Spherical aberrations in the thermal-wave model for luminosity estimates in particle accelerators

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An approach for estimating the luminosity in linear colliders in the presence of spherical aberrations is developed within the framework of the recently proposed thermal-wave model for relativistic-charged-particle-beam propagation. By taking into account a quadrupolelike lens with octupole deviations, the transverse beam motion is governed by a two-dimensional Schrödinger-like equation, with an anharmonic potential. To first order in perturbation theory and in the thin-lens approximation, we analytically find the transverse beam density, the spot size, and the luminosity reduction factor at the interaction point in terms of the initial conditions. Some numerical estimates are also given.

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I. INTRODUCTION

The propagation of a relativistic-charged-particle beam concerns a number of topics in particle accelerators and in plasma physics [1]. The beam motion is generally affected by the presence of the surrounding medium. As an example, when a relativistic-charged-particle beam passes through an optical system, as occurs in an accelerating machine, a transverse electromagnetic force acts on the particles, changing their initial trajectories in such a way as to produce a macroscopic beam focusing, defocusing, bending, and/or having more complicated effects [1].

In the conventional machines these effects are produced by different devices such as bending dipoles, quadrupoles, sextupoles, octupoles, etc. In particular, in an ideal quadrupole the force on the beam depends linearly on the transverse particle coordinates ($\mathbf{F}_\perp = \hat{\mathbf{x}} K_x x + \hat{\mathbf{y}} K_y y$). Other higher-multipole devices produce a force which depends on a high-order power of the transverse coordinates [2,3].

In plasma-based optical devices (plasma lenses [4–6]), the expression of the transverse force is composed of a main part, which is quadrupole-like, and a secondary part, which is high-order-multipole-like. However, there are some regimes in which the force is quasilinear [5].

The high-order multipole components of the transverse force introduce the aberrations in the optical accelerator lattice (transverse aberrations, which, when there is a cylindrical symmetry, are often called spherical aberrations), but usually they are only relatively small corrections of the quadrupole ones.

Since the transverse-density profile of the beam is involved in the transverse beam dynamics, the spherical aberrations cause a dependence of the focal length on the transverse density profile of the beam. This means that the determination of the real transverse density, after passing through an optical device, is required, because it would be quite valuable for estimating the aberration corrections and, consequently, the luminosity at the interaction point in the colliders. Thus an analysis of changes that occur on the transverse density profile of a relativistic-charged-particle beam, when it propagates through a quadrupolelike device with small high-order multipole corrections, is needed.

Recently a *thermal-wave model for relativistic-charged-particle-beam propagation* has been proposed [7]. It successfully recovers the usual results of the relativistic-charged-particle-beam optics, and it seems to be useful for describing several problems in particle accelerators, such as luminosity estimates and the self-interaction (nonlinear interaction) of the beam with the surrounding medium (e.g., the wake-field interaction in conventional accelerators as well as in the current accelerator scheme). In particular, this model has been successfully applied to an ideal quadrupolelike lens in terms of a wave description (of thermal nature) of the transverse beam dynamics, with a harmoniclike potential [7]. Moreover, this wave model has also been applied to the nonlinear beam-plasma interaction, in terms of a self-consistent description of the interaction between the plasma wake fields and the driving relativistic electron (positron) beam in a collisionless, unmagnetized, overdense plasma [8]. Remarkably, in this framework the main results for the beam filamentation threshold and the self-pinching equilibrium condition were reproduced and, at the same time, the reaction of the wake field and the spatial evolution of the beam were considered in a self-consistent way [8].

In this paper we use the *thermal wave model* in order to study the spherical aberration deviations to the octupole order, when a relativistic-charged-particle beam passes through an optical system, such as a quadrupolelike lens with small octupole deviation. In Sec. II we briefly summarize the main features of the thermal-wave model. In Sec. III we define our problem, which is formally analogous to a time-dependent quantum problem, in terms of the so-called *beam wave function* (BWF), whose squared modulus gives the transverse density profile of the beam at any location. The aberrationless limit is reviewed and discussed in Sec. IV. In Sec. V, by taking into account the spherical aberrations, we employ the usual perturbation theory to first order in the perturbative expansion, giving a solution in the thin-lens approximation. Finally, Sec. VI summarizes the conclusions.

II. A BRIEF PRESENTATION OF THE THERMAL WAVE MODEL

This model has been proposed in order to describe the dynamics of a relativistic-charged-particle beam that interacts with the surrounding medium and, at the same time, suffers the beam-emittance spreading (thermal effect). In particular, if ϵ is the transverse emittance of a relativistic beam traveling along the z axis with velocity βc ($\beta \approx 1$), under the action of a potential $u(r,z,\phi)$ (r, z, and ϕ being the cylindrical coordinates), the transverse beam dynamics is governed by the following Schrödinger-like equation for a complex wave function $\Psi(r,z,\phi)$, called the beam wave function [7]:

$$i\epsilon\frac{\partial\Psi}{\partial z} = -\frac{\epsilon^2}{2}\nabla_\perp^2\Psi + U(r,z)\Psi \ , \tag{1}$$

where $\nabla_\perp$ is the two-dimensional transverse gradient, and

$$U(r,z,\phi) = \frac{u(r,z,\phi)}{m_0\gamma\beta^2c^2} \ , \tag{2}$$

m_0 and $\gamma = (1-\beta^2)^{-1/2}$ being the particle rest mass and the relativistic gamma factor, respectively.

Let us denote by $\sigma(r,z,\phi)$ and N the transverse density number of the beam (i.e., the number of particles per unit transverse section) and the total number of particles, respectively; thus the meaning of Ψ is given by the following relationship:

$$\sigma(r,z,\phi) = N|\Psi(r,z,\phi)|^2 \ , \tag{3}$$

where the following normalization for Ψ has been provided:

$$\int_0^{2\pi} d\phi \int_0^\infty |\Psi|^2 r\, dr = 1 \ . \tag{4}$$

Equation (3) means that $|\Psi|^2$ gives the transverse density profile of the beam.

In general, (1) must be coupled with the field-force equation of the system:

$$\mathbf{F}_\perp = -m_0\gamma\beta^2c^2\nabla_\perp U = -\nabla_\perp u \ . \tag{5}$$

The pair of coupled equations (1) and (5) describes the evolution of the particle beam, and also represents a *wave description* for the charged-particle-beam optics in paraxial approximation, as recently pointed out [7].

Two kinds of remarkable analogies between this wave description, on one hand, and the nonrelativistic quantum mechanics and electromagnetic beam optics in paraxial approximation, on the other hand, have been pointed out [7]. In the quantum analogy ϵ and z represent Planck's constant and time, respectively, while in the electromagnetic analogy ϵ and U correspond to the inverse of the wave number k (diffraction parameter) and the refractive index, respectively.

Furthermore, by defining the beam radius

$$R(z) = \left[\int_0^{2\pi} d\phi \int_0^\infty r^2|\Psi|^2 r\, dr\right]^{1/2} , \tag{6}$$

and the averaged total linear momentum

$$P(z) = \left[\frac{\epsilon^2}{2}\int_0^{2\pi} d\phi \int_0^\infty |\nabla_\perp\Psi|^2 r\, dr\right]^{1/2} , \tag{7}$$

an *uncertainty principle,* fully similar to the uncertainty principle noted in quantum mechanics, holds [7]:

$$PR \geq \epsilon \ . \tag{8}$$

By introducing the radial and azimuthal components of P, P_r, and P_ϕ, respectively, defined as

$$P_r(z) = \frac{dR}{dz} \ , \tag{9}$$

$$P_\phi(z) = (P^2 - P_r^2)^{1/2} \ , \tag{10}$$

we also have (for more details see [7])

$$PR \geq P_\phi R = \text{const} \ . \tag{11}$$

We observe that in general $P_\phi R \geq \epsilon$. According to a well-known quantum theorem, $P_\phi R = \epsilon$ is possible only if the transverse beam profile is Gaussian (the aberrationless case discussed in Sec. IV); for any other profile, $P_\phi R > \epsilon$. As has been already pointed out, if a pure Gaussian beam enters an optical system and at the exit it verifies the last inequality, this means that aberrations have occurred and the entropy of the system has been increased [7]. Correspondingly, when a particle beam enters a lens, the aberrations, which occur when the force on the beam is anharmonic, cause an increase of the radius at the exit with respect to the aberrationless radius value.

Spherical aberrations play a very important role in the luminosity estimates during the final focusing in linear colliders. In the framework of the thermal-wave model, we define the *local luminosity* (the number of events per unit cross section and per unit time) at the interaction point ($z = z^*$) for two identical colliding beams by

$$\Lambda^* \equiv \frac{2\nu N_1 N_2}{4}|\Psi(r,\phi,z^*)|^2 \ , \tag{12}$$

where ν is the repetition rate, N_1 and N_2 are the beam populations, and $\Psi(r,\phi,z^*)$ is the total BWF of the system at the interaction point. In (12) the factor 2 before ν is due to the relative longitudinal motion of the two colliding beams, while the numerical factor 4 in the denominator accounts for the overlapping of the beam wave functions. Consequently, the spherical aberrations can be straightforwardly described by means of $|\Psi|^2$, which, in turn, is found by solving an appropriate Schrödinger-like equation with an anharmoniclike potential $U(r,z)$, as done in Sec. V.

III. DEFINITION OF THE PROBLEM

We want to describe, in cylindrical symmetry, the aberration effect when a pure Gaussian charged-particle beam enters an optical system where an anharmonic potential $U(r,z)$ acts on the beam, such as

$$U(r,z) = \tfrac{1}{2}Kr^2 - \lambda r^4, \quad \text{for } 0 \leq z \leq l \tag{13}$$

where l is the longitudinal length of the optical system, K

is the focusing strength of a harmonic potential (aberrationless potential), and $-\lambda r^4$ accounts for the aberration deviation due to a nonlinear term in the transverse force acting on the beam (*octupole deviation* of a quadrupole).

At the exit of such an optical system, the output transverse beam density $\sigma(r,z>l)$ has to be found. We will assume that the thermal-wave model for relativistic-charged-particle-beam propagation is valid. This model was synthetically described in the preceding section. We will consider a charged-particle beam passing through an optical system of longitudinal length l. Consequently, while the beam is traveling along z, the potential has the following z dependence:

$$U(r,z)=\begin{cases}\frac{1}{2}Kr^2-\lambda r^4 \quad \text{for } 0\le z\le l & (14)\\ 0 \quad \text{for } z<0,\quad z>l\ . & (15)\end{cases}$$

In the quantum analogy, (1) can be solved with the well-known perturbative techniques. The application limit of the perturbation theory is discussed below.

IV. ABERRATIONLESS CASE

First of all, let us consider for simplicity the case $\lambda=0$ (unperturbed case). Outside the lens ($z<0$ and $z>l$) the beam propagates in vacuum. In this region, the solutions for BWF, with given initial beam radius $[R(z=0)\equiv R_0]$ and wave-front curvature radius at $z=0$ $[\rho(z=0)\equiv\rho_0]$, are Gaussian-like and given in [7]. On the other hand, inside the lens we have to solve the following equation:

$$i\epsilon\frac{\partial}{\partial z}\Psi^{(0)}=-\frac{\epsilon^2}{2}\nabla_\perp^2\Psi^{(0)}+\tfrac{1}{2}Kr^2\Psi^{(0)}\ . \tag{16}$$

The superscript (0) denotes $\lambda=0$. Due to the presence of the harmonic potential $U(r)=\frac{1}{2}Kr^2$, the possible solutions for the BWF are modified with respect to the propagation in vacuum, and, while remaining Gaussian-like, they turn out to be

$$\Psi_n^{(0)}(r,z)=\frac{\exp\left[-\dfrac{r^2}{2R^2(z)}\right]}{\sqrt{\pi}R(z)}L_n\left[\frac{r^2}{R^2(z)}\right]\times\exp\left[i\frac{r^2}{2\epsilon\rho(z)}\right]\exp[i\phi_n(z)]\ , \tag{17}$$

where $L_n(x)$ are the normalized Laguerre polynomials with $n=0,1,2,3,\ldots$,

$$R^2(z)=R_0^2\left[\left[\cos\sqrt{K}z+\frac{1}{\sqrt{K}\rho_0}\sin\sqrt{K}z\right]^2+\frac{\epsilon^2}{KR_0^4}\sin^2\sqrt{K}z\right]\ , \tag{18}$$

and

$$\phi_n(z)=-(2n+1)\left\{\arctan\left[\left[\frac{R_0^2}{\epsilon\sqrt{K}\rho_0^2}+\frac{\epsilon}{\sqrt{K}R_0^2}\right]\times\tan(\sqrt{K}z)+\frac{R_0^2}{\epsilon\rho_0}\right]-\arctan\left[\frac{R_0^2}{\epsilon\rho_0}\right]\right\}\ , \tag{19}$$

and

$$\frac{1}{\rho(z)}=\frac{1}{2R^2(z)}\frac{dR^2(z)}{dz} \tag{20}$$

is the curvature radius of the wave front of the BWF. Note that in (19) we have set $\phi_n(z=0)=0$.

We assign the form of the BWF at $z=0$ (initial condition) and then solve (16) inside the lens ($z\le l$). Let us choose as an initial condition a simple Gaussian-like beam traveling along the z axis in the positive direction. This means

$$\Psi^{(0)}(r,0)=\frac{\exp\left[-\dfrac{r^2}{2R_0^2}\right]}{\sqrt{\pi}R_0}\exp\left[i\frac{r^2}{2\epsilon\rho_0}\right]\ , \tag{21}$$

where R_0 and ρ_0 are given. With this initial condition, the solution for the BWF inside the lens is given by $\Psi_0^{(0)}(r,z)$.

It is worth observing that the BWF remains Gaussian. This property is connected to the linearity of the transverse force, i.e., no distortions are introduced in the transverse profile (aberrationless case). Moreover, by using (18) and (20) we get the values of the beam radius and the curvature radius at the exit of the lens ($z=l$) in the thin-lens approximation:

$$R_1\equiv R(z=l)\approx R_0\ ,$$

$$\rho_1\equiv\rho(z=l)\approx\rho_0\frac{1+2\dfrac{l}{\rho_0}}{1+\dfrac{l}{\rho_0}+\dfrac{l\rho_0}{\beta_0^2}-\rho_0Kl}\ . \tag{22}$$

In Eq. (22), β_0 stands for the ratio R_0^2/ϵ. Note that $|\rho_1|$ represents the corresponding value of the focal length f (a negative sign for ρ_1 corresponds to a focusing lens). Let us choose for the following, $\rho_1<0$. Taking the limit as $\rho_0\to\infty$ and $\epsilon\to0$, we easily recover the well-known thin-lens equation [3].

Outside the lens ($z>l$), in vacuum (v), the BWF remains functionally unmodified with respect to $\Psi_0^{(0)}(r,z)$, the only difference being that the expression for the beam radius is given by

$$R_v^2(z)=R_0^2\left[\left[1+\frac{\beta_0^2}{f^2}\right]\frac{(z-l)^2}{\beta_0^2}-\frac{(z-l)}{f}+1\right]\ . \tag{23}$$

Note that the next waist of the beam caustic (interaction point) would be obtained by imposing the condition $(dR_v/dz)_{z=z^*}=0$, which gives

$$z^* = l + \frac{\dfrac{\beta_0^2}{f}}{1+\dfrac{\beta_0^2}{f^2}} . \tag{24}$$

Thus, (12) gives

$$\Lambda^{(0)*} = \frac{\nu N_1 N_2}{2\pi R^{*2}} \exp\left[-\frac{r^2}{R^{*2}} \right] , \tag{25}$$

where $R^* \equiv R_v(z=z^*)$ is the beam radius at the interaction point.

It is useful to introduce the *effective luminosity (averaged luminosity),* which is defined as

$$\mathcal{L}^{(0)*} \equiv \langle \Lambda^{(0)*} \rangle = 2\pi \int_0^\infty \Lambda^{(0)*}(r,z^*) |\Psi^{(0)}(r,z^*)|^2 r\, dr . \tag{26}$$

In particular, Eq. (25) gives

$$\mathcal{L}^{(0)*} = \frac{\nu N_1 N_2}{4\pi R^{*2}} . \tag{27}$$

This result recovers the effective luminosity at the interaction point for two identical Gaussian beams reported in [9].

V. OCTUPOLE CORRECTIONS

A. Inside the lens

Using the quantum notation, we now write the solutions of (1), with $U(r,z)$ given by (14) for $0 \le z \le l$ and for $\lambda \neq 0$, as the following expansion:

$$\Psi(r,z) = \sum_{n=0}^{\infty} c_n(z) \Psi_n^{(0)}(r,z) , \tag{28}$$

where $\Psi_n^{(0)}(r,z)$ is the BWF of the nth mode for the unperturbed case ($\lambda=0$, the aberrationless case) given by (17). Since (1) cannot be solved exactly, we apply a perturbation method to solve for the BWF.

By substituting the expansion (28) in (1) and following the usual time-dependent perturbation method for quantum mechanics [10], we easily get, up to first-order approximation,

$$c_n(z) \approx \delta_{n,0} + \left[\frac{i\lambda}{\epsilon} \right] \int_0^z \langle n | r^4 | 0 \rangle dz' , \tag{29}$$

where

$$\langle n | r^4 | m \rangle = 2\pi \int_0^\infty \Psi_n^{(0)*} r^4 \Psi_m^{(0)} r\, dr . \tag{30}$$

By using Eqs. (17)–(20), Eq. (30) for $m=0$ becomes

$$\langle n | r^4 | 0 \rangle = R^4(z) \exp\{ -i[\phi_n(z) - \phi_0(z)] \} \times (2\delta_{n,2} - 4\delta_{n,1} + 2\delta_{n,0}) . \tag{31}$$

This allows us to compute the coefficients of the expansion (28) to the first-order approximation, at the end of the lens, in the thin-lens approximation ($\sqrt{K}\, l \ll 1$):

$$\begin{aligned} c_0(l) &\approx 1 + 2i\tau , \\ c_1(l) &\approx -4i\tau , \\ c_2(l) &\approx 2i\tau , \\ c_n(l) &= 0 \quad \text{for } n > 2 , \end{aligned} \tag{32}$$

where the parameter τ is defined by the equation $\tau = \lambda l R_0^4/\epsilon$. This way, we can find the BWF at the exit of the lens (output beam wave function):

$$\Psi(r,l) = \frac{1}{\mathcal{N}} \sum_{n=0}^{2} c_n(l) \Psi_n^{(0)}(r,l) , \tag{33}$$

with $\mathcal{N}^2 = \sum_{n=0}^{2} |c_n(l)|^2 = 1 + 24\tau^2$. Thus, for a thin lens, we get

$$\begin{aligned} \Psi(r,l) &= \frac{\exp\left[-\dfrac{r^2}{2R_1^2} \right]}{\sqrt{\pi} R_1 (1+24\tau^2)^{1/2}} \exp\left[-i \frac{r^2}{2\epsilon f} \right] \times \left[(1+2i\tau) L_0 \left[\frac{r^2}{R_1^2} \right] - 4i\tau L_1 \left[\frac{r^2}{R_1^2} \right] + 2i\tau L_2 \left[\frac{r^2}{R_1^2} \right] \right] \\ &= \frac{\exp\left[-\dfrac{r^2}{2R_1^2} \right]}{\sqrt{\pi} R_1 (1+24\tau^2)^{1/2}} \exp\left[-i \frac{r^2}{2\epsilon f} \right] \left[1 + i\tau \frac{r^4}{R_1^4} \right] . \end{aligned} \tag{34}$$

The application limit of the perturbative expansion can be easily obtained by requiring $|\lambda| R_0^4 \ll K R_0^2/2$. Thus, by using (22), we obtain the condition $|2\tau f/\beta_0| \ll 1$.

B. Outside the lens

For $z > l$, the beam propagates again in vacuum and the corresponding BWF must satisfy Eq. (1) with initial condition represented by (34). Thus we get

$$\Psi(r,z) = \frac{\exp\left[-\dfrac{r^2}{2R_v^2(z)} \right]}{\sqrt{\pi} R_v(z) [1+24\tau^2]^{1/2}} \exp\left[i \frac{r^2}{2\epsilon \rho_v(z)} + i\phi_v(z) \right] \times \left[(1+2i\tau) L_0 \left[\frac{r^2}{R_v^2(z)} \right] - 4i\tau \exp[i2\phi_v(z)] L_1 \left[\frac{r^2}{R_v^2(z)} \right] + 2i\tau \exp[i4\phi_v(z)] L_2 \left[\frac{r^2}{R_v^2(z)} \right] \right] , \tag{35}$$

where

$$\phi_v(z)=\arctan\left[\frac{\beta_0}{f}-\left[1+\frac{\beta_0^2}{f^2}\right]\frac{(z-l)}{\beta_0}\right]-\arctan\left[\frac{\beta_0}{f}\right],$$

$$\frac{1}{\rho_v(z)}=\frac{1}{R_v(z)}\frac{dR_v(z)}{dz}. \tag{36}$$

By substituting (24) in (36), we obtain the phase ϕ_v at the interaction point:

$$\phi_v^*\equiv\phi_v(z=z^*)=-\arctan\left[\frac{\beta_0}{f}\right]. \tag{37}$$

Figure 1 shows the density profile at the interaction point $\pi R^{*2}|\Psi(r,z^*)|^2$, considered as a function of r/R^*, and computed for several values of τ and for $f/\beta_0=0.5$. Note that at this point the beam is no longer purely Gaussian, because the octupole component of the force pushes the particles outward in such a way as to modify the transverse-density profile.

By analogy with (26), and by using Eq. (35), we introduce the effective luminosity in the presence of spherical aberrations ($\lambda\neq0$) as

$$\mathcal{L}^*\equiv 2\pi\int_0^\infty \Lambda^*(r,z^*)|\Psi(r,z^*)|^2 r\,dr , \tag{38}$$

where

$$\Lambda^*=\frac{\nu N_1 N_2}{2}|\Psi(r,z^*)|^2 . \tag{39}$$

In order to estimate both the variation of the beam radius and the variation of the effective luminosity connected with the presence of the aberrations, we define the spot size

$$\left[\frac{\Delta R_v}{R_v}\right]_{z=z^*}\equiv\frac{\left[2\pi\int_0^\infty r^2|\Psi(r,z^*)|^2 r\,dr\right]^{1/2}}{R^*}-1 \tag{40}$$

and the luminosity reduction factor

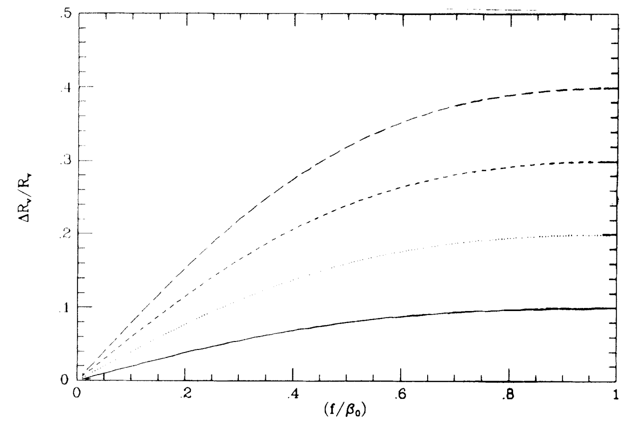

FIG. 2. Spot size vs f/β_0 for $\tau=0.025$ (solid line), $\tau=0.05$ (dotted line), $\tau=0.075$ (short-dashed line), $\tau=0.1$ (long-dashed line).

$$\mathcal{R}=\frac{\mathcal{L}^*}{\mathcal{L}^{(0)*}} . \tag{41}$$

Consequently, by using (23), (35), and (37), we obtain

$$\left[\frac{\Delta R_v}{R_v}\right]_{z=z^*}\approx 8\tau\frac{f/\beta_0}{1+(f/\beta_0)^2}, \tag{42}$$

and

$$\mathcal{R}\approx 1-8\tau\frac{(f/\beta_0)[1+5(f/\beta_0)^2+7(f/\beta_0)^4+3(f/\beta_0)^6]}{[1+(f/\beta_0)^2]^4} . \tag{43}$$

The expressions (42) and (43) have been given in a way that is consistent with a first-order approximation in perturbation theory.

Figures 2 and 3 plot the spot size and the luminosity reduction factor, respectively, at the interaction point versus (f/β_0) for several values of τ. We can immediately understand that the enhancement of the beam radius

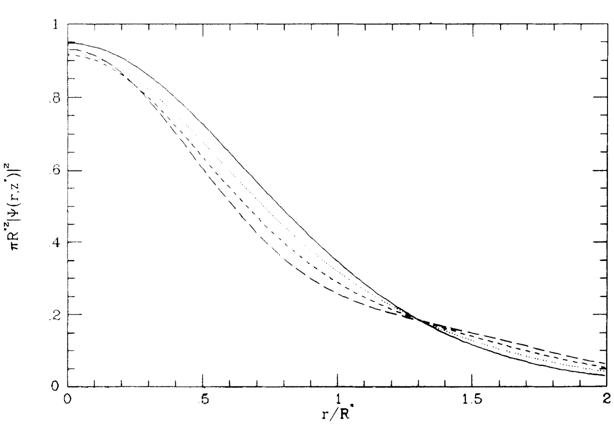

FIG. 1. Beam density profile at the interaction point vs r/R^*, for $f/\beta_0=0.5$ and $\tau=0.025$ (solid line), $\tau=0.05$ (dotted line), $\tau=0.075$ (short-dashed line), $\tau=0.1$ (long-dashed line).

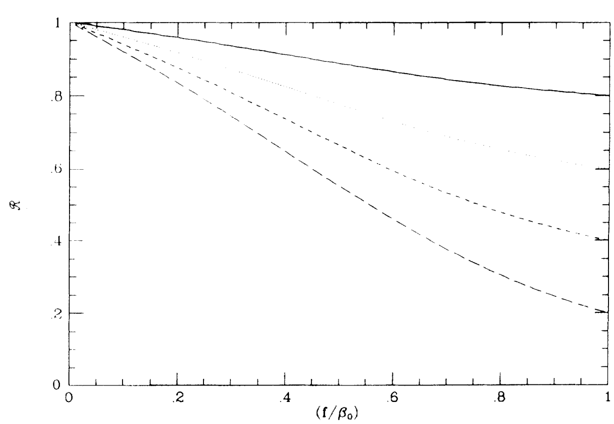

FIG. 3. Luminosity-reduction factor vs f/β_0 for $\tau=0.025$ (solid line), $\tau=0.05$ (dotted line), $\tau=0.075$ (short-dashed line), $\tau=0.1$ (long-dashed line).

and the corresponding reduction of the luminosity are due to the displacement of a part of the total beam particles outward because of the aberrations.

VI. CONCLUSIONS

In this paper, we have presented an application of the recently proposed thermal-wave model for relativistic-charged-particle-beam propagation [7] to the luminosity estimates in a linear collider when spherical aberrations are taken into account. We have considered the propagation of a relativistic-electron beam through a thin quadrupolelike lens with octupole deviations. Such propagation has been described by a Schrödinger-like equation, with an anharmonic potential well, for a complex wave function whose squared modulus gives the transverse-density profile of the beam. The solution of this equation has been found starting from a pure Gaussian initial-particle space distribution, and by following the usual time-dependent perturbation theory of quantum mechanics up to first order. Consequently, we have determined the proper transverse space-distribution function of the beam particles at the interaction point. Due to the presence of spherical aberrations, it results in a superposition of the initial fundamental mode plus the first two excited modes of the aberrationless case. This has allowed us to calculate, for the final focusing stage of a linear collider, both the spot size and the luminosity-reduction factor (with respect to the aberrationless case) at the interaction point.

We remark that, according to the perturbation theory, our results should be considered under the condition $|2\tau f/\beta_0| \ll 1$, which guarantees the convergence of the perturbative expansion and the reliability of the first-order results. We stress that the theoretical approach presented in this paper concerns a purely transverse dynamics. A more careful analysis of the luminosity should also take into account chromatic aberrations, which are considered negligible in this approach because they involve the longitudinal dynamics. Thus a three-dimensional extended thermal-wave model is needed for a more general treatment of the aberration effects.

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