_y(1+\tfrac{1}{2}k_w^2x^2)\sin k_w z]\ , \tag{A3}$$

where $\mathbf{i}_x$ and $\mathbf{i}_y$ are unit vectors in the x and y directions, respectively. We derive an expression of $\mathbf{A}_r$ in Appendix B. All the other notations are as follows: c is the speed of light, e is the electron charge, and γ is the electron energy in units of its rest mass energy mc^2.

It is convenient to choose z, the distance from the wiggler entrance, as the independent variable. The new Hamiltonian is simply p_z:

$$p_z = [m^2c^2\gamma^2 - m^2c^2 - (\mathbf{p}_\perp - e\,\mathbf{A})^2]^{1/2}\ . \tag{A4}$$

Equations of motion for an electron are then given by Hamilton's equations:

$$\frac{d\mathbf{x}}{dz} = -\frac{\partial p_z}{\partial \mathbf{p}_\perp},\quad \frac{d\mathbf{p}_\perp}{dz} = \frac{\partial p_z}{\partial \mathbf{x}} \tag{A5}$$

and

$$\frac{dt}{dz} = \frac{\partial p_z}{\partial H},\quad \frac{dH}{dz} = -\frac{\partial p_z}{\partial t}\ . \tag{A6}$$

The variables $\mathbf{x}$ and $\mathbf{p}_\perp$ include the fast-oscillating helical motion. As variables to be used in the Vlasov equation, we define slowing varying new transverse variables $\mathbf{x}_\beta$ and their canonical momentum conjugates $\mathbf{p}_\beta$ as the average of $\mathbf{x}$ and $\mathbf{p}_\perp$ over the wiggler period:

$$\mathbf{x}_\beta = \frac{1}{\lambda_w}\int_z^{z+\lambda_w}\mathbf{x}\,dz,\quad \mathbf{p}_\beta = \frac{1}{\lambda_w}\int_z^{z+\lambda_w}\frac{\mathbf{p}_\perp}{mc\gamma}dz\ , \tag{A7}$$

where we have introduced the normalization factor $mc\gamma$ that makes $\mathbf{p}_\beta$ dimensionless. When the rapidly oscillating radiation fields and the nonlinear terms that are of order of $1/\gamma^2$ or higher are ignored, the transverse part of Hamilton's equations (A5) becomes equations of a simple harmonic oscillator:

$$\frac{d\mathbf{x}_\beta}{dz} = \mathbf{p}_\beta,\quad \frac{d\mathbf{p}_\beta}{dz} = -k_\beta^2\mathbf{x}_\beta\ , \tag{A8}$$

where

$$k_\beta^2 = K^2\frac{k_w^2}{2\gamma^2} \tag{A9}$$

is the betatron wave number in the absence of external focusing and $K=eA_w/(mc)$ is the peak wiggler parameter.

The equation of motion of $\tau = t - z/v_r$ is obtained by carrying out the partial derivative in the first equation of Eq. (A6), where v_r is the longitudinal velocity of the reference electron with the zero transverse oscillation amplitude. It is approximately given by

$$\frac{d\tau}{dz} \approx \frac{1}{c}\left[1+\frac{1+K^2}{2\gamma^2}+\frac{1}{2}(\mathbf{p}_\beta^2+k_\beta^2\mathbf{x}_\beta^2)\right] - \frac{1}{c}\left[1+\frac{1+K^2}{2\gamma_r^2}\right]$$
$$\approx \frac{1}{c}\left[-\frac{1+K^2}{\gamma_r^2}\frac{\gamma-\gamma_r}{\gamma_r}+\frac{1}{2}(\mathbf{p}_\beta^2+k_\beta^2\mathbf{x}_\beta^2)\right]\ . \tag{A10}$$

So far, the equations of motion of $\mathbf{x}_\beta$, $\mathbf{p}_\beta$, and τ were derived by taking into account the wiggler field only. To derive an equation of motion of the energy γ, it is essential to consider the interaction of the electron's helical motion and the radiation field. Hamilton's equation of γ becomes

$$mc^2\frac{d\gamma}{dz} = -e\frac{\partial \mathbf{A}_r}{\partial t}\frac{(\mathbf{p}_\perp - e\,\mathbf{A})}{p_z}\ . \tag{A11}$$

Note that $\mathbf{A}_w = \mathbf{A}_w(\mathbf{x},z)$ has no time dependence. Using Hamilton's equation of $\mathbf{x}$,

$$\frac{d\mathbf{x}}{dz} = \frac{\mathbf{p}_\perp - e\,\mathbf{A}}{p_z}\ , \tag{A12}$$

Equation (A11) can be written as

$$mc^2\frac{d\gamma}{dz} = -e\frac{d\mathbf{x}}{dz}\frac{\partial \mathbf{A}_r}{\partial t}$$
$$\approx -e\frac{d\mathbf{x}_h}{dz}\frac{\partial \mathbf{A}_r}{\partial t}\ , \tag{A13}$$

where we have retained only the fast-oscillating helical motion $\mathbf{x}_h$ in $d\mathbf{x}/dz$, as the first-order approximation.

APPENDIX B: DERIVATION OF $\mathbf{A}_r(\mathbf{r},t)$ AND ENERGY CHANGE

In this appendix we derive Eq. (25) for $\mathbf{A}_r(\mathbf{r},t)$, and Eq. (29) for the energy change by the radiation field. The starting equation is Eq. (24). First, we introduce the Fourier transforms of ρ_1 and $\mathbf{G}$ over $\mathbf{x}_\beta$ and τ as

$$\rho_\omega(\mathbf{k}_\perp,z)=\left[\frac{1}{2\pi}\right]^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_1(\mathbf{x}_\beta,\tau;z)e^{-i\mathbf{k}_\perp\cdot\mathbf{x}_\beta}\times e^{i\omega\tau}d^2\mathbf{x}_\beta d\tau \qquad (B1)$$

and

$$\mathbf{G}_{\omega\mathbf{k}_\perp}(z|z')=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{G}(\mathbf{r},t|\mathbf{r}',t')e^{-i\mathbf{k}_\perp\cdot(\mathbf{x}-\mathbf{x}')}e^{i\omega(t-t')}\times d^2(\mathbf{x}-\mathbf{x}')d(t-t')\ . \qquad (B2)$$

From Eq. (18), we have an explicit form of $\mathbf{G}_{\omega\mathbf{k}_\perp}$ in free space:

$$\mathbf{G}_{\omega\mathbf{k}_\perp}(z|z')=\frac{1}{(2\pi)^4}\overleftrightarrow{\mathbf{I}}\int_{-\infty}^{\infty}\frac{e^{ik_z(z-z')}}{\mathbf{k}^2-\left[\dfrac{\omega}{c}\right]^2}dk_z\ . \qquad (B3)$$

We carry out the integration over k_z using the residue theorem. The contour of integration goes from the negative infinity to the positive infinity along the real axis and closed in the upper half-plane. It goes above a pole on the negative real k_z axis and below a pole on the positive real k_z axis. The result is

$$\mathbf{G}_{\omega\mathbf{k}_\perp}(z|z')=i\frac{\pi}{(2\pi)^4}\overleftrightarrow{\mathbf{I}}\frac{e^{i\hat{k}_z(z-z')}}{\hat{k}_z}\ , \qquad (B4)$$

where

$$\hat{k}_z=[k^2-(k_x^2+k_y^2)]^{1/2}\ . \qquad (B5)$$

If we neglect the small $\mathbf{x}_\beta$ dependence of $v_\parallel$ in the helical radius r_h, we find that the convolution law can be applied to Eq. (24) between $\mathbf{G}(\mathbf{r},t|\mathbf{r}',t')$ and $\rho_1(\mathbf{x}_\beta,\tau;z)$ over $\mathbf{x}'_\beta$ and τ' integrals. It follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{G}(\mathbf{r},t|\mathbf{r}',t')\rho_1(\mathbf{x}'_\beta,\tau';z')d^2\mathbf{x}'_\beta d\tau' =\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{G}_{\omega\mathbf{k}_\perp}(z|z')\rho_\omega(\mathbf{k}_\perp,z')e^{i\mathbf{k}_\perp\cdot(\mathbf{x}-\mathbf{x}'_h)}\times e^{-i\omega(t-z'/v_r)}d^2\mathbf{k}_\perp d\omega\ , \qquad (B6)$$

where we have used $\mathbf{x}'=\mathbf{x}'_\beta+\mathbf{x}'_h$ and $t'=\tau'+z'/v_r$.

We also introduce the Laplace transform of $\rho_\omega(\mathbf{k}_\perp,z)$ with respect to z defined by

$$\rho_{\omega q}(\mathbf{k}_\perp)=\int_0^{\infty}\rho_\omega(\mathbf{k}_\perp,z)e^{-qz}dz\ , \qquad (B7)$$

where the Laplace-transformed function $\rho_{\omega q}(\mathbf{k}_\perp)$ is defined only for $\mathrm{Re}(q)\ge q_0$. The inverse Laplace transform is given by

$$\rho_\omega(\mathbf{k}_\perp,z)=\frac{1}{2\pi i}\int_{q_0-i\infty}^{q_0+i\infty}\rho_{\omega q}(\mathbf{k}_\perp)e^{qz}dq\ . \qquad (B8)$$

Inserting Eqs. (B6) and (B8) into Eq. (24), we have

$$\mathbf{A}_r(\mathbf{r},t)=e\mu_0\int_{-\infty}^{\infty}\left[\frac{1}{2\pi i}\int_{q_0-i\infty}^{q_0+i\infty}\left[\int_{-\infty}^{\infty}\mathbf{H}_{\omega q}(\mathbf{k}_\perp,z)\rho_{\omega q}(\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}}d^2\mathbf{k}_\perp\right]e^{qz}dq\right]e^{-i\omega\tau}d\omega\ , \qquad (B9)$$

where we have defined the integral

$$\mathbf{H}_{\omega q}(\mathbf{k}_\perp,z)=\int_0^z\mathbf{G}_{\omega\mathbf{k}_\perp}(z|z')\frac{d\mathbf{x}'_h}{dz'}e^{-i\mathbf{k}_\perp\cdot\mathbf{x}'_h}\times e^{i\omega(z'-z)/v_r}e^{q(z'-z)}dz'\ . \qquad (B10)$$

Now, our task is to carry out the integration in Eq. (B10). If we insert Eqs. (3) and (B3) into Eq. (B10), we obtain

$$\mathbf{H}_{\omega q}(\mathbf{k}_\perp,z)=-i\frac{K}{\gamma_r}\frac{\pi}{(2\pi)^4}\frac{\mathbf{I}_{\omega q}(\mathbf{k})}{\hat{k}_z}e^{-(q+i\omega/v_r-i\hat{k}_z)z}\ , \qquad (B11)$$

where

$$\mathbf{I}_{\omega q}(\mathbf{k})=\int_0^z(\mathbf{i}_x\cos k_w z'+\mathbf{i}_y\sin k_w z')\times e^{(q+i\omega/v_r-i\hat{k}_z)z'}e^{ik_x r_h\sin k_w z'}\times e^{-ik_y r_h\cos k_w z'}dz'\ , \qquad (B12)$$

and we have replaced $c/v_\parallel$ by 1 in the helical radius to a good accuracy. By using the expansion formulas [6]

$$e^{ik_x r_h\sin k_w z'}=\sum_{m=-\infty}^{\infty}(-1)^m J_m(k_x r_h)e^{-imk_w z'} \qquad (B13)$$

and

$$e^{-ik_y r_h\cos k_w z'}=\sum_{n=-\infty}^{\infty}i^n J_n(k_y r_h)e^{-ink_w z'}\ , \qquad (B14)$$

the integration in Eq. (B12) can be carried out, with the result,

$$\mathbf{I}_{\omega q}(\mathbf{k})=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^m i^n J_m(k_x r_h)J_n(k_y r_h)\times\left[\mathbf{i}_x\frac{I_+^{(m+n)}+I_-^{(m+n)}}{2}+\mathbf{i}_y\frac{I_+^{(m+n)}-I_-^{(m+n)}}{2i}\right]\ , \qquad (B15)$$

where

$$I_\pm^{(m+n)}=\frac{e^{\{i[(\pm1-m-n)k_w-\hat{k}_z+\omega/v_r]+q\}z}-1}{i[(\pm1-m-n)k_w-\hat{k}_z+\omega/v_r]+q}\ . \qquad (B16)$$

If we notice that

$$I_+^{(m+n+1)}=I_-^{(m+n-1)} \qquad (B17)$$

and change the index from m to $p=m+n+1$, Eq. (B15) can be rewritten as

$$\mathbf{I}_{\omega q}(\mathbf{k})=\sum_{p=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left\{\mathbf{i}_x\frac{1}{2}[J_{p-n+1}(k_x r_h)+J_{p-n-1}(k_x r_h)]+\mathbf{i}_y\frac{1}{2i}[J_{p-n+1}(k_x r_h)-J_{p-n-1}(k_x r_h)]\right\}(-1)^{p-1}(-i)^n J_n(k_y r_h)I_0^{(p)}\ , \tag{B18}$$

where

$$I_0^{(p)}=\frac{e^{[i(-pk_w-\hat{k}_z+\omega/v_r)+q]z}-1}{i(-pk_w-\hat{k}_z+\omega/v_r)+q} \tag{B19}$$

can be approximated in the high-gain regime by

$$I_0^{(p)}\approx\frac{e^{[i(-pk_w-\hat{k}_z+\omega/v_r)+q]z}}{i(-pk_w-\hat{k}_z+\omega/v_r)+q}\ . \tag{B20}$$

The double summation in Eq. (B18) can be reduced to a single summation by using Graf's additive theorem [6]:

$$J_\nu[(z^2+\xi^2-2z\xi\cos\theta)^{1/2}]=\left[\frac{z-\xi e^{i\theta}}{z-\xi e^{-i\theta}}\right]^{\nu/2}\sum_{n=-\infty}^{\infty}J_{\nu+n}(z)J_n(\xi)e^{in\theta}\ . \tag{B21}$$

The result is

$$\mathbf{I}_{\omega q}(\mathbf{k})=\sum_{p=-\infty}^{\infty}(-1)^{p-1}e^{-ip\theta_k}\left\{\mathbf{i}_x\frac{1}{2}[e^{-i\theta_k}J_{p+1}(k_\perp r_h)+e^{i\theta_k}J_{p-1}(k_\perp r_h)]+\mathbf{i}_y\frac{1}{2i}[e^{-i\theta_k}J_{p+1}(k_\perp r_h)-e^{i\theta_k}J_{p-1}(k_\perp r_h)]\right\}I_0^{(p)}\ , \tag{B22}$$

where $k_\perp=(k_x^2+k_y^2)^{1/2}$ and $\theta_k=\tan^{-1}(k_y/k_x)$.

Inserting the above equation (B22) into Eq. (B11) and replacing $\hat{k}_z$ by $(k^2-k_\perp^2)^{1/2}$, we have

$$\mathbf{H}_{\omega q}(\mathbf{k}_\perp,z)=-i\frac{\pi}{(2\pi)^4}\frac{K}{\gamma_r}\frac{1}{(k^2-k_\perp^2)^{1/2}}\sum_{p=-\infty}^{\infty}\frac{e^{-ipk_w z}\mathbf{V}_p(\mathbf{k}_\perp)}{q+i\left[-pk_w-(k^2-k_\perp^2)^{1/2}+\dfrac{w}{v_r}\right]}\ , \tag{B23}$$

where

$$\mathbf{V}_p(\mathbf{k}_\perp)=(-1)^{p-1}e^{-ip\theta_k}\left\{\mathbf{i}_x\frac{1}{2}[e^{-i\theta_k}J_{p+1}(r_h k_\perp)+e^{i\theta_k}J_{p-1}(r_h k_\perp)]+\mathbf{i}_y\frac{1}{2i}[e^{-i\theta_k}J_{p+1}(r_h k_\perp)-e^{i\theta_k}J_{p-1}(r_h k_\perp)]\right\}\ . \tag{B24}$$

Next, let us calculate the energy change by the radiation field. Substituting Eqs. (3) and (B9) into Eq. (8), we have

$$\frac{d\gamma}{dz}=\frac{e}{mc^2}\frac{K}{\gamma_r}\frac{c}{v_r}(\mathbf{i}_x\cos k_w z+\mathbf{i}_y\sin k_w z)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_h}\times\left\{-ie\mu_0\int_{-\infty}^{\infty}\left[\frac{1}{2\pi i}\int_{q_0-i\infty}^{q_0+i\infty}\left[\int_{-\infty}^{\infty}\omega\mathbf{H}_{\omega q}(\mathbf{k}_\perp,z)\rho_{\omega q}(\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\beta}d^2\mathbf{k}_\perp\right]e^{qz}dq\right]e^{-i\omega\tau}d\omega\right\}\ . \tag{B25}$$

The term

$$\mathbf{L}=(\mathbf{i}_x\cos k_w z+\mathbf{i}_y\sin k_w z)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_h} \tag{B26}$$

can be expanded in terms of the Bessel functions, and the resulting double series can be reduced to a single series as $\mathbf{I}_{\omega q}(\mathbf{k})$. We find that

$$\mathbf{L}=\sum_{l=-\infty}^{\infty}(-1)^{-(l-1)}e^{il\theta_k}\left\{\mathbf{i}_x\frac{1}{2}[e^{i\theta_k}J_{-(l+1)}(k_\perp r_h)+e^{-i\theta_k}J_{-(l-1)}(k_\perp r_h)]-\mathbf{i}_y\frac{1}{2i}[e^{i\theta_k}J_{-(l+1)}(k_\perp r_h)-e^{-i\theta_k}J_{-(l-1)}(k_\perp r_h)]\right\}e^{ilk_w z}\ . \tag{B27}$$

If we insert Eq. (B27) into Eq. (B25) and retain only the slowing varying term $p = l$, we obtain

$$\frac{d\gamma}{dz}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi i}\int_{q_0-i\infty}^{q_0+i\infty}\left[\int_{-\infty}^{\infty}P_{\omega q}(k_\perp)\rho_{\omega q}(\mathbf{k}_\perp)e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\beta}d^2\mathbf{k}_\perp\right]e^{qz}dq\right]e^{-i\omega\tau}d\omega\ , \tag{B28}$$

where

$$P_{\omega q}(k_\perp)=\sum_{p=-\infty}^{\infty}\frac{r_e}{2\pi c}\left[\frac{K}{\gamma_r}\right]^2\frac{(-1)^p\left[J_p^2(k_\perp r_h)\left[\frac{p}{k_\perp r_h}\right]^2+J_p'^2(k_\perp r_h)\right]}{[1-(k_\perp/k)^2]^{1/2}\left[q+i\left[-pk_w-(k^2-k_\perp^2)^{1/2}+\frac{w}{v_r}\right]\right]}\ , \tag{B29}$$

where $r_e = e^2/(4\pi\epsilon_0 mc^2)$ is the classical electron radius, ϵ_0 is the permittivity of free space, and $J_p'(x)$ is the derivative of the Bessel function. Here, we have used the recurrence formulas [6]

$$\begin{aligned}J_{p-1}(x)+J_{p+1}(x)&=2J_p(x)\frac{p}{x}\ ,\\ J_{p-1}(x)-J_{p+1}(x)&=2J_p'(x)\ .\end{aligned} \tag{B30}$$

APPENDIX C: ORTHOGONAL FUNCTIONS

1. Hollow beam

$$f_{0\perp}(r^2)=\frac{1}{(\pi R_0^2 k_\beta)^2}\delta\left[1-\left[\frac{r}{R_0}\right]^2\right]\ .$$

We choose the weight function to be

$$W_\perp(r^2)=\delta\left[1-\left[\frac{r}{R_0}\right]^2\right]\ , \tag{C1}$$

where $r = (r_x^2 + r_y^2)^{1/2}$ is the amplitude of the electron position in four-dimensional transverse phase space. The normalization constant then becomes

$$C=(\pi R_0^2 k_\beta)^2\ . \tag{C2}$$

Any perturbation on the hollow beam will have to take place around the ring $r = R_0$, where electrons populate. As a result, all $R_{\omega q}^{(m,n)}(r_x,r_y)$ degenerate into δ functions, i.e.,

$$R_{\omega q}^{(m,n)}(r_x,r_y)\propto\delta\left[1-\left[\frac{r}{R_0}\right]^2\right]r_x^{|m|}r_y^{|n|}\ . \tag{C3}$$

Thus, $f_k^{(|m|,|n|)}(r_x,r_y)$ is a nonzero constant for $k = 0$, and vanishes otherwise. By introducing the polar coordinate as

$$r_x = r\cos\phi,\quad r_y = r\sin\phi\ , \tag{C4}$$

the orthogonality relationship (47) for $k = l = 0$ is written as

$$\int_0^\infty\delta\left[1-\left[\frac{r}{R_0}\right]^2\right](f_0^{(|m|,|n|)})^2 r^{2|m|+2|n|+3}dr\int_0^{\pi/2}\cos^{2|m|+1}\phi\,\sin^{2|n|+1}\phi\,d\phi=1\ . \tag{C5}$$

The constant $f_0^{(|m|,|n|)}$ is found to be

$$f_0^{(|m|,|n|)}=\frac{1}{R_0^{|m|+|n|+2}}\left[\frac{2}{\alpha_{|m|,|n|}}\right]^{1/2}\ , \tag{C6}$$

where

$$\begin{aligned}\alpha_{|m|,|n|}&=\int_0^{\pi/2}\cos^{2|m|+1}\phi\,\sin^{2|n|+1}\phi\,d\phi\\ &=\frac{(2|m|)!!(2|n|)!!}{(2|m|+2|n|+2)!!}\ .\end{aligned} \tag{C7}$$

Equation (49) then becomes

$$\begin{aligned}C_{|m|,|n|,0}(k_x,k_y)&=\frac{1}{R_0^{|m|+|n|+2}}\left[\frac{2}{\alpha_{|m|,|n|}}\right]^{1/2}\int_0^{\pi/2}\int_0^\infty J_{|m|}(k_x r\cos\phi)J_{|n|}(k_y r\sin\phi)\\ &\quad\times\delta\left[1-\left[\frac{r}{R_0}\right]^2\right]r^{|m|+|n|+3}dr\cos^{|m|+1}\phi\,\sin^{|n|+1}\phi\,d\phi\\ &=\frac{R_0^2}{\sqrt{2\alpha_{|m|,|n|}}}\frac{J_{|m|+|n|+1}(k_\perp R_0)}{(k_\perp R_0)^{|m|+|n|+1}}(k_x R_0)^{|m|}(k_y R_0)^{|n|}\ ,\end{aligned} \tag{C8}$$

where we have used the integration formula [6]

$$\int_0^{\pi/2} J_\nu(z_1\cos x)J_\mu(z_2\sin x)\cos^{\nu+1}x\,\sin^{\mu+1}x\,dx = \frac{z_1^\nu z_2^\mu}{[(z_1^2+z_2^2)^{1/2}]^{\nu+\mu+1}} J_{\nu+\mu+1}(z_1^2+z_2^2)^{1/2} . \quad \text{(C9)}$$

2. Gaussian beam

$$f_{0\perp}(r^2)=\frac{1}{(2\pi\sigma_x^2 k_\beta)^2} e^{-\frac{1}{2}(r/\sigma_x)^2} .$$

In this case, $f_{0\perp}(r^2)$ can be factorized as

$$f_{0\perp}(r^2)=f_{0x}(r_x)f_{0y}(r_y) . \quad \text{(C10)}$$

Accordingly, the weight function $W_\perp(r^2)$ also can be factorized as

$$W_\perp(r^2)=W_x(r_x)W_y(r_y) , \quad \text{(C11)}$$

where each weight function is defined by

$$W_x(r_x)=e^{-\frac{1}{2}(r_x/\sigma_x)^2}, \quad W_y(r_y)=e^{-\frac{1}{2}(r_y/\sigma_x)^2} , \quad \text{(C12)}$$

respectively, and we have chosen the normalization constant to be

$$C=(2\pi\sigma_x^2 k_\beta)^2 . \quad \text{(C13)}$$

It follows from the orthogonality relationship (47) that the orthogonal functions also can be factorized as

$$f_k^{(|m|,|n|)}(r_x,r_y)=g_j^{(|m|)}(r_x)g_i^{(|n|)}(r_y) , \quad \text{(C14)}$$

where the orthogonal functions $g_j^{(|m|)}(r_x)$ satisfy the orthogonality relationship

$$\int_0^\infty e^{-\frac{1}{2}(r_x/\sigma_x)^2} g_j^{(|m|)}(r_x)g_l^{(|m|)}(r_x)r_x^{2|m|+1}dr_x=\delta_{jl} , \quad \text{(C15)}$$

and $g_i^{|n|}(r_y)$ satisfy a similar equation where r_x, $|m|$, and j are replaced by r_y, $|n|$, and i, respectively. The orthogonal functions are given by

$$g_j^{(|m|)}(r_x)=D_j^{(|m|)}L_j^{(|m|)}\left[\frac{r_x^2}{2\sigma_x^2}\right] , \quad \text{(C16)}$$

where $L_j^{(|m|)}(x)$ is the generalized Laguerre polynomials [6] and

$$D_j^{(|m|)}=\frac{1}{\sigma_x^{|m|+1}}\left[\frac{j!}{2^{|m|}(|m|+j)!}\right]^{1/2} . \quad \text{(C17)}$$

The functions $f_k^{(|m|,|n|)}(r_x,r_y)$ can be expressed as

$$f_k^{(|m|,|n|)}(r_x,r_y)=D_j^{(|m|)}D_i^{(|n|)}L_j^{(|m|)}\left[\frac{r_x^2}{2\sigma_x^2}\right]L_i^{(|n|)}\left[\frac{r_y^2}{2\sigma_x^2}\right] , \quad \text{(C18)}$$

where k and (j,i) are related by

$$k=\frac{(j+i+1)(j+i)}{2}+i, \quad i,j=0,1,2,\ldots . \quad \text{(C19)}$$

We then have

$$C_{|m|,|n|,k}(k_x,k_y)=\int_0^\infty J_{|m|}(k_x r_x)e^{-\frac{1}{2}(r_x/\sigma_x)^2}D_j^{(|m|)}L_j^{(|m|)}\left[\frac{r_x^2}{2\sigma_x^2}\right]r_x^{|m|+1}dr_x \times\int_0^\infty J_{|n|}(k_y r_y)e^{-\frac{1}{2}(r_y/\sigma_x)^2}D_i^{(|n|)}L_i^{(|n|)}\left[\frac{r_y^2}{2\sigma_x^2}\right]r_y^{|n|+1}dr_y$$
$$=\frac{\sigma_x}{\sqrt{j!(|m|+j)!}}e^{-\frac{1}{2}k_x^2\sigma_x^2}\left[\frac{k_x\sigma_x}{\sqrt{2}}\right]^{|m|+2j}\frac{\sigma_x}{\sqrt{i!(|n|+i)!}}e^{-\frac{1}{2}k_y^2\sigma_x^2}\left[\frac{k_y\sigma_x}{\sqrt{2}}\right]^{|n|+2i} . \quad \text{(C20)}$$

APPENDIX D: ASYMMETRIC FOCUSING IN A PLANAR WIGGLER WITH PARABOLIC POLE FACE

For a small transverse displacement from the wiggler axis, the vector potential of a planar wiggler with a parabolic pole face, $\mathbf{A}_w$, can be approximated by

$$\mathbf{A}_w=A_w\left[\mathbf{i}_x\frac{k_{wy}}{k_{wz}}(1+\tfrac{1}{2}k_{wx}^2x^2+\tfrac{1}{2}k_{wy}^2y^2)\sin k_w z -\mathbf{i}_y\frac{k_{wx}}{k_{wz}}k_{wx}k_{wy}xy\,\sin k_w z\right] , \quad \text{(D1)}$$

where

$$k_{wx}^2+k_{wy}^2=k_{wz}^2 . \quad \text{(D2)}$$

The transverse trajectory of the electron consists of the betatron motion and the wiggler motion. The betatron oscillations are governed by the equations of motion for a simple harmonic oscillator with the betatron wave number $k_{\beta x}$ and $k_{\beta y}$ in the x and y planes, respectively [in the absence of external focusing, $k_{\beta x}=Kk_{wx}/(\gamma\sqrt{2})$ and $k_{\beta y}=Kk_{wy}/(\gamma\sqrt{2})$], and the wiggler motion $\mathbf{x}_w$ is expressed by

$$x_w=r_w\cos k_{wz}z ,$$
$$y_w\sim 0 . \quad \text{(D3)}$$

Here, $r_w = K/(\gamma k_{wz})$ is the radius of the wiggler motion and

$$K = eA_w k_{wy}/(mck_{wz}) = eB/(mc^2 k_{wz})$$

is the peak value of the wiggler parameter where B is the peak magnetic field on axis.

In contrast with the helical wiggler case, the longitudinal velocity of the electron has the longitudinal modulation with the wave number $2k_{wz}$:

$$v_\parallel = \bar{v}_\parallel + \frac{1}{4c}\left[\frac{K}{\gamma}\right]^2 \cos 2k_{wz} z \ , \quad \text{(D4)}$$

where the overbar denotes an average over one wiggler period. As the result, the arrival time t of an electron at position z is also modulating:

$$t = \int_0^z \frac{dz}{\bar{v}_\parallel} - \frac{1}{8k_{wz}c}\left[\frac{K}{\gamma}\right]^2 \sin 2k_{wz} z \ . \quad \text{(D5)}$$

From now on, we simply denote k_{wz} as k_w.

If we insert the above equations (D3)–(D5) into Eqs. (8) and (24) and follow the procedure shown in Appendix B, we obtain an expression for the energy change $d\gamma/dz$ similar to Eq. (29), where $P_{\omega q}(k_\perp)$ should be replaced by $P_{\omega q}(k_x,k_y)$ given by Eq. (89).

We again assume that the focusing in the wiggler is matched to the electron beam so that f_0 is a function of $x_\beta^2 + (p_{\beta x}/k_{\beta x})^2$, $y_\beta^2 + (p_{\beta y}/k_{\beta y})^2$, and γ only, and we also assume for simplicity that f_0 can be factorized as:

$$f_0 = f_{0\perp}(x_\beta^2 + (p_{\beta x}/k_{\beta x})^2, y_\beta^2 + (p_{\beta y}/k_{\beta y})^2) f_{0\parallel}(\gamma) \ . \quad \text{(D6)}$$

Now, the (Fourier-Laplace-transformed) linearized Vlasov equation is given by [cf. Eq. (32)]

$$\left[q - i\omega\frac{d\tau}{dz}\right] f_{\omega q} + p_{\beta x}\frac{\partial f_{\omega q}}{\partial x_\beta} - k_{\beta x}^2 x_\beta \frac{\partial f_{\omega q}}{\partial p_{\beta x}} + p_{\beta y}\frac{\partial f_{\omega q}}{\partial y_\beta} - k_{\beta y}^2 y_\beta \frac{\partial f_{\omega q}}{\partial p_{\beta y}} = -f_{0\perp}\frac{df_{0\parallel}}{d\gamma}\int P_{\omega q}(k_x,k_y)\rho_{\omega q}(\mathbf{k}_\perp) e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\beta} d^2\mathbf{k}_\perp \ . \quad \text{(D7)}$$

Let us introduce the transverse polar coordinates as

$$x_\beta = r_x\cos\phi_x, \quad y_\beta = r_y\cos\phi_y \ , \quad \text{(D8)}$$

$$\frac{p_{\beta x}}{k_{\beta x}} = r_x\sin\phi_x, \quad \frac{p_{\beta y}}{k_{\beta y}} = r_y\sin\phi_y \ . \quad \text{(D9)}$$

Then, the second and third terms in the LHS of the linearized Vlasov equation, Eq. (D7), are written as

$$p_{\beta x}\frac{\partial f_{\omega q}}{\partial x_\beta} - k_{\beta x}^2 x_\beta\frac{\partial f_{\omega q}}{\partial p_{\beta x}} + p_{\beta y}\frac{\partial f_{\omega q}}{\partial y_\beta} - k_{\beta y}^2 y_\beta\frac{\partial f_{\omega q}}{\partial p_{\beta y}} = -k_{\beta x}\frac{\partial f_{\omega q}}{\partial\phi_x} - k_{\beta y}\frac{\partial f_{\omega q}}{\partial\phi_y} \ . \quad \text{(D10)}$$

The matching condition, Eq. (D6), can be written in terms of r_x and r_y as

$$f_0 = f_{0\perp}(r_x,r_y) f_{0\parallel}(\gamma) \ . \quad \text{(D11)}$$

The rest of the procedure closely follows the formulation described in Secs. IV and V. One important difference is that the unperturbed transverse distribution (and also the weight function) is a function of both r_x and r_y, not $r = (r_x^2 + r_y^2)^{1/2}$ only. Therefore, for example, the orthogonality relationship, Eq. (47), should be modified as

$$\int_0^\infty\int_0^\infty W_\perp(r_x,r_y) f_k^{(|m|,|n|)}(r_x,r_y) f_l^{(|m|,|n|)}(r_x,r_y) r_x^{2|m|+1} r_g^{2|n|+1} dr_x dr_y = \delta_{kl} \ . \quad \text{(D12)}$$

Finally, we arrive at the dispersion relation

$$\det(\mathbf{I} + \boldsymbol{\beta}\mathbf{M}) = 0 \ , \quad \text{(D13)}$$

where the matrix elements of $\beta_{k,l}^{m,n}$ and $M_{m',n',j}^{m,n,l}$ are given by

$$\beta_{k,l}^{m,n} = \int_1^\infty\int_0^\infty\int_0^\infty \frac{W_\perp(r_x,r_y) f_k^{(|m|,|n|)}(r_x,r_y) f_l^{(|m|,|n|)}(r_x,r_y) r_x^{2|m|+1} r_y^{2|n|+1}}{q - i\omega\frac{d\tau}{dz}(r_x,r_y,\gamma) - i(k_{\beta x}m + k_{\beta y}n)} \frac{df_{0\parallel}}{d\gamma} dr_x dr_y d\gamma \quad \text{(D14)}$$

and

$$M_{m',n',j}^{m,n,l} = i^{|m|+|n|-(|m'|+|n'|)} \frac{(2\pi)^2 k_{\beta x} k_{\beta y}}{C} \int_{-\infty}^\infty\int_{-\infty}^\infty P_{\omega q}(k_x,k_y) C_{|m|,|n|,l}(k_x,k_y) C_{|m'|,|n'|,j}(k_x,k_y) dk_x dk_y \ , \quad \text{(D15)}$$

respectively.

If we retain only the lowest-order term $m = n = k = 0$ in the azimuthal and radial expansions, as we did in Sec. VI, the dispersion relation (D13) can be written in a general form as

$$1=i\frac{k}{k_1}\frac{r_e}{c}\left[\frac{K}{\gamma_r}\right]^2\frac{k_w}{\gamma_r}\frac{[JJ]^2}{4\pi}\int_0^\infty\int_0^\infty\int_1^\infty\frac{f_{0\parallel}(\gamma)d\gamma f_{0\perp}(r_x,r_y)(2\pi)^2k_{\beta x}k_{\beta y}r_xdr_xr_ydr_y}{\left[q+2i\frac{k}{k_1}k_w\frac{\gamma-\gamma_r}{\gamma_r}-i\frac{1}{2}k(k_{\beta x}^2r_x^2+k_{\beta y}^2r_y^2)\right]^2}$$
$$\times\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{dk_xdk_y}{q+ik_w\frac{k-k_1}{k_1}+i\frac{k_x^2+k_y^2}{2k}}$$
$$\times\left[\int_0^\infty\int_0^\infty f_{0\perp}(r_x,r_y)J_0(k_xr_x)J_0(k_yr_y)(2\pi)^2\times k_{\beta x}k_{\beta y}r_xdr_xr_ydr_y\right]^2, \quad \text{(D16)}$$

where $f_{0\perp}(r_x,r_y)$ is normalized such that

$$\int_0^\infty\int_0^\infty f_{0\perp}(r_x,r_y)(2\pi)^2k_{\beta x}k_{\beta y}r_xdr_xr_ydr_y=1 \ . \quad \text{(D17)}$$

Here, we have used the approximated expression of $P_{\omega q}(k_x,k_y)$, Eq. (92), and have retained only the fundamental harmonic term of the forward radiation $p=1$. For a Gaussian beam,

$$f_{0\perp}(r_x,r_y)=\frac{1}{(2\pi)^2k_{\beta x}\sigma_x^2k_{\beta y}\sigma_y^2}e^{-r_x^2/2\sigma_x^2-r_y^2/2\sigma_y^2}\ , \quad \text{(D18)}$$

$$f_{0\parallel}(\gamma)=\frac{N}{\hat{\tau}}\frac{1}{\sqrt{2\pi}\sigma_\gamma\gamma_r}e^{-(\gamma-\gamma_r)^2/2\gamma_r^2\sigma_\gamma^2}\ , \quad \text{(D19)}$$

the above dispersion relation can be written in a scaled form as

$$1=i\frac{\left[\frac{k_{\beta x}}{k_wD}\right]^{1/2}\left[\frac{k_{\beta y}}{k_wD}\right]^{1/2}}{4\pi\sqrt{2\pi}\sqrt{2k_1\epsilon_x}\sqrt{2k_1\epsilon_y}}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\frac{e^{-(s^2+u^2+v^2)/2}dsu\,duv\,dv}{\left[\frac{q}{k_wD}+2i\frac{\sigma_\gamma}{D}s-\frac{i}{4}\left[2k_1\epsilon_x\frac{k_{\beta x}}{k_wD}u^2+2k_1\epsilon_y\frac{k_{\beta y}}{k_wD}v^2\right]\right]^2}$$
$$\times\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{e^{-\alpha^2-\beta^2}d\alpha\,d\beta}{\frac{q}{k_wD}+i\frac{k-k_1}{k_1D}+i\left[\frac{\alpha^2}{2k_1\epsilon_x}\frac{k_{\beta x}}{k_wD}+\frac{\beta^2}{2k_1\epsilon_y}\frac{k_{\beta y}}{k_wD}\right]}\ , \quad \text{(D20)}$$

where we have replaced k by k_1 except in the detuning term $(k-k_1)/(k_1D)$ to a good approximation. The scaled growth rate $\mathrm{Re}(q)/(k_wD)$ is a function of the six scaling parameters:

$$\frac{\mathrm{Re}(q)}{k_wD}=F\left[2k_1\epsilon_x,2k_1\epsilon_y,\frac{\sigma_\gamma}{D},\frac{k_{\beta x}}{k_wD},\frac{k_{\beta y}}{k_wD},\frac{k-k_1}{k_1D}\right]\ . \quad \text{(D21)}$$

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