Acceleration of particles by an asymmetric Hermite-Gaussian laser beam

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The application of a focused laser beam to accelerate particles using the longitudinal electric-field component is investigated one step beyond the paraxial-ray approximation. Vacuum acceleration to high energies along the axis of an asymmetric Hermite-Gaussian beam is in principle possible, but the interaction distance is short (one Rayleigh length on each side of focus). The use of a gas can increase the energy gain per focal passage by a factor of 2.4, while permitting arbitrary spacing of drift tubes and lenses of a lens waveguide. Drift tubes are therefore not needed. A beam loaded with a graded-index gas, in which phase and particle velocities are equal over the interaction trajectory, permits three-dimensionally stable particle trajectories. This property is explained by the anisotropy of the medium in a comoving reference frame. The functions of acceleration, focusing, and bunching of particles can thus be performed simultaneously by a single optical beam that is guided in a lens waveguide.

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I. INTRODUCTION

In a conventional radio-frequency linear particle accelerator (linac), charged particles are accelerated by the longitudinal electric field in a TM mode of a traveling waveguide. This concept was proposed in 1924 by Ising (See Ref. [1]). Since the invention of the laser, a number of methods for using it to accelerate particles have been proposed. The first to suggest the application of a laser in this way was probably Shimoda [2]. (The inverse effect, which is the production of radiation by stimulated emission in a particle beam, involved the use of periodic structures, as in the Smith-Purcell [3] and free-electron lasers [4]). More recently, a number of accelerator proposals have been suggested, such as that based on the inverse Smith-Purcell effect [5], inverse Cerenkov radiation [6], inverse free-electron laser [7], laser-plasma interaction [8], and the use of focused laser beams [9]. All of these have problems that make experimental realization difficult.

The laser accelerator scheme discussed here is based on the idea of a focused laser beam described in Refs. [10,11]. It differs from the method proposed in Ref. [9], in which the particle traverses the Gaussian laser beam at an oblique angle, in that the acceleration takes place only by the action of the *longitudinal* electric field, the motion being along the beam axis. This scheme has several advantages based on the linearity of the particle trajectories: (a) the energy loss to radiation is negligible, (b) the interaction length can be extended so that lower field strength is needed for achieving a given gain, and (c) gas "loading" [12] of the linac for phase-matching purposes is a practical possibility. The existence of the longitudinal field in a focused beam can be understood by expansion of the beam mode in traveling plane waves, of which all, except for the plane wave propagating parallel to the axis, have a nonvanishing longitudinal field component. An estimate of the magnitude of this longitudinal component of the beam is obtained by integrating Gauss's law along the propagation axis z, yielding

$$E_z = -\int \nabla_t \cdot \mathbf{E}_t dz \ , \tag{1a}$$

$$\approx \frac{i}{k} \nabla_t \cdot \mathbf{E}_t \tag{1b}$$

$$\approx \frac{i}{kw} E_t \ , \tag{1c}$$

where Eq. (1b) applies in the paraxial-ray approximation and Eq. (1c) gives an order of magnitude estimate in terms of the beam width w. The form of Eq. (1c) indicates, by the presence of w as a divisor, that acceleration is greatest at the beam focus. We shall therefore introduce the "energy gain per focal passage," or GFP, which in the case of vacuum propagation will be denoted VGFP. To avoid ambiguity, it will be understood that these terms are meant to be the maximum available energy gain of a particle that is accelerated through the central acceleration region centered on focus. In either case, this quantity is proportional to the square root of the laser power, the other parameters assumed fixed.

Equation (1c) also shows that the transverse- and longitudinal-field components are in phase for a beam that is symmetric in the field as the fundamental Gaussian mode, and they are 90° out of phase for an asymmetric beam, as the Hermite-Gaussian mode of order (1,0). Hence, it is generally not possible to both have net acceleration and focusing of particle beams in the case of the latter. The exception is when, through graded loading, the phase is held constant (Sec. III).

The phase velocity of a plane wave propagating at an angle θ with the z axis equals $c/(n \cos\theta)$, where n is the refractive index of the medium. Hence, a particle moving with velocity v along the z axis will slip out of phase with the wave unless v equals the phase velocity of the wave.

This requires that n satisfies the *synchronization,* or *Cherenkov* condition

$$n=\frac{c}{v\cos\theta} . \tag{2}$$

For this reason the acceleration mechanism is sometimes called the inverse Cherenkov effect [13]. In a laser beam the phase velocity varies in the direction of propagation, as well as transversely. The use of a spatially varying index would therefore be advantageous, although it is not known at present if that is achievable.

The energy gradient of conventional rf linear accelerators is typically about 1 MeV/m. For comparison, a VGFP of 31 MeV is predicted feasible [14] for a radially polarized 1-TW Gaussian beam, and well over 1 GeV in a gas-loaded "diffraction-free" beam [15] accelerator. In order to optimize the gain, the accelerator must meet a number of requirements. The most important of these are (1) the injection of particles (most likely electrons) must be synchronized to the phase of the field at its optimum value by prebunching and control of the injection time, (2) the phase velocity of the field, and possibly the pulse velocity as well, should approximate the particle velocity, which, as discussed above, is achieved by "loading" the accelerator with a dispersive gas or vapor, (3) losses of the beam should be sufficiently low and the beam quality be maintained so as to permit its reuse as an accelerator, e.g., in a lens waveguide or ring-cavity configuration, and (4) the interaction distance must be long enough that drift tubes or other hardware used to inject or remove electrons into or from the field are not located in regions of high field intensity.

The achievement of the above requirements depends largely on the degree to which certain technical obstacles can be overcome. Gas loading of the linac will resolve some issues, but also introduce other problems. For example, efficient energy transfer is possible in a vacuum linac using a tightly focused beam, but the interaction distances are typically short, requiring short gap distances between drift tubes. The gas-loaded linac, on the other hand, can yield significant gain increase and much longer interaction lengths, although requiring an injection energy that is approximately twice as great as that in a vacuum linac operated at the same wavelength. The loaded accelerator also permits matching of phase and particle velocities well below the speed of light, yielding arbitrarily low kinetic injection energy. However, to be able to perform this last maneuver, one must be able to grade the refractive index of the medium in the direction of the axis, at least in the case of the Gaussian beam. A more serious problem is that the presence of the gas imposes upper limits on the field strength, as determined by ionization, and the gas density is absorption—and collision—limited. Finally, there are problems due to the heating of the gas, and nonlinear properties such as self-focusing. These problems are minimized by decreasing the focusing strength and increasing the injection energy.

In this paper we discuss the properties of the linearly polarized Hermite-Gaussian beam of order (1,0), which has an on-axis maximum of the longitudinal electric field and vanishing transverse field there. Expressions for the GFP are derived for the vacuum linac and the two types of gas-loaded devices. In the gas-loaded accelerator we calculate the gain for interaction in the central acceleration region and for the entire beam length. Expressions for the best constant gas index and the functional form of the best graded index are derived. The principal parameters to be determined are the energy gain and the interaction length, both of which should be as large as possible, and the minimum injection energy, which is preferably low. The longitudinal and transverse stabilities of the particle trajectories are also discussed in each case. In a succeeding paper, these calculations are carried out for the "diffraction-free" beam pioneered by Durnin, Micelli, and Eberly [15].

II. ASYMMETRIC VACUUM HERMITE-GAUSSIAN BEAM LINAC

A. Energy gain in single focal passage (VGFP)

The Gaussian beam has the advantage over other beam geometries in that it usually constitutes the fundamental laser mode, and therefore is generally available. It has other useful properties, such as it can be brought to a tight focus and it remains Gaussian when transmitted by lenses or reflected by spherical mirrors. The latter property makes it useful for application in lens waveguides, or, generally, whenever beam recovery is required. The lowest-order Hermite-Gaussian mode, i.e., the fundamental Gaussian beam, is not suitable for particle acceleration, as it does not have a maximum on the axis in the longitudinal field. On the other hand, the plane polarized, Hermite-Gaussian TM 1,0 mode has its maximum value of the longitudinal field on the axis. The expression for this mode is [16]

$$E_x(x,y,z,t)=-\frac{4w_0E_0}{\sqrt{2}\pi w(z)^2}x\exp\left\{i\left[kz-\omega t+\frac{k}{2}\mu(z)(x^2+y^2)-2\psi(z)\right]\right\} . \tag{3}$$

In this paper the convention is adhered to that the physical field is given by the real part of the complex function. The normalization constants in Eq. (3) were chosen so that the expression for the laser power has a simple form, and the negative sign was chosen in order to simplify the force equations (where the constant of charge e is defined positive). The constant $k=2\pi/\lambda=\omega/c$, in Eq. (3), is the wave vector of the light defined in terms of the wavelength λ, or the angular frequency ω. The complex function $\mu(z)$ is the beam parameter defined by

$$\mu(z)=\frac{1}{z-iz_q} , \tag{4a}$$

where z_q is the Rayleigh length of the Gaussian beam.

The beam width $w(z)$ is given by

$$w(z)^2 = w_0^2\left[1+\frac{z^2}{z_q^2}\right], \tag{4b}$$

and the diffractive phase $\psi(z)$ is

$$\tan\psi = \frac{z}{z_q} . \tag{4c}$$

The beam width $w(z)$ is the radius at which E_x is a maximum. The radius of the beam at the focus, or beam waist w_0, is related to z_q by $z_q = \frac{1}{2}kw_0^2$. The on-axis longitudinal electric-field component is found by Eq. (1) to be

$$E_z(0,0,z,t) = -\frac{4iE_0}{\sqrt{2\pi}kw_0}\frac{z_q^2}{z^2+z_q^2}e^{i\phi(z,t)} , \tag{5}$$

where $\phi(z)=\phi'-2\psi(z)$ is the on-axis phase, and ϕ' is the phase of a plane wave propagating in the z direction: $\phi'=kz-\omega t+\phi_0'$, where a constant ϕ_0' has been added. The on-axis phase velocity of the wave is obtained by setting $d\phi=0$, yielding

$$v_0(z) = \frac{\omega}{k-2\psi'} = \omega \Big/ \left[k - \frac{2z_q}{z^2+z_q^2}\right] . \tag{6}$$

Note that $v_0(z) > c$, and at the focus, where v_0 is largest, $v_0(0) \approx c(1+2/kz_q)$, to first order in $1/kz_q$. The z dependence of v_0 is a consequence of the diffraction of the Hermite-Gaussian beam. It is an undesirable property for particle acceleration.

The injection point and range, or path length over which continuous acceleration or positive energy transfer to the particle occurs, are easily determined for ultrarelativistic particles [17]. Then $z \approx ct$, so that $\phi' \approx \phi_0'$ yields the largest acceleration at the beam focus when $\phi_0' = \frac{1}{2}\pi$. The phase $\psi(z)$ varies from $-\pi/4$ at $z=-z_q$ to $+\pi/4$ at $z=+z_q$. The optimum point for injecting a relativistic particle is thus at a Rayleigh length before the focus with $\phi=\pi$. At $z=+z_q$ the z component of the force changes sign, so that the interval $(-z_q, z_q)$ constitutes the interaction (positive acceleration) range of the particle. Note that the phase drifts in the negative direction for any physically allowed particle velocity, which is consistent with the property that the phase velocity v_0 is greater than the speed of light c.

The VGFP is found by taking $\phi_0' = \frac{1}{2}\pi$, for maximum acceleration, and integrating, $E_z(0,0,z)$, over the interaction interval $(-z_i, +z_i)$, located symmetrically about the focus. Using Eq. (4) for the field, this gives

$$\begin{aligned} W_g &= e\int_{-z_i}^{z_i} E_z(0,0,z)dz \\ &= \frac{4eE_0}{\sqrt{2\pi}kw_0}\int_{-z_i}^{z_i}\frac{z_q^2}{z^2+z_q^2}\cos[2\psi(z)]dz \\ &= \frac{2eE_{0z}z_q^2 z_i}{z_i^2+z_q^2} . \end{aligned} \tag{7}$$

In the limit that $z_i \to \infty$ the gain vanishes. This is a special case of the Lawson-Woodward theorem [18], proved in Appendix A. The gain attains a maximum value when $z_i = z_q$, in agreement with the conclusion of the previous paragraph, given by

$$W_{0g} = 2e\left[\frac{P}{\pi\epsilon_0 c}\right]^{1/2} = eE_{0z}z_q , \tag{8}$$

where $P = \frac{1}{2}\epsilon_0 c w_0^2 E_0^2$ is the total beam power. Taking $P = 1$ TW, as a numerical example, we have $W_{0g} = 22$ MeV.

The VGFP will fall short of that predicted by Eq. (8) unless the injection energy is sufficient to assure that phase slippage is small. We obtain a rough estimate of this energy in the following way. Let the velocity of the particle be v, and assume that its magnitude changes little over the interaction length L. The change in the phase ϕ' is then given approximately by

$$\Delta\phi' = kL - \omega L/v . \tag{9}$$

Assuming $\Delta\phi' \approx 0.5$ rad, and using the relation between v and the Lorentz factor γ, valid when $\beta = v/c \to 1$

$$\gamma^{-2} \approx 2(1-\beta) , \tag{10}$$

we obtain from (9) and (10) for the minimum value of γ the expression

$$\gamma_{\min} \approx \sqrt{kL} . \tag{11}$$

Equation (11) differs from the minimum Lorentz factor derived in Ref. [11]. The reason is that in [11] the beam is implicitly assumed nondiffracting and propagates with the phase velocity $v_0 = c$, in which case the particle can be captured by the wave. This means, in this case, that the phase evolves asymptotically to a final constant, which does not occur for a diffracting beam.

Taking as a numerical example, $\lambda = 10.6$ μm and $z_q = 1$ cm, we obtain 54 MeV for the minimum injection energy of electrons. At such a tight focus the vaporization problems of drift tubes are likely to occur. By injecting at higher energy, say above 200 MeV, larger values of z_q are permitted, but the gain remains at 22 MeV. In general, by defocusing the problems associated with high field magnitudes are alleviated, but the price is that higher injection energy is required while the gain remains the same.

A fortunate feature of the rough result of Eq. (11) is that $\gamma_{\min}$ is independent of the mass of the particle, unlike the result of Ref. [11] where it is directly proportional to the mass. Hence, for Hermite-Gaussian accelerators, at least, it appears to be not much more difficult to accelerate protons than electrons.

B. Stability of particle trajectories in vacuum Gaussian beam

The conditions for the transverse stability of the particle trajectory in the plane-polarized Hermite-Gaussian beam are discussed in Refs. [11] and [14]. Here we extend the analysis of [11] to include diffractive effects of the beam. The force acting on the particle consists of both the electric and magnetic fields' contributions. Thus the components in rectangular coordinates are

$F_x = e(E_x + v_y B_z - v_x B_y)$, $F_y = e(E_y + v_z B_x - v_x B_z)$, and $F_x = e(E_z + v_y B_x - v_x B_y)$. We are now only interested in particles whose trajectories lie sufficiently close to the z axis that terms of higher order than the first in the x and y coordinates are negligible. In this approximation $B_x = 0$, and the remaining magnetic-field components are given by

$$B_x \approx \frac{k}{\omega}\mu E_x^{(0)}(0,0,z,t)y \ , \tag{12a}$$

$$B_y \approx -\frac{k}{\omega}(ik + \mu)E_z^{(0)}(0,0,z,t)x \ , \tag{12b}$$

where $E_z^{(0)}$ is the zero-order longitudinal field, without diffractive correction, as given in Eq. (5). A proof of Eqs. (12), and a demonstration of their self-consistency, is given in Appendix B. A diffractive correction to $E_z^{(0)}$ is also derived in Appendix B. This leads to the following equations for the transverse force components:

$$F_x = -k\left[1 - \beta\left[1 + \frac{z_q/k}{z^2 + z_q^2}\right]\right]\frac{eE_{0z}z_q^2}{z^2 + z_q^2}x\cos\phi + \beta\frac{eE_{0z}z_q^2}{(z^2 + z_q^2)^2}xz\sin\phi \ , \tag{13a}$$

$$F_y = \beta(z\sin\phi + z_q\cos\phi)\frac{eE_{0z}z_q^2}{(z^2 + z_q^2)^2}y \ , \tag{13b}$$

where $v = c\beta$ denotes the z component of the particle velocity.

The equations of motion in the transverse direction are obtained from the relativistic equation of motion and Eqs. (13). We assume that the particle is injected near the point $z = -z_q$ on the axis with energy $\gamma_{in} m_0 c^2$. Since $\phi' \approx 0$, $\phi = \pi/2 - 2\psi(z)$. Equating the force to the time rate-of-change of momentum, introducing the energy gained by a particle propagating along the axis $W_{0g} = eE_{0z}z_q$, and, finally, the dimensionless coordinates, defined by

$$\xi = \left[\frac{x}{z_q}, \frac{y}{z_q}\right], \quad Z = \frac{z}{z_q} \ , \tag{14}$$

we obtain the following damped harmonic-oscillator equation with "time"-dependent damping and spring constants

$$\frac{d^2\xi}{dZ^2} + \Gamma(Z)\frac{d\xi}{dZ} + K(Z)\xi = 0 \ , \tag{15}$$

where the damping constant $\Gamma(Z)$ is given by

$$\Gamma(Z) = \kappa(Z)\frac{1 - Z^2}{(1 + Z^2)^2} \ , \tag{16}$$

and K represents a diagonal spring-constant matrix having components in the x and y directions, given by

$$K_x = -\kappa(Z)Z\left[\beta\frac{3 - Z^2}{(1 + Z^2)^3} - 3kz_q\frac{(1 - \beta)}{(1 + Z^2)^2}\right] \ , \tag{17a}$$

$$K_y = -\beta\kappa(Z)Z\frac{3 - Z^2}{(1 + Z^2)^3} \ . \tag{17b}$$

The slowly varying function of Z, $\kappa(Z)$, is the ratio of the VGFP in the relativistic limit, given in Eq. (8) to the particle energy at Z, or,

$$\kappa(Z) = \frac{W_{0g}}{\gamma(Z)m_0c^2} \ . \tag{18}$$

The presence of the damping term is related to the fact that as the transverse velocity is increased, so does the energy in the rest frame, and therefore the relativistic mass increases also. This mass increase appears as a resistance to the motion, to first order, in the form of a linear damping constant.

Note that the second term inside the brackets in Eq. (17a) is negligibly small unless the energy is at least that satisfying Eq. (11), where L is of the order of z_q. Henceforth, in this paper, the terminology "ultra-relativistic limit" will mean that $\beta \to 1$, and $\gamma \geq \gamma_{\min}$, where $\gamma_{\min}$ is defined in Eq. (11). An exception is in Sec. III B, where there is no lower bound on the injection energy.

In the ultrarelativistic limit, K_x and K_y both approach the same expression, given by

$$K_t(Z) = -\kappa(Z)Z\frac{3 - Z^2}{(1 + Z^2)^3} \ . \tag{19}$$

The spatial dependence of the Lorentz factor, $\gamma(Z)$, can be found in the same approximation. From Eq. (5) we obtain

$$\gamma(Z) = \gamma_{\text{in}} + \frac{W_{0g}}{m_0c^2}\frac{1 + Z}{1 + Z^2} \ , \tag{20}$$

subject to $Z_{\text{in}} = -1$. The Lorentz factor $\gamma(Z)$ can be approximated by its input value, as a first approximation, so that $\kappa(Z)$, also, is slowly varying. The magnitudes of the independent terms in Eqs. (15) and (16) are typically in the range of 0.1 to 1. Thus all terms are potentially important.

The stability properties of Eq. (15) are summarized by the requirement that $\Gamma(Z)$ and $K_t(Z)$ are both positive.

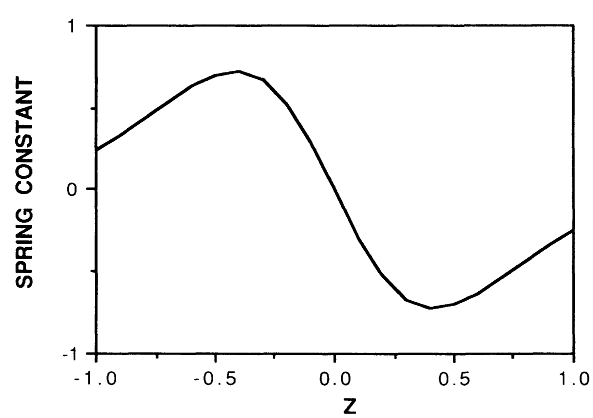

FIG. 1. The dimensionless normalized "spring constant" $K_t(Z)/\kappa(Z)$ for the transverse force acting on an ultrarelativistic particle traveling near the axis of a vacuum $TM_{1,0}$ Hermite-Gaussian accelerator, where $Z = z/z_q$, and z_q is the Rayleigh length. The function $\kappa(Z)$ is nearly constant.

The condition on $\Gamma(Z)$ is always satisfied in acceleration regions, since $\Gamma(Z)$ is proportional to E_z. The proportionality to E_z is fortuitous, especially for the gas-loaded accelerator, in which acceleration can occur far beyond the Rayleigh distance from the focus. The spring constant K_t is plotted in Fig. 1. It changes sign at the focus, implying that the transverse force is towards the axis when $Z<0$ and away from it when $Z>0$. We cannot, therefore, speak of overall stability, the more so because both negative- and positive-Z ranges must be utilized. Useful trajectories necessarily have initial values of position and angle close to those of an axially propagating particle.

The longitudinal stability, or the property of the phase of the longitudinal force to remain constant, is likewise affected by the phase drift in the case of the vacuum Gaussian beam accelerator. These stability properties are not significantly affected by the use of a gas, unless this is of the graded index type to be discussed.

III. GAS-LOADED LINAC

A. Uniformly gas-loaded accelerator

A rare gas has been used already in free-electron lasers [19,20] for phase-matching purposes. Consider first the case of a uniform medium. Since the phase velocity of the Hermite-Gaussian beam decreases monotonically with longitudinal distance from the focus, it is not possible to obtain a perfect match with the particle velocity except, at most, two isolated points. The question is thus whether to match at the focus, where the field is strongest, or at some distance from it.

For an axially propagating particle there are three important parameters: the GFP, the interaction length, and the minimum injection energy. The first two can be obtained from the expression for the energy gain in the relativistic limit, while the third is more correctly determined if electron dynamics are considered. The interaction range, which is defined as being the distance over which continuous acceleration takes place, is larger than the vacuum value of $2z_q$. On the other hand, net gain is now also achieved over the *full* length of the beam.

Let us first consider acceleration only over the interaction length, as defined above. The starting point is the general expression for the gain

$$W_g=eE_{z0}\int_{z_1}^{z_2}\frac{z_q^2}{z^2+z_q^2}\cos[nkz-\omega t-2\psi(z)]dz\ , \tag{21}$$

where we took $\phi_0'=\pi/2$ for maximum acceleration at $z=ct=0$, and n is the index of refraction. For a relativistic particle $t=z/c$, and $t=z/v$ serves as an approximation in which the particle velocity v is approximated by its value at injection. This ignores the effect of the field on the particle trajectory, so that we are treating with a version of the Born approximation. Using Eq. (8), and changing the integration variable to ψ, the above expression simplifies to

$$W_g=W_{0g}\int_{\psi_1}^{\psi_2}\cos Q\, d\psi\ , \tag{22}$$

where $Q=\delta\tan\psi-2\psi$, and $\delta=(n-c/v)kz_q$. Equation (12) assumes a maximum value if the force changes sign at the limits of integration, or at these points

$$Q(\psi_i)=\pm(N+\tfrac{1}{2})\pi,\quad i=1,2,\quad N=0,1,2,\ldots\ . \tag{23}$$

Denoting the limits of the central acceleration region ($N=0$) by $\pm\psi_1$, we have

$$\delta\tan\psi_1-2\psi_1=\frac{\pm\pi}{2}\ . \tag{24}$$

Equation (24) gives the interaction length as function of n by means of the substitution $\tan\psi_1=z_1/z_2$, where $-z_1$ is the input coordinate. If $\delta<\delta_0$, then the negative (positive) sign in Eq. (24) corresponds to the lower (upper) limit of integration, and if $\delta>\delta_0$ this identification is reversed. The value of δ_0 is obtained by solution of

$$\sqrt{\delta_0(2-\delta_0)}-2\cos^{-1}\sqrt{\delta_0/2}+\frac{\pi}{2}=0\ , \tag{25}$$

which yields $\delta_0\approx 0.33$. A possible value of n is obtained by arguing that W_g is large if $dQ/d\psi=0$ at the focus. The solution for n is then

$$n=\frac{c}{v}=\frac{2}{kz_q}\ . \tag{26}$$

This index value does not yield the maximum GFP, but it is the value that results in equal particle and phase velocities at the beam focus, to first order in $1/kz_q$. Using Eq. (23), the numerical solution of Eq. (24) yields for the injection and extraction coordinates the solution

$$-z_1=z_2=1.86z_q\ . \tag{27}$$

The corresponding GFP, in the limit $v\to c$, is

$$W_g=W_{0g}\int_{-1.08}^{1.08}\cos[2\tan\psi-2\psi]d\psi =1.9W_{0g}\ . \tag{28}$$

The injection coordinate z_1 and W_g are plotted as functions of n in Figs. 2(a) and 2(b). The maximum gain occurs at the value of n given by

$$n=\frac{c}{v}+\frac{1}{kz_q}\ , \tag{29}$$

which falls, for $v\to c$, just halfway between its vacuum value, $n=1$, and that given in Eq. (26). With this value of n the phase velocity equals c at a Rayleigh length from the focus. The corresponding GFP is given by

$$W_g=2.4W_{0g}\ , \tag{30}$$

and the injection point is

$$z_1=4.25z_q\ . \tag{31}$$

Both the gain and the interaction length constitute a considerable increase over the vacuum values.

The "switchover" point, determined by $\delta=0.33$, is also interesting. Figure 2 shows that there is a jump discontinuity here in both the distance from focus of the injection point and the GFP. The gain at this value of δ constitutes only a modest improvement over the vacuum

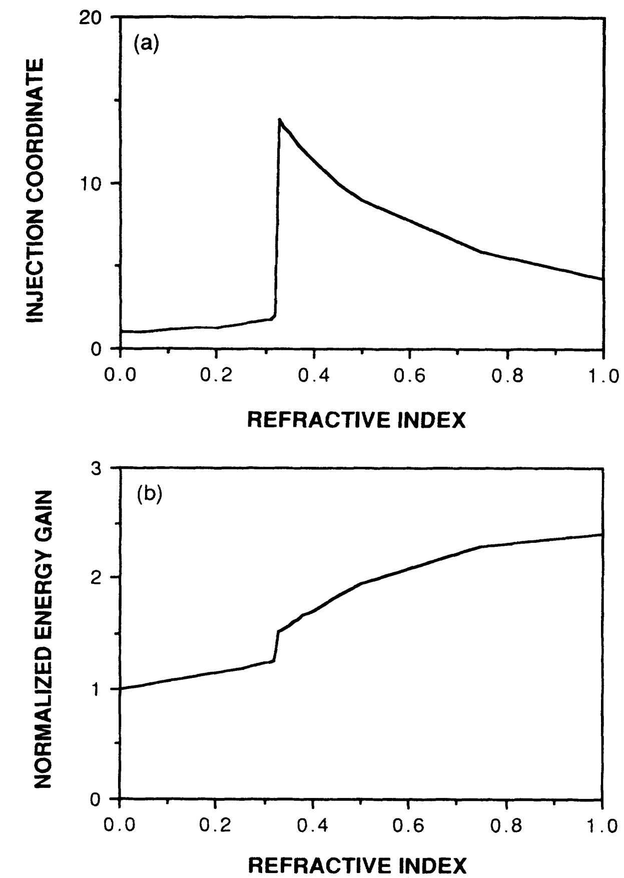

FIG. 2. Properties of particles accelerated in a gas-loaded Hermite-Gaussian accelerator plotted as functions of the parameter $\delta=(n-1)kz_q$, where n is the refractive index of the gas. (a) The plotted function is the negative of the injection coordinate z_i in units of z_q. (b) The plotted function is the energy gain in units of the maximum vacuum gain $W_{0g}=E_{0z}z_q$. The discontinuity in the graphs is explained in the text.

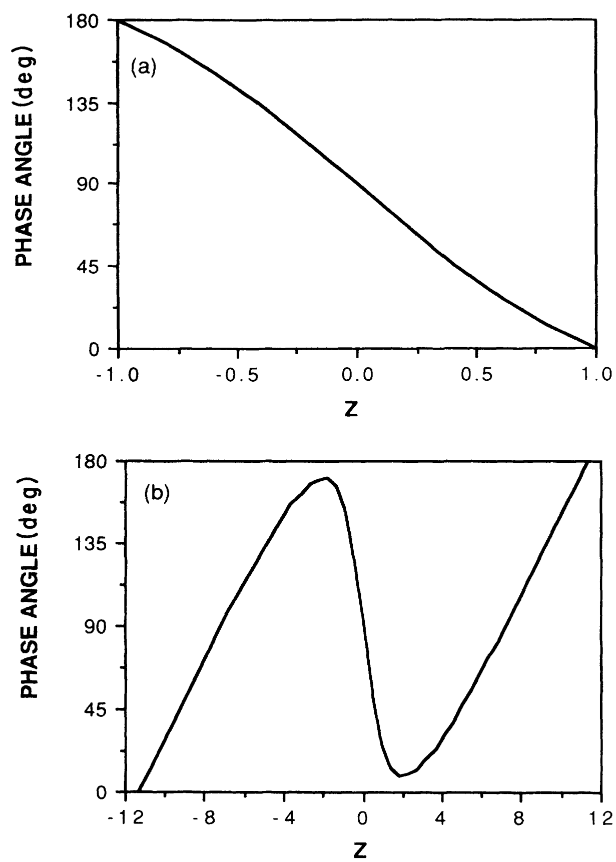

FIG. 3. The phase ϕ of the longitudinal electric field at the position of an ultrarelativistic axially propagating particle. (a) Vacuum ($\delta=0$), and (b) gas-loaded beam ($\delta=0.33$).

value

$$W_g=1.5W_{0g} , \tag{32}$$

but the interaction length attains a maximum here, with

$$z_1\approx 14z_q . \tag{33}$$

The origin of the jump discontinuity lies in part in the property that, as n increases, a point is reached where the two points on the trajectory of equal phase and particle velocities (turning points of the phase) enter the central acceleration zone. The phase drift then reverses twice during the interaction, resulting in considerable increase of the interaction length. The function $\phi(z)$ is illustrated in Figs. 3(a) and 3(b), where Fig. 3(a) shows the vacuum case and Fig. 3(b) corresponds to $\delta=0.35$. The latter shows the reversals in the phase drift. From this figure the explanation for the jump discontinuity in Fig. 2 is immediately evident. In the range $\delta<\delta_0$, the two neighboring regions of deacceleration to the central accelerating region shrink as δ approaches δ_a. Had we taken the injection point of the particle to be in the next accelerating region, and allowed it to drift through the short (when $\delta\approx\delta_0$) deacceleration region, a smooth transition rather than a jump discontinuity connecting the two regions would have been obtained. This also shows that the calculated gain to the left of the discontinuity (except at exactly $n=1$) is less than the maximum that is obtainable.

Figure 4 shows the result of computer calculations of the GFP as a function of the injection energy for the different index values considered in this section. The interaction range is that of positive acceleration, as treated up until now. The laser power is $P=1$ TW, and the beam parameter $z_q=0.1$ m. The high-injection-energy limit corresponds to the value given by the formulas given above in each case. Note that the minimum injection energy is smallest for the vacuum linac, followed by the highest index case. The rough approximation for this quantity, Eq. (11), is quite well satisfied by these curves. The intermediate index ($\delta=1$) yields the largest GFP, but only at high injection energy, while the switchover point ($\delta=0.33$) requires the highest injection energy of all the cases studied. In fact, in all cases the advantage of the long interaction length is offset by the high minimum injection energy. This is of course predicted by Eq. (11), according to which the injection energies are identical for two cases only if the interaction lengths are the same. The only way to get around that is to extend the interaction beyond the central region, since then Eq. (11) does not apply.

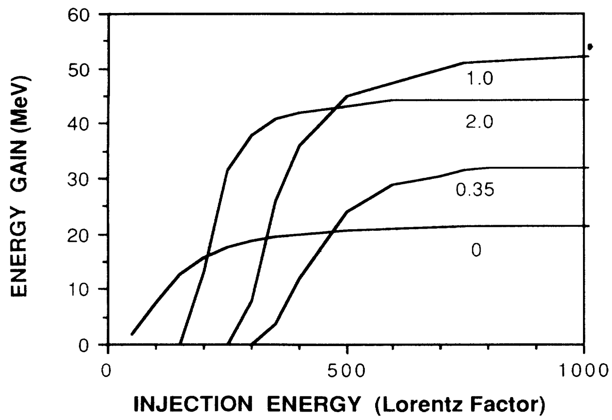

FIG. 4. The energy gain of electrons moving along the beam axis as a function of the Lorentz factor $\gamma=1/(1-v^2/c^2)^{1/2}$ at injection, where v is the electron velocity. The curves correspond to different values of gas index, where the corresponding value of δ is indicated. The Rayleigh length $z_q=0.1$ m, the beam power $P=1$ TW.

Thus the question of possibly prolonging the interaction, although not necessarily with additional gain, is of considerable interest. Suppose that the particle is allowed to interact with the infinite length of beam. It is shown in Appendix A that a net transfer of energy occurs in the presence of a gas, although not in the vacuum. The solution of Eq. (22) yields in the limit $z_1\to\infty$ for the energy transferred to the particle, the simple result

$$W_g^{\infty\,\text{beam}}=2\pi\delta e^{-\delta}W_{0g}\ . \tag{34}$$

This expression has a maximum when $\delta=1$, given by

$$W_g^{\infty\,\text{beam}}=2.31\,W_{0g}\ , \tag{35}$$

which is only slightly less than the gain for the finite interaction length at the same index value, given in Eq. (30). On the other hand, we have the significant advantage of the present configuration in that it allows arbitrarily wide spacing of lenses and drift tubes. Specifically, the drift tubes can be eliminated entirely, resolving the heat damage problem of the vacuum accelerator. Computer integration of the relativistic equation of motion, for $\delta=1$, over interaction lengths up to 20 Rayleigh lengths on each side of focus, have shown that the minimum-injection energy is roughly the same as that found for the central acceleration region. One can explain this by recognizing that what happens to the particle before it reaches the focal region is not too important, provided it arrives there with enough energy and the right phase, due to the rapid decline of the field strength according to $1/z^2$ in the gas-loaded linac.

Summarizing this section, in a linac of length just equal to the range of positive acceleration, a uniformly loaded beam yields an improvement factor of about 2 over the vacuum beam if the index n is chosen so that the phase velocity of light equals exactly c at the beam focus, and an improvement factor of 2.4 is obtained if the phase velocity in the presence of the gas at focus falls halfway between c and the vacuum phase velocity at the focus. In the former case the interaction length is $3.72z_q$, and in the latter is $8.5z_q$. Nevertheless, since the beam radius at the extremes of the trajectory is still only a factor of about three times the radius at the focus, the vaporization problem of drift tubes remains without a satisfactory solution at this point.

Acceleration over the *entire beam length,* yielding a net improvement factor of 2.31 over energy gain of the vacuum linac and with minimum injection energy being approximately that for acceleration over the central acceleration region ($N=0$), has been found possible, resolving the drift-tube problem. The important question of beam stability in these cases will be taken up in Sec. III D 1.

B. Optimally graded-index accelerator

It is possible, in principle, for a traveling wave to "capture" a fraction of the particles that are characterized by common energy and phase. This is achieved by varying the phase velocity in such a way that at each point of the trajectory it just equals the velocity of a representative particle. Thus, for a relativistic particle beam a medium of index $n_0(z)=c/v_0(z)$ is required. Such index grading may possibly be realized by flowing a gas through a tube that is heated at the ends. Alternatively, the index of the gas could be controlled optically, through saturation by resonant absorption of photons from a laser beam.

For a subrelativistic particle, the gas index must be graded more steeply in order to make the field accompany the slower particle. In principle, it would be possible in this way to accelerate a particle starting from rest, so that the concept of minimum injection energy is no longer valid as an imposed condition. A second advantage is that the interaction length of the captured particles would also be arbitrarily large in such an ideal medium, an advantage shared by the constant-index medium accelerator. And finally, we shall find that the stability properties are more favorable than in the constant-index case.

We determine the refractive index function $n(z)$ for which the phase slippage will be zero, given the injection characteristics of a reference particle. The solution is a function of the beam geometry, which in turn generally depends on $n(z)$. We make the simplifying assumptions that n is close to unity and slowly varying. The latter condition is equivalent to the inequality $\eta\ll 1$, where $\eta=eE_z/m_0c\omega$. Then the principal effect of the varying index will be on the z dependence of the phase ϕ, which is now given by

$$\phi(z)=k\int^z n(\phi')dz'-2\psi(z)-\omega t\ . \tag{36}$$

Differentiating ϕ with respect to time, and using $dz/dt=v$, we obtain for the condition that $\phi=$const at the coordinate following the particle:

$$n(z)=\frac{c}{v(z)}+2\psi'(z)/k\ . \tag{37}$$

This states that the particle velocity equals the phase velocity in the medium's presence.

Using Eq. (5), we find the energy at z of a particle injected at z_{in} with energy $\gamma_{\text{in}} m_0 c^2$,

$$\gamma(z)=\gamma_{\text{in}}+\frac{W_{0g}}{m_0c^2}[\psi(z)-\psi(z_{\text{in}})] \ . \tag{38}$$

Some insight is gained from the expansion of $n(z)$ to first order about the origin. Defining the velocity $v(0)$, the normalized field $\eta(0)$, and the Lorentz factor $\gamma(0)$ at the focus, we obtain

$$n(z)\approx\frac{c}{v(0)}+\frac{2}{kz_q}-\frac{\eta(0)}{[\gamma(0)-1]^{3/2}}kz \ . \tag{39}$$

The peak refractive index $n(0)$ and its first derivative increase with increasing sharpness of focusing and decreasing injection energy. The gradient is also proportional to the field strength. This is expected because as the particle's velocity increases the difference with the phase velocity of the beam decreases, hence the negative sign of the third term on the right-hand side.

C. Energy gain in graded-index medium

For optimum $n(z)$ the maximum transfer of energy is obtained by integrating the longitudinal electric-field amplitude from $z=-\infty$ to $+\infty$. The phase ϕ is now a constant. The result of the integration is

$$W_g^{\text{GI}}=2e\left[\frac{\pi P}{\epsilon_0 c}\right]^{-1/2}\sin\phi \ . \tag{40}$$

Note that $W_g^{\text{GI}}(\phi=\pi/2)=\pi W_{0g}$. The principal advantage of this type of gas-loaded Gaussian beam linac, however, is that lower injection energy is possible, and we shall also see that it has transverse-focusing and particle-bunching properties. In any case, the former property already permits the spacing of drift tubes to be arbitrarily large, and it is even possible to do without them completely.

D. Stability of particle trajectories in gas-loaded Gaussian beam

The stability of the particle beam must be reexamined when a gas is present in view of the fact that the on-axis phase velocity of the optical beam can now be smaller than the particle velocity. This causes reversals in the sign of force components, in the transverse directions as well as in the longitudinal direction. One effect of this was, as was seen, to increase the interaction length. A second effect is that the transverse force components that canceled in the ultrarelativistic limit now no longer cancel, which has important implications for the stability of the particle beam. The cases of uniform and graded gas loading are discussed individually below.

1. Constant-index accelerator

In the presence of a medium of refractive index n, B_y is modified by replacing k by nk in Eq. (12b) and the phase ϕ. Substituting Eq. (5) for E_z on the axis yields then

$$B_y=-\frac{1}{c}\left[\left[nk+\frac{z_q}{z^2+z_q^2}\right]\cos\phi+\frac{z\sin\phi}{z^2+z_q^2}\right]\frac{E_{0z}z_q^2}{z^2+z_q^2}x \ . \tag{41}$$

The force in the x direction becomes accordingly

$$F_x=-ek\left[1-n\beta-\frac{vz_q/\omega}{z^2+z_q^2}\right]\frac{E_{0z}z_q^2}{z^2+z_q^2}x\cos\phi + e\beta\frac{eE_{0z}z_q^2}{(z^2+z_q^2)^2}zx\sin\phi \ , \tag{42}$$

while the y component is obtained from F_x by replacing the term $1-n\beta$ inside the square brackets by zero. The expression in the ultrarelativistic limit for the spring constant K_x is then given by

$$K_x=-\kappa(Z)\left\{\left[\delta+\frac{1}{1+Z^2}\right]\left[\frac{2Z\cos\delta Z-(1-Z^2)\sin\delta Z}{(1+Z^2)^2}\right] + Z\frac{(1-Z^2)\cos\delta Z+2Z\sin\delta Z}{(1+Z^2)^3}\right\} , \tag{43}$$

and K_y is now obtained by replacing δ inside the first pair of square brackets with zero. The damping constant Γ is the same as before, given in Eq. (16).

In Figs. 5(a)–5(c) we see the spring constants as functions of position for three values of δ. Since both are odd functions of Z there is no focusing tendency, in accordance with the comment below Eq. (1): particles leave the focal region making the same angle with the axis as they had entering the focal region. In the case of a weak field, at least for $\delta\leq 1$, there is a *net contraction* of a beam of such particles, due to the sign reversal of the spring constant from positive before the focus to negative after it. The contraction is caused by the inertia of the particles, which gain a component of velocity towards the axis in the attractive field before the focus. This transverse momentum is overcome by the repulsive force past the focus, but not without leaving a net displacement towards the axis. The here described effect of beam collimation is certainly favorable [14], and could find useful application.

2. Optimally graded index: Transverse stability of particle motion

If the medium is described by a graded refractive index of the form discussed above, the possibility exists that the particle motion is stable with respect to a small longitudinal displacement from a comoving equilibrium point. The reference particle propagates along the axis with the varying phase velocity $v_0(z)$. We consider the stability of its trajectory with respect to small displacements in both the transverse and longitudinal directions. Since these motions are separable to first order in the displacement, we consider them individually, beginning with the transverse. The expressions for the force components derived in the preceding section are valid here as well, with n being the given function of z, derived above, and the phase ϕ equal to a constant along the equilibrium trajectory. Assuming the Born approximation, for which

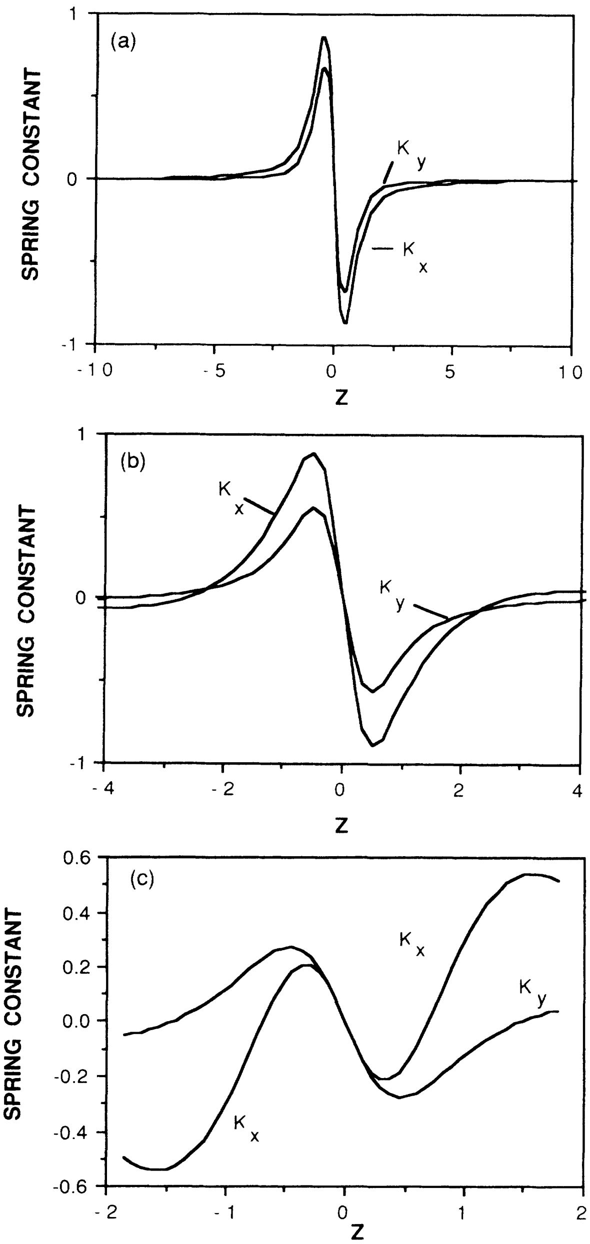

FIG. 5. The dimensionless normalized spring constants $K_x(Z)/\kappa(Z)$ and $K_y(Z)/\kappa(Z)$ in the transverse directions parallel and perpendicular, respectively, to the direction of polarization of the plane-polarized Hermite-Gauss beam; (a) $\delta=0.35$, (b) $\delta=1$, (c) $\delta=2$.

$n(z)=v_0(z)/v(z)$, so that to first order in $1/kz_q$

$$1-n\beta=-\frac{2/kz_q}{1+Z^2}\ , \tag{44}$$

we obtain different spring constants in the x and y directions, given by

$$K_x=-\beta\kappa(Z)\frac{3\cos\phi+Z\sin\phi}{(1+Z^2)^2}\ , \tag{45a}$$

$$K_y=-\beta\kappa(Z)\frac{\cos\phi+Z\sin\phi}{(1+Z^2)^3}\ , \tag{45b}$$

and the damping constant

$$\Gamma(Z)=\frac{\kappa(Z)}{1+Z^2}\sin\phi\ , \tag{46}$$

where Z can have any value in the range $-\infty$ to $+\infty$ and ϕ can have any fixed value between 0 and π for positive acceleration. Consider three values of ϕ as examples. When $\phi=0$ then both K_x and K_y are negative for all values of Z, indicating repulsion from the axis. At the phase angle of maximum acceleration, $\phi=\frac{1}{2}\pi$, both K_x and K_y are positive for $Z<0$ and negative for $Z>0$. Finally, as ϕ approaches π, both transverse coordinates are stable for all values of Z.

E. Longitudinal stability of trajectories: particle bunching

Bunching of particles occurs in a loaded laser linac with graded index if the particles are attracted to stable equilibrium points. We investigate here the conditions under which this may occur. The equation for the energy gain per unit time is

$$\frac{1}{v}\frac{d\gamma}{dt}=k\eta(z(t))\sin\phi(z(t))\ , \tag{47}$$

where $\eta(z(t))=e\mathcal{E}_z(z(t))/m_0c\omega$ is the dimensionless longitudinal electric field, $\mathcal{E}_z$ being defined by $E(z(t))\sin\phi(z(t))$. Assume that the motion of the particle under consideration deviates slightly from the equilibrium trajectory $z_0(t)$. Thus let $z(t)=z_0(t)+\delta z(t)$ and likewise for the velocity $v(t)=v_0(t)+\delta v(t)$, where $v_0(t)$ is the velocity of the equilibrium point. The left-hand side of Eq. (42), after expansion to first order in δv, becomes

$$\frac{1}{v}\frac{d\gamma}{dt}=\frac{1}{v_0}\frac{d\gamma_0}{dt}+3\frac{\gamma_0^2}{c^2}\frac{d\gamma_0}{dt}\delta v+\frac{\gamma_0^3}{c^2}\frac{d\delta v}{dt}\ . \tag{48}$$

Similarly, expansion of the right-hand side yields

$$k\eta(z)\sin\phi(z)=k\eta_0\sin\phi_0+k\left[\frac{d\eta_0}{dz}\sin\eta_0\cos\phi_0\delta\phi\right]\ , \tag{49}$$

where

$$\delta\phi=\phi(z)-\phi_0=[kn(z_0)-2\psi'(z_0)]\delta z=\frac{\omega}{v_0}\delta z\approx k\delta z\ . \tag{50}$$

Combining Eqs. (42) and (43) with (41), we obtain to first order in δz another damped oscillator equation,

$$\frac{d^2\zeta}{dz_0^2}+\Gamma_z(Z_0)\frac{d\zeta}{dZ_0}+K_z(Z_0)\zeta=0\ , \tag{51}$$

where $Z_0=z_0/z$ and $\zeta=\delta z/z_q$. The damping and spring constants are given by

$$\Gamma_z(z_0)=3\kappa(Z_0)\sin\phi_0/(Z_0^2+1)\ , \tag{52}$$

and

$$K_z(Z_0) = -\frac{\gamma(Z_0)^{-2}}{Z_0^2+1} k z_q \cos\phi_0 \kappa(Z_0) , \tag{53}$$

respectively. The term in the spatial derivative of the field was omitted, being of the order of $1/kz_q$ compared to the remaining term.

The function K_z is plotted in Fig. 6(b). The stability condition reduces to the simple requirement on the phase angle: $\cos\phi > 0$. In Sec. III D 2 it was found that if $\phi > \frac{1}{2}\pi$, K_x and K_y are also positive for at least $Z < 0$, so that the particle beam is stable in all three dimensions over the distance that these conditions are satisfied. As ϕ approaches π the range of complete stability approaches the entire length of the beam. The price is that the longitudinal force also approaches zero.

It may be objected that the property of three-dimensional stability contradicts Earnshaw's theorem [21] of electrostatics, which applies in a reference frame that is comoving with the field's phase. Earnshaw's theorem states that a system of electric charges cannot exist in stable equilibrium under the influence of static electric fields alone. That this theorem does not automatically apply to the present situation follows from the fact that the mere existence of dielectric media also requires the existence of nonelectrostatic forces. Specifically, we note that the velocity of light in the comoving frame is not isotropic, but depends on the direction of propagation. Therefore, we must have $\nabla \cdot \mathbf{E} \neq 0$ in that frame. Arguments based on the presence of the quasistatic magnetic field, and the fact that the medium is a spatially and temporally varying dielectric in the moving frame, are irrelevant, as the true cause is the one given.

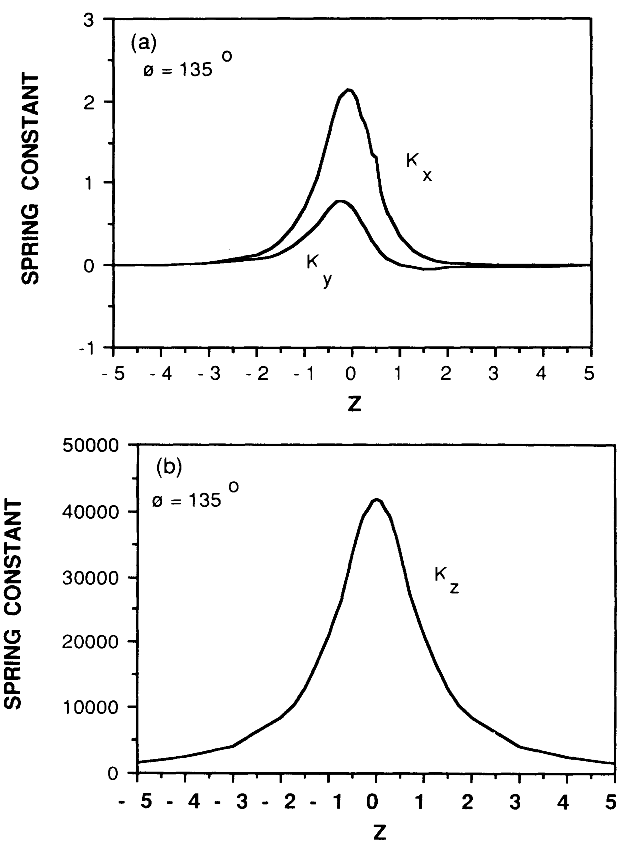

FIG. 6. The normalized spring constants in dimensionless units in the transverse x,y and longitudinal z directions for a beam with graded index $n(z)$. The phase angle $\phi = 135°$; (a) spring constants in the transverse directions, (b) in the longitudinal direction.

The stability of the particle beam is limited by the fact that the laser field is concentrated in the focal region, and away from it decays as the inverse square power of the distance. The decay of transverse forces is even faster: they decay as the inverse fourth power. In the vacuum and uniformly loaded (of constant index) cases, a particle will leave the focal region at the same angle as it entered. A particle that is initially parallel to the axis will therefore remain parallel upon leaving the focal region, provided the net deflections in the negative- and positive-Z region are small. We may speak of "stability" in these cases. In the graded-index case net focusing or defocusing exists. However, if the conclusion of stability were warranted in the case of net attraction towards the axis, following the conventional definition of stability, then the particle would execute damped oscillations about the axis. Instead, the transverse component of momentum gained by the particle upon passage of the laser focus is too large to be reversed by the rapidly weakening transverse force, so that while being attracted to the axis, the particle bypasses and then moves perpetually away from it. Thus instead of being "stable" we should consider this situation as being "unstable." On the other hand, the particle beam can be recollimated if the optical beam is refocused, provided that the lateral displacement of the particles is still smaller than the half-width $w(z)$ of the optical beam. In this case, of the graded-index accelerator, it is useful to make a Gaussian optics analogy for the particle beam, and to treat the focal regions of the optical beam as if they were thin lenses for the particle beam, at least to a first approximation.

Further justification for this point of view is obtained from calculating the transverse deflection in the Born approximation. If we assume that the condition of paraxial incidence is satisfied as well, then the connection to Gaussian optics can be made. The focal distances of the particle trajectories, f_x, f_y, and f_z, are then derived by integrating the corresponding spring constants K_x, K_y, and K_z over the entire Z range. Neglecting the Z dependence of κ and γ, we find

$$z_q/f_x = \int_{-\infty}^{+\infty} K_x(Z)dZ = -\tfrac{3}{2}\pi\kappa\cos\phi , \tag{54a}$$

$$z_q/f_y = \frac{1}{3}\frac{z_q}{f_x} = -\tfrac{1}{2}\pi\kappa\cos\phi , \tag{54b}$$

$$z_q/f_x = \int_{-\infty}^{+\infty} K_z(Z)dZ = -\tfrac{1}{2}\pi\gamma^{-2}kz_q\kappa\cos\phi . \tag{54c}$$

These equations show that in the present situation the optical beam is capable of both focusing and defocusing the particle beam, depending on the phase ϕ. The magnitudes of f_x, f_y, and f_z are likewise controlled by ϕ. In case that it should be necessary for the focal lengths in the x and y directions to be equal, then this can be accomplished with a radially polarized laser beam, as discussed in Sec. V. Moreover, we note that the focal distance for bunching also approximately equals the focal lengths in the transverse directions when the energy is in the neighborhood of that given in Eq. (11), i.e., when $\gamma \approx \sqrt{kz_q}$. The importance of all this certainly is in the construction of a lens waveguide accelerator, as proposed in Ref. [10]. In addition to demonstrating the possibility of the concept, we have extended it by showing that while the lenses guide the optical beam, the focal regions of the beam function as lenses for the particle beam in the usual sense, and finally, also bunch the particles.

IV. RADIALLY POLARIZED HERMITE-GAUSSIAN LINAC

A radially polarized beam can be constructed by superposing two plane-polarized beams of the form

$$\mathbf{E}_1 = 2^{-1/2}\mathcal{E}(r)x\hat{\mathbf{i}} \ , \tag{55a}$$

$$\mathbf{E}_2 = 2^{-1/2}\mathcal{E}(r)y\hat{\mathbf{j}} \ , \tag{55b}$$

where $\mathcal{E}(r)$ is the field defined in Eq. (3) divided by x. Then

$$\mathbf{E}^{\mathrm{rad}} = \mathbf{E}_1 + \mathbf{E}_2 = 2^{-1/2}\mathcal{E}(r)\mathbf{r} \ . \tag{56}$$

The resulting field is a radially polarized Laguerre-Gaussian mode of order (1,0). From Eq. (1b), one has

$$E_z^{\mathrm{rad}} = \sqrt{2}E_z^{\mathrm{plane}} \ , \tag{57}$$

given equal beam powers of the two beams. Thus the GFP is a factor of $\sqrt{2}$ greater for the radial beam.

The transverse forces corresponding to the field of Eq. (56) are determined from those of Eqs. (55) by a 90° rotation about the z axis, and the transverse force of the radially polarized field is obtained by superposition. Defining, as in Sec. III, $K_{x,y}$ to be the spring constants associated with small displacements from the axis in the x and y directions in the plane-polarized beam, the corresponding constants of the radially polarized beam are given by

$$K_x^{\mathrm{rad}} = K_y^{\mathrm{rad}} = 2^{-1/2}(K_x + K_y) \ . \tag{58}$$

V. GAS-BREAKDOWN CONSIDERATIONS

The complexity of the gas-breakdown process permits at best a qualitative estimate, as it depends not only on the nature of the gas, its pressure, the wavelength, and pulse length, but also on the presence of impurities and surface effects of container walls. In addition, experimental data are still lacking in this area, particularly at the CO_2 laser wavelength [13].

The probability of ionization $P(t)$ is obtained by integrating the ionization rate $w(t)$. The ionization rate in the quasistatic limit is found from dc tunneling theory [22],

$$w(t) = 4\omega_0 \left[\frac{E_i}{E_h}\right]^{5/2} \frac{E_a}{E(t)} \exp\left[-\frac{2}{3}\left[\frac{E_i}{E_h}\right]^{3/2} \frac{E_a}{E(t)}\right] , \tag{59}$$

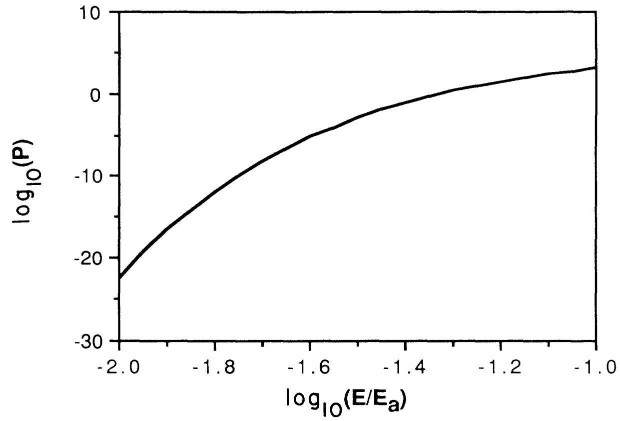

FIG. 7. The calculated ionization probability of a hydrogen atom in the field of a Gaussian pulse $E(t) = E\exp(-t^2/\tau^2)$ as function of the normalized peak field amplitude, where $E_a = 600$ GeV/m, and $\tau = 1$ ps.

where $\omega_0 = me^4/(4\pi\epsilon_0)^2\hbar^3$ and E_h are the frequency and ionization potential of hydrogen, E_i is the ionization potential of an atom in the medium, and $E_a = 4\pi\epsilon_0 m^2 e^5 \hbar^4$ is the atomic unit of the electric field. In Fig. 7 is plotted as an example the ionization probability of a hydrogen atom in the field of a Gaussian pulse $E(t) = E\exp(-t^2/\tau^2)$ as a function of the maximum field strength E, where $\tau = 1$ ps.

VI. PROOF OF PRINCIPLE EXPERIMENT AND SUMMARY

In this paper we derived the energy gain for both a vacuum and a gas-loaded Hermite-Gaussian laser linac. The field near the axis of Hermite-Gauss beams is weak compared to other beam geometries, yielding relatively small energy gradients. Acceleration in the vacuum is impractical because the interaction length equals at most two Rayleigh distances, which implies that drift tubes or other equipment must be placed in high-intensity regions of the beam. However, the more important result is that in a uniform density medium the interaction length is arbitrarily large, while the minimum injection energy is no larger than when the interaction is restricted to the principal (centered on focus) acceleration region, although this is about twice that of the vacuum linac. Hence, no drift tubes are required, and it should be possible to build an extended laser accelerator consisting of a focusing lens waveguide, as proposed in Ref. [10].

Another result obtained here is that the minimum injection energy is directly proportional to the particle mass, and is independent of the field strength. This opposes a conclusion of Ref. [11], where the injection energy was found to be proportional to the square of the mass and inversely proportional to the field. This disagreement is attributed to the difference in beam geometries studied in the two papers. The implication is that laser accelerators of protons will be easier to construct than was anticipated previously.

Finally, we evaluated the stability of the particle beam, or better, the *focusing* properties of the optical beam, in the various cases of vacuum and gas-loaded laser beams. In the vacuum and uniformly loaded Hermite-Gauss beams regions of transverse stability are counterbalanced by regions of instability, thus in these cases there is no net real focusing. On the other hand, the focusing properties of a longitudinally *graded*-index medium were found to be significant. In particular, the sign and magnitude of the focal lengths in all three dimensions, was found to be controlled by the phase. It is thus possible to construct a lens waveguide that simultaneously guides the optical and particle beams, while also acting as an accelerator and bunching device for the latter.

It may also be worth repeating that the graded-index beam requires no minimum injection energy. This property may enhance the feasibility of low-energy and ion accelerators.

A proof-of-principle experiment is simplest to perform in a constant-index medium. Assuming a pulsed carbon dioxide laser of peak power 1 TW and a Rayleigh range $z_q=5$ cm, then the gas index corresponding to $\delta=1$ is $n-1\approx 1.5\times 10^{-5}$, and the minimum injection energy is about 150 MeV. The energy gain corresponding to the given power would be about 30 MeV in a radially polarized beam. If the gas is chosen to be molecular hydrogen, then using data from Ref. [23] we obtain about 100 Torr for the pressure. The peak intensity of the beam is about 10^{14} W/cm^2, which should be below the ionization threshold of hydrogen if the laser pulse length is shorter than about 20 ps [24]. For a pulse length of 1 ps, corresponding to a pulse energy of 1 J, the peak field strength amounts to 65 GeV/m. Figure 7 predicts that this is above the ionization threshold of hydrogen. On the other hand, experimental data [14] (at $\lambda=10.6\mu$) indicate that this need not be the case. In the worst scenario, a less strongly focused laser beam could be used, at lower gas pressure, but requiring higher injection energy.

A design for a waveguide accelerator consists of a series of lenses of focal length f, separated a distance D such that $D<4f$, where for a sharp focus $D\approx 4f$. In the Gaussian optics approximation [16] the beam waist at the midway points between the lenses is given by

$$w_0=\left[\frac{\lambda D}{2\pi}\left[\frac{4f}{D}-1\right]^{1/2}\right]^{1/2} \tag{60}$$

and the half-width of the beam at the location of the lenses is

$$w_1=\left[\frac{2\lambda f}{\pi\sqrt{4f/D-1}}\right]^{1/2} . \tag{61}$$

Using the relation $z_q=\frac{1}{2}kw_0^2$, and Eqs. (54), applicable to a lens guide with graded-index medium, the focal lengths of the effective wave guide for the electrons can be determined.

In conclusion, we list some of the most important obstacles anticipated by us before a laser linac can become operational: (1) Cherenkov and transition radiation by electrons in passing through holes in lenses. (2) Heating and optical saturation of the gas medium. (3) The choice of best gas, laser source, etc., so as to minimize collision, absorption, and other effects. (4) What is the limit in laser power as determined by ionization of the gas, subject to parameters such as pulse duration, pressure, wavelength detuning from resonance, etc. (5) Is it possible, and if so, what is the best way, to attain z-dependent index grading of the gas?

It is well to remember that some of the listed obstacles are dependent on beam geometry. For example, obstacle (5) is of little concern for a diffraction-free Bessel beam, which, on the other hand, has problems peculiar to its own geometry.

APPENDIX A: PROOF OF THE LAWSON-WOODWARD THEOREM

According to the Lawson-Woodward theorem no net energy can be transferred by a beam to a particle if the interaction time is infinite. We considered a beam, for convenience it was plane polarized in the x direction, propagating through a medium of dielectric constant ϵ. The wave equation satisfied by the Fourier component at angular frequency ω

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}=-\epsilon\frac{\omega^2}{c^2}E_x \tag{A1}$$

has solutions which may be expanded in plane waves propagating in the positive z direction, i.e.,

$$E_x=i\int dk_x dk_y k_z A(k_x,k_y)\times\exp[i(k_x x+k_y y+k_z z-\omega t)] , \tag{A2}$$

where $k_z=(\epsilon\omega^2/c^2-k_x^2-k_y^2)^{1/2}$. From Gauss's law, in the absence of free charges, the z component of the field is given by

$$E_z=\int dk_x dk_y A(k_x,k_y)\exp[i(k_x x+k_y y+k_z z-\omega t)] . \tag{A3}$$

Consider a particle moving along the z axis with velocity c. The work done by the field between z_1 and z_2 is

$$\begin{aligned}W&=e\int_{z_1}^{z_2}E_z dz=e\int_{z_1}^{z_2}dz\int dk_x dk_y A(k_x,k_y)\exp\left[i\left[k_x x+k_y y+k_z z-\frac{\omega}{c}z\right]\right]\\&=-ie\pi\int dk_x dk_y A(k_x,k_y)\exp[i(k_x x+k_y y)]D(k_z,z) ,\end{aligned} \tag{A4}$$

where $2\pi D(k_{z,z})$ is the result of the integration over z. If the limits of this integral approach $\pm\infty$, then $D=\delta(k_z-\omega/c)$, and for an azimuthally symmetric beam we obtain

$$W=\mathrm{Re}\left[2\pi e\sqrt{\epsilon-1}\frac{\omega}{c}A(0)\right] . \tag{A5}$$

This vanishes for vacuum propagation, since $A(0)$ is finite. Note that the LW theorem is not valid in the presence of a gas.

If the field satisfies the boundary condition that $E_x(x,y,0,t)$ is a given function of x,y,t then it is easy to show that the function $A(k_x,k_y)$, defined in Eq. (A2), is independent of ϵ, so that its effect is to replace ω by $n\omega$, where $n=\sqrt{\epsilon/\epsilon_0}$ in the spatial part of the Fourier representation of the field.

APPENDIX B: DERIVATION OF E_z AND B

In this appendix a derivation of Eqs. (12) is given, and their self-consistency with Maxwell's equations demonstrated. As in Appendix A, we make a plane-wave expansion of E_x,

$$E_x=\int dk_x dk_y\, A(k_x,k_y)\exp(i\phi) , \tag{B1}$$

where $\phi=k_x x+k_y y+k_z z-\omega t$, and k_z is defined as in Appendix A. We have absorbed the factor ik_z in Eq. (A2) into the function A of the transverse-wave vector k_x,k_y.

This is an exact solution of the wave equation, the corresponding paraxial-ray approximation is obtained by expansion of k_z, i.e.,

$$k_z\approx k[1-\tfrac{1}{2}k^{-2}(k_x^2+k_y^2)] . \tag{B2}$$

In order to indicate that (B2) has been applied to the phase ϕ, we write ϕ'.

From Gauss's law, we obtain the exact expression for E_z,

$$E_z=-\int dk_x dk_y\, A(k_x,k_y)\exp(i\phi)k_x/k_z . \tag{B3}$$

which in the paraxial-ray approximation becomes

$$E_z=-\int \frac{dk_x dk_y\, A(k_x,k_y)\exp(i\phi')k_x}{k[1+\frac{1}{2}k^{-1}(k_x^2+k_y^2)]} . \tag{B4}$$

Using the solution Eq. (3) of the corresponding paraxial form of Eq. (B1), Eq. (B4) can similarly be solved, yielding near the axis

$$E_z\approx(i/k-2\mu/k^2)E_x/x . \tag{B5}$$

The magnetic-field components are obtained from Faraday's law. Thus

$$B_x=-\frac{i}{\omega}\frac{\partial E_x}{\partial y}=\frac{k}{\omega}\mu y E_z . \tag{B6}$$

$$B_y=-\frac{i}{\omega}\left[\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right] . \tag{B7}$$

Substituting the plane-wave expansions for E_x and E_z,

$$\begin{aligned}B_y&=\frac{1}{\omega}\int dk_x dk_y\, A(k_x,k_y)\exp(i\phi)k_z^{-1}(k^2-k_y^2)\\&\approx\frac{1}{\omega}\int dk_x dk_y\, A(k_x,k_y)\exp(i\phi')[k+\tfrac{1}{2}k^{-1}(k_x^2-k_y^2)]\\&=\frac{1}{\omega}(k-i\mu)E_x .\end{aligned} \tag{B8}$$

Using Eq. (B5), we have to first order in μ (the diffractive contribution)

$$B_y\approx-\frac{k}{\omega}(ik+3\mu)xE_z . \tag{B9}$$

Similarly, for B_z, we obtain

$$B_z\approx-\frac{k}{\omega}\mu y E_x . \tag{B10}$$

Note that although B_z is of second order in the transverse coordinates, its derivative with respect to y, which is needed below, is of first order.

Lastly, we show that these solutions are consistent with Ampere's law. Thus,

$$-i\frac{k^2}{\omega}E_x=\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z} . \tag{B11}$$

Using Eq. (B9), along with Eq. (4a), we find

$$\frac{\partial B_y}{\partial z}\approx\omega^{-1}(ik^2-k\mu)E_x . \tag{B12}$$

Finally, substituting Eqs. (B9) and (B12) into Eq. (B11), we see that the latter is indeed satisfied to first order in μ.

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