Fluctuation-induced forces between manifolds immersed in correlated fluids

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We develop a general field-theoretical approach for computing fluctuation-induced forces between manifolds immersed in a correlated fluid. Both isotropic, e.g., a superfluid or critical binary mixture, and anisotropic, e.g., nematic or smectic liquid crystals, are considered. The effects of different types of boundary conditions are explored, and in particular the role of deformations from perfect geometries is studied. Specific results include the following: The Casimir force between a flat and a self-affinely rough surface acquires a correction term that decays with the average separation of the plates through an exponent related to the roughness of the boundary. Surface fluctuations in films of correlated fluids may be enhanced, or suppressed by such forces, depending on whether the boundaries are alike or different. We also compute the resulting two-body force between a line (directed polymer) and a surface, and the three-body interaction (repulsive) between three lines.

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I. INTRODUCTION AND SUMMARY

It is naturally quite common to find simple particles or more extended objects, such as polymers or membranes, in a complex medium. The fluctuations of the medium are then modified by the presence of such objects, and in turn induce effective interactions among them. Curiously, the most well-known manifestation of this phenomenon is the Casimir force [1] between parallel conducting plates, due to *quantum fluctuations* of the electromagnetic field in vacuum. However, similar forces arise due to *thermal fluctuations* for external bodies immersed in a classical fluid. At a temperature T, the magnitude of such interactions is proportional to $k_B T$, while their range depends on the extent of correlations in the fluid. For a fluid with long-range correlations (e.g., a binary mixture close to the critical point, a superfluid, or a liquid crystal [2,3]), the forces are long ranged with *universal* amplitudes. We shall generically label all such fluctuation-induced interactions Casimir forces. Thermal Casimir effects are closely related to finite-size corrections to the free energy [4] and have also been discussed in connection with wetting close to a tricritical point [5], unbinding of fluid membranes in liquid crystals [3,6], and the elongation of surface domains during epitaxial growth [7]. (We wish to contrast forces induced by fluctuations of the surrounding medium to the closely related forces due to fluctuations of a component of the external body. For example, the van der Waals or London dispersion forces are due to quantum fluctuations in the molecular dipoles [8]; their thermal analogs are the Keesom forces in colloidal physics [9]. There are also entropically induced interactions due to fluctuations of surfaces [10] or lines [11].)

Again somewhat curiously, the most familiar classical example of Casimir forces appears in the rather complicated context of critical finite-size effects. Following the original work of Fisher and de Gennes [12], several authors [13,14] have considered the free energy of a critical system (e.g., a binary mixture at T_c) between two boundaries separated by H. On dimensional grounds it is natural to expect a contribution to the free energy per unit area $f(H)$ of the form $f(H)/kT=\Delta/H^{d-1}$, in d space dimensions. The interesting feature of this expression is that the amplitude Δ is a universal number, depending only on the universality class of the critical system and the type of boundary conditions imposed. The strength of the boundary couplings is irrelevant. In a simple lattice-gas description, the fluid can be mapped to a spin system (e.g., an Ising model for the binary mixture), and the boundary couplings are described by magnetic fields h_1 and h_2 applied to the two edges. Then, by symmetry, there are only four possible values for Δ: $\Delta_{++}=\Delta_{--}$ for $h_1 h_2>0$; Δ_{00} for $h_1=h_2=0$; Δ_{+-} for $h_1 h_2<0$; and $\Delta_{0+}=\Delta_{0-}$ for $h_1=0$, $h_2\neq 0$, independent of the magnitudes of h_1 and h_2. The amplitudes Δ are closely related to conformal charges of the critical theories, and in two dimensions their exact values can be obtained by employing techniques of conformal field theories [14]. In higher dimensions, they can be estimated numerically [15], and by $\epsilon=4-d$ expansions [16]. In general, Δ is negative for like boundaries ($++$, $--$, and 00), while positive for unlike boundaries ($+-$, $+0$, and -0), implying that like boundaries attract, while unlike boundaries repel, a general feature of fluctuation-induced forces.

In this paper we consider the more general problem of fluctuation-induced forces between extended objects (polymers, membranes, etc.) immersed in a correlated fluid. The two elements that enter this problem are the following: (1) The correlated fluid does not have to be a critical mixture, a rather singular condition, as any system with a continuously broken symmetry and the associated Goldstone modes will suffice. Possible examples are a superfluid, or a liquid crystal, as described in the next paragraph. (2) The external objects interact with the fluid that envelops them and change its fluctuations in their vicinity. An important aspect of our study is going beyond the simple geometries (straight edges and flat

plates) considered in previous studies by looking at rough and deformed structures. Such nonstandard geometries have also been examined by a multiple-scattering approach [17], but the resulting perturbative series is not particularly illuminating. Here we develop a field-theoretical approach that allows us to compute fluctuation-induced forces in a relatively simple way. This approach has a number of advantages. First, different manifolds (with arbitrary intrinsic and embedding dimensions) in various correlated fluids can be treated in a similar fashion. Second, the boundary conditions are quite easily implemented. Finally, the corrections due to "roughness" can be computed perturbatively in the deformations.

The first element requires a description for the correlated fluid. The simplest example is an isotropic fluid with a broken continuous symmetry. A superfluid, with an order parameter

$$\psi(\mathbf{r})=\sqrt{\rho(\mathbf{r})}\exp[i\phi(\mathbf{r})] ,$$

provides an ideal candidate. At low temperatures the local density $\rho(\mathbf{r})$ is approximately constant, while the phase angle $\phi(\mathbf{r})$ is a massless Goldstone mode which in a d-dimensional space is subject to the simple Hamiltonian

$$\mathcal{H}_0[\phi]=\int d^d\mathbf{r}\frac{K}{2}(\nabla\phi(\mathbf{r}))^2 . \tag{1.1}$$

In general, the number of Goldstone modes and their interactions depend on the nature of the broken symmetry, but their fluctuations are quite generically long ranged. A fluid at criticality (e.g., a binary mixture at its demixing point) is more complicated and not describable by a quadratic Hamiltonian, but we expect it to exhibit the same *qualitative* features in its response to boundaries. Liquid crystals provide anisotropic examples of correlated fluids due to broken symmetry. They also provide an easily accessible system, since experiments can be performed at room temperature, and they require no fine tuning to achieve criticality. The order parameter of a *nematic* liquid crystal is a director field $\mathbf{n}(\mathbf{r})$, characterizing the local preferred direction of the long axis of the molecules [18]. The energy cost of fluctuations of the nematic director $\mathbf{n}(\mathbf{r})$ is given by [18]

$$\mathcal{H}_N=\tfrac{1}{2}\int d^3\mathbf{r}[\kappa_1(\nabla\cdot\mathbf{n})^2+\kappa_2(\mathbf{n}\cdot\nabla\times\mathbf{n})^2 +\kappa_3(\mathbf{n}\times\nabla\times\mathbf{n})^2] . \tag{1.2}$$

In a smectic liquid crystal, the molecules segregate into layers which are fluidlike. The fluctuations of these layers from perfect stacking are described by a scalar deformation $u(\mathbf{x},z)$, which in d dimensions is subject to a Hamiltonian

$$\mathcal{H}_S=\tfrac{1}{2}\int d^d\mathbf{r}\left[B\left[\frac{\partial u}{\partial z}\right]^2+\kappa(\nabla^2 u)^2\right] . \tag{1.3}$$

Both Hamiltonians are explicitly free of a "mass" term, and hence lead to long-range correlations. The fluctuation-induced forces between two parallel plates due to such liquid crystals were previously calculated by Mikheev [2] and by Ajdari, Peliti, and Prost [3], who indeed find interesting power-law interactions.

For the second element of our description, we need to specify how the external bodies modify the fluid fluctuations. The effects of such couplings are implemented as boundary conditions, and we distinguish between the following two possibilities: In type-I boundaries the fluid fluctuations are suppressed, e.g., by strong anchoring for liquid crystals, or by substrates that prefer one of the coexisting fluids in a critical mixture (a magnetic field in the spin analogy). Type-II boundaries correspond to the suppression of the normal gradient of fluctuations, as on an open surface. Type-II boundaries are also appropriate for a superfluid, where the normal velocity vanishes at the substrate. Anticipating the universality of fluctuation-induced forces with respect to the strength of the boundary couplings, we implement their effects by requiring either the field ϕ (type I), or its normal derivative $\partial_\perp\phi$ (type II), to vanish on the surface of the external bodies. Most computations of Casimir forces are for simple geometries, e.g., between two parallel plates. It is natural to consider how these forces are modified by the roughness that is present in most "random" surfaces. The interactions can in turn alter the thermal fluctuations of a fluid surface, or a liquid-crystal film. The main purpose of this work is to develop a general approach for computing the corrections to fluctuation-induced forces in the presence of deformations.

Many commonly encountered forms of "randomness" in nature in fact exhibit self-similar or fractal [19,20] scaling. A particularly amenable form of rough boundary for our study is the self-affine surface which can be described by a single-valued height function $h(\mathbf{x})$ with respect to the transverse coordinate $\mathbf{x}$. A random self-affine surface is described by a probability distribution that is invariant under a transformation $h\to\lambda h$, $\mathbf{x}\to\lambda^{\zeta_S}\mathbf{x}$, i.e., the surface is self-similar under anisotropic rescaling. The self-similar fluctuations in the height grow with distance as

$$\overline{[h(\mathbf{x})-h(\mathbf{y})]^2}=A_S|\mathbf{x}-\mathbf{y}|^{2\zeta_S} , \tag{1.4}$$

i.e., characterized by a *roughness exponent* $0<\zeta_S<1$ and an amplitude A_S. Equilibrium thermal fluctuations of films or membranes, subject to surface tension or bending energies, quite generally lead to such height-height correlations [21]. Many solid surfaces produced by rapid growth [22] or etching processes [23] also exhibit self-affine correlation. There is indeed both theoretical [24] and numerical [25] support for the appearence of self-affinity in nonequilibrium growth processes. In both equilibrium and nonequilibrium circumstances, the theoretical models predict a *universal* exponent ζ_S which reflects the underlying physics, but a nonuniversal, material-dependent amplitude A_S in Eq. (1.4). It is thus of both practical and theoretical importance to measure the roughness exponent ζ_S. The most direct method of measuring this exponent is by direct imaging of the surface with scanning tunneling microscopy (STM) [23,26]. However, the size of such scanned images is usually limited, and it is useful to complement this approach with more macroscopic probes such as examining adsorption

isotherms [22,27]. In this paper we propose accurate measurements of fluctuation-induced forces between rough surfaces as an alternative macroscopic probe of self-affinity. We find a correction to the leading Casimir force that decays with the average separation of the two plates through an exponent related to ζ_S. The force generated by this term is within the sensitivity of current force apparatus [28] and can potentially be used to measure the roughness exponent.

In addition to the fixed surfaces of a rough solid substrate, we can also examine the spontaneous deformations of a fluid-air interface. Since such deformations also modify the correlations of the fluid, their energy cost is modified by the Casimir forces. We find that for "like" boundaries, as in the case of a free-standing liquid-crystal film, the interface fluctuations are enhanced to the lowest order, while for "unlike" boundaries, e.g., a smectic liquid-crystal film on a solid substrate, they are suppressed. A brief description of our method and these results appeared earlier [29].

The paper is organized as follows. In Sec. II we derive a general formula for the effective free energy of n immersed manifolds with arbitrary intrinsic and embedding dimensions, subject to type-I boundary conditions. This expression is obtained by directly integrating out the fluid degrees of freedom, with the given constraint at the boundaries. Explicit results are then computed for an isotropic fluid confined between two surfaces, one deformed. Particular emphasis is given to the interaction between self-affine and flat surfaces. In Sec. III we generalize the derivation to include different types of boundary conditions and correlated fluids. Explicit results for anisotropic fluids (liquid crystals in nematic or smectic phase) are presented. We discuss how the calculations can be improved for a critical fluid which cannot be described by a Gaussian Hamiltonian. In Sec. IV we consider other geometries, such as a line and a surface, and also discuss fluctuation-induced many-body interactions. We conclude by pointing out some other problems that may be studied by the methods developed in this paper.

II. THE METHOD

In this section we first derive a general formula for fluctuation-induced forces between arbitrary manifolds immersed in a correlated fluid. The fluid fluctuations are described by a one-component field ϕ subject to the quadratic action in Eq. (1.1). The external manifolds suppress these fluctuations as in type-I boundary conditions. After the details of our approach have been described for this case, their generalization to more complex conditions, as presented in Sec. III, is relatively straightforward.

Consider n manifolds embedded in a fluid, each described by coordinates $\mathbf{r}_\alpha(x_\alpha)$. Here, x_α is a D_α-dimensional internal coordinate for the αth manifold ($D_\alpha=1$ for a polymer and $D_\alpha=2$ for a membrane), and $\mathbf{r}_\alpha$ indicates its position in the d-dimensional fluid [30]. The fluctuation-induced interactions between the manifolds are obtained by integrating over all configurations of the field ϕ, with the constraints imposed by the external manifolds. Type-I boundary conditions correspond to the constraints $\phi(\mathbf{r}_\alpha(x_\alpha))=0$, for $\alpha=1,2,\ldots,n$, which can be imposed by inserting δ functions. Using the integral representation of δ function, we obtain

$$\exp\left[-\frac{\mathcal{H}_{\rm eff}[\mathbf{r}_\alpha(x_\alpha)]}{kT}\right]=\frac{1}{Z_0}\int \mathcal{D}\phi(\mathbf{r})\prod_{\alpha=1}^{n}\mathcal{D}\psi_\alpha(x_\alpha)\exp\left[-\mathcal{H}_0[\phi]+i\int dx_\alpha\psi_\alpha(x_\alpha)\phi(\mathbf{r}_\alpha(x_\alpha))\right], \tag{2.1}$$

where Z_0 is the partition function for the unperturbed fluid and $\psi_\alpha(x_\alpha)$ are the auxiliary fields defined on the n manifolds, acting as sources coupled to ϕ. After integrating over the field ϕ, we obtain the long-range interactions between the sources as

$$\exp\left[-\frac{\mathcal{H}_{\rm eff}}{kT}\right]=\int\prod_{\alpha=1}^{n}\mathcal{D}\psi_\alpha(x_\alpha)\exp\{-\mathcal{H}_1[\psi_\alpha(x_\alpha)]\}\ . \tag{2.2}$$

The action $\mathcal{H}_1[\psi_\alpha(x_\alpha)]$ for the n-component field $\Psi\equiv(\psi_1,\psi_2,\ldots,\psi_n)$ is given by

$$\mathcal{H}_1[\Psi]\equiv\Psi M\Psi^T=\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\int dx_\alpha dx_\beta\psi_\alpha(x_\alpha)G^d(\mathbf{r}_\alpha(x_\alpha)-\mathbf{r}_\beta(x_\beta))\psi_\beta(x_\beta)\ , \tag{2.3}$$

where $G^d(\mathbf{r})\equiv\langle\phi(\mathbf{r})\phi(0)\rangle_0$ is the two-point correlation function of ϕ in free space. Finally, the effective interaction between the manifolds is obtained as

$$\mathcal{H}_{\rm eff}[\mathbf{r}_\alpha(x_\alpha)]=\frac{kT}{2}\ln\mathrm{Det}\{M[\mathbf{r}_\alpha(x_\alpha)]\}\ . \tag{2.4}$$

The matrix M is a functional of $\mathbf{r}_\alpha(x_\alpha)$ and its determinant is in general difficult to evaluate. It is possible, however, to perturbatively calculate the corrections due to small deformations around simple geometries. As an explicit example, we compute the interaction between two D-dimensional surfaces in $d=D+1$ dimensions, with average separation H and one plate deformed by $h(\mathbf{x})$, i.e., $\mathbf{r}_1(\mathbf{x})=(\mathbf{x},0)$ and $\mathbf{r}_2(\mathbf{x})=(\mathbf{x},H+h(\mathbf{x}))$. The matrix M can be read off from Eq. (2.3) as

$$M(\mathbf{x},\mathbf{y})=\begin{bmatrix} G^d(\mathbf{x}-\mathbf{y},0) & G^d(\mathbf{x}-\mathbf{y},H+h(\mathbf{y})) \\ G^d(\mathbf{x}-\mathbf{y},H+h(\mathbf{x})) & G^d(\mathbf{x}-\mathbf{y},h(\mathbf{x})-h(\mathbf{y}))\end{bmatrix}\ . \tag{2.5}$$

For small deformations, the above matrix can be perturbatively evaluated by expanding the Green's function in terms of deformation $h(\mathbf{x})$:

$$M(\mathbf{x},\mathbf{y})=M_0(\mathbf{x},\mathbf{y})+\delta M(\mathbf{x},\mathbf{y}) ,$$

where $M_0(\mathbf{x},\mathbf{y})$ is the matrix for two flat surfaces and $\delta M(\mathbf{x},\mathbf{y})$ is the correction due to deformations. The matrix $M_0(\mathbf{x},\mathbf{y})$ depends only on the difference $(\mathbf{x}-\mathbf{y})$:

$$M_0(\mathbf{x},\mathbf{y})=\begin{bmatrix} G^d(\mathbf{x}-\mathbf{y},0) & G^d(\mathbf{x}-\mathbf{y},H) \\ G^d(\mathbf{x}-\mathbf{y},H) & G^d(\mathbf{x}-\mathbf{y},0) \end{bmatrix} . \tag{2.6}$$

Hence, it can be diagonalized by transforming to Fourier space, where

$$\tilde{M}_0(\mathbf{p},\mathbf{q})=\begin{bmatrix} \tilde{G}^d(\mathbf{p}) & \tilde{G}^d(\mathbf{p},H) \\ \tilde{G}^d(\mathbf{p},H) & \tilde{G}^d(\mathbf{p}) \end{bmatrix}(2\pi)^D\delta^D(\mathbf{p}+\mathbf{q}) , \tag{2.7}$$

with the Fourier-transformed Green's functions

$$\begin{aligned} \tilde{G}^d(\mathbf{p})&=\int G^d(\mathbf{x},0)\exp(i\mathbf{p}\cdot\mathbf{x})d^D\mathbf{x} , \\ \tilde{G}^d(\mathbf{p},H)&=\int G^d(\mathbf{x},H)\exp(i\mathbf{p}\cdot\mathbf{x})d^D\mathbf{x} . \end{aligned} \tag{2.8}$$

Using Eq. (2.4), the effective free energy can be decomposed as $\mathcal{H}_{\rm eff}=\mathcal{H}_{\rm flat}+\mathcal{H}_{\rm corr}$, where $\mathcal{H}_{\rm flat}=\frac{1}{2}\ln \mathrm{Det}\tilde{M}_0$ and

$$\mathcal{H}_{\rm corr}=\tfrac{1}{2}\ln \mathrm{Det}(1+\tilde{M}_0^{-1}\delta\tilde{M}) .$$

The determinant of $\tilde{M}_0$ can be easily computed, leading to

$$\begin{aligned} \frac{\mathcal{H}_{\rm flat}}{kTA}&=\int\frac{d^D\mathbf{p}}{(2\pi)^D}\ln\tilde{G}^d(\mathbf{p}) \\ &\quad+\tfrac{1}{2}\int\frac{d^D\mathbf{p}}{(2\pi)^D}\ln\left[1-\left[\frac{\tilde{G}^d(\mathbf{p},H)}{\tilde{G}^d(\mathbf{p})}\right]^2\right] , \end{aligned} \tag{2.9}$$

where A is the D-dimensional area. The correction $\mathcal{H}_{\rm corr}$ can be evaluated perturbatively in powers of $h(\mathbf{x})$. At second order we find (for details see the Appendix)

$$\begin{aligned} \frac{\mathcal{H}_{\rm corr}}{kT}&=-(A_1+A_2+A_3)\int d^D\mathbf{x}\frac{h^2(\mathbf{x})}{2} \\ &\quad+\tfrac{1}{4}\int d^D\mathbf{x}d^D\mathbf{y}[h(\mathbf{x})-h(\mathbf{y})]^2 \\ &\quad\times\left\{\frac{\partial^2 G}{\partial z^2}(\mathbf{x}-\mathbf{y},0)F_1(\mathbf{x}-\mathbf{y})\right. \\ &\quad\left.+F_2^2(\mathbf{x}-\mathbf{y})+F_3^2(\mathbf{x}-\mathbf{y})\right\} , \end{aligned} \tag{2.10}$$

with the H-dependent coefficients

$$\begin{aligned} A_1&=\int\frac{d^D\mathbf{p}}{(2\pi)^D}\frac{\tilde{G}^d(\mathbf{p},H)}{\mathcal{N}(\mathbf{p})}\frac{\partial^2\tilde{G}^d(\mathbf{p},H)}{\partial H^2} , \\ A_2&=\int\frac{d^D\mathbf{p}}{(2\pi)^D}\left[\frac{\tilde{G}^d(\mathbf{p},H)}{\mathcal{N}(\mathbf{p})}\frac{\partial\tilde{G}^d(\mathbf{p},H)}{\partial H}\right]^2 , \\ A_3&=\int\frac{d^D\mathbf{p}}{(2\pi)^D}\left[\frac{\tilde{G}^d(\mathbf{p})}{\mathcal{N}(\mathbf{p})}\frac{\partial\tilde{G}^d(\mathbf{p},H)}{\partial H}\right]^2 \end{aligned} \tag{2.11}$$

and the three functions defined as

$$\begin{aligned} F_1(\mathbf{x})&=\int\frac{d^D\mathbf{p}}{(2\pi)^D}\frac{\tilde{G}^d(\mathbf{p})}{\mathcal{N}(\mathbf{p})}e^{i\mathbf{p}\cdot\mathbf{x}} , \\ F_2(\mathbf{x})&=\int\frac{d^D\mathbf{p}}{(2\pi)^D}\frac{\tilde{G}^d(\mathbf{p})}{\mathcal{N}(\mathbf{p})}\frac{\partial\tilde{G}^d(\mathbf{p},H)}{\partial H}e^{i\mathbf{p}\cdot\mathbf{x}} , \\ F_3(\mathbf{x})&=\int\frac{d^D\mathbf{p}}{(2\pi)^D}\frac{\tilde{G}^d(\mathbf{p},H)}{\mathcal{N}(\mathbf{p})}\frac{\partial\tilde{G}^d(\mathbf{p},H)}{\partial H}e^{i\mathbf{p}\cdot\mathbf{x}} , \end{aligned} \tag{2.12}$$

where

$$\mathcal{N}(p)=[\tilde{G}^d(\mathbf{p})]^2-[\tilde{G}^d(\mathbf{p},H)]^2 .$$

The above results are generally valid in $d=D+1$ dimensions and for any fluid described by a quadratic action. We now focus on the specific example of two surfaces in $d=3$, immersed in an isotropic fluid described by the Hamiltonian in Eq. (1.1). In this case, the Green's function is given by

$$G^3(\mathbf{x},z)=\int\frac{d^3p}{(2\pi)^3}\frac{\exp i(\mathbf{p}\cdot\mathbf{x}+p_z z)}{K(p^2+p_z^2)} . \tag{2.13}$$

Using this expression, we can explicitly evaluate the two Fourier transforms defined in Eq. (2.8) as

$$\begin{aligned} \tilde{G}^3(\mathbf{p})&=\frac{1}{2Kp} , \\ \tilde{G}^3(\mathbf{p},H)&=\frac{\exp(-pH)}{2Kp} . \end{aligned} \tag{2.14}$$

Substituting the above into Eq. (2.9), we obtain

$$\frac{\mathcal{H}_{\rm flat}}{kTA}=\int\frac{d^2p}{(2\pi)^2}\ln\left[\frac{1}{2Kp}\right]-\frac{\zeta(3)}{16\pi}\frac{1}{H^2} , \tag{2.15}$$

which is the Casimir interaction per unit area of two flat plates. The first term in Eq. (2.15) is a contribution to the surface tension which depends on a lattice cutoff. The second term, decaying as $1/H^2$, has a universal amplitude

$$-\zeta(3)/16\pi\approx-0.023\,91 .$$

The energy cost of the deformations is obtained using Eq. (2.10) as

$$\begin{aligned} \frac{\mathcal{H}_{\rm corr}}{kT}&=-\frac{3\zeta(3)}{16\pi H^4}\int d^2\mathbf{x}\,h^2(\mathbf{x}) \\ &\quad+\tfrac{1}{4}\int d^2\mathbf{x}\,d^2\mathbf{y}[h(\mathbf{x})-h(\mathbf{y})]^2 \\ &\quad\times\left\{\frac{1}{8\pi^2|\mathbf{x}-\mathbf{y}|^6}-\frac{1}{2\pi|\mathbf{x}-\mathbf{y}|^3H^3}K_1(t)\right. \\ &\quad\left.+\frac{1}{H^6}[K_1^2(t)+K_2^2(t)]\right\} , \end{aligned} \tag{2.16}$$

where $t\equiv|\mathbf{x}-\mathbf{y}|/H$ and the two kernels are given by

$$\begin{aligned} K_1(t)&\equiv\int_0^\infty du\frac{u^2}{2\pi(e^{2u}-1)}J_0(tu) , \\ K_2(t)&\equiv\int_0^\infty du\frac{u^2e^u}{2\pi(e^{2u}-1)}J_0(tu) . \end{aligned} \tag{2.17}$$

There is an implicit short-distance cutoff a for the power

laws in Eq. (2.16). The first term in Eq. (2.16) represents an instability to deformations due to the attraction between the plates. The effect of this term on surface fluctuations of a thin liquid film will be discussed in Sec. III. The coefficient of this term is obtained by adding the three constants given in Eqs. (2.11). Due to the complexity of these expressions, it is quite remarkable that the final result is the same as replacing the $1/H^2$ term in Eq. (2.15) with $1/(H+h(\mathbf{x}))^2$ and averaging over the position $\mathbf{x}$. The second correction represents long-range interactions between deformations induced by the fluctuations of the field. The first term in the curly brackets is the conformation energy of the deformed surface in the absence of the second plate, and is independent of H. The remaining terms represent correlations due to the presence of the second plate. Both $K_1(t)$ and $K_2(t)$ approach a constant as $t\to 0$. As $t\to\infty$, $K_1(t)\sim 1/t^3$ and $K_2(t)\sim\exp(-bt)$, with $b\approx 3.3$. The large-t behaviors of $K_1(t)$ and $K_2(t)$ determine the long-range interactions between height fluctuations.

Equation (2.16) can be used to calculate the Casimir force between a flat and a fixed rough surface. For a self-affine surface, the averaging over the random heights can be performed by using Eq. (1.4). The quench averaged free energy per unit area is

$$\frac{f(H)}{kT}=-\frac{\zeta(3)}{16\pi}\frac{1}{H^2}-\frac{3\zeta(3)}{16\pi}\frac{A_S L^{2\zeta_S}}{H^4}+\frac{C_1}{4}\frac{A_S}{H^{4-2\zeta_S}}\ , \tag{2.18}$$

where L is the extent (upper cutoff) of the self-affine structure, satisfying $\Delta H\equiv A_S^{1/2}L^{\zeta_S}\ll H$ to avoid contact between plates. The coefficient C_1 in Eq. (2.18) is given by

$$C_1=\int_{a/H}^{L/H}\{-t^{2\zeta_S-2}K_1(t)+2\pi t^{2\zeta_S+1}[K_1(t)]^2 +2\pi t^{2\zeta_S+1}[K_2(t)]^2\}dt \tag{2.19}$$

and weakly depends on the ratio L/H, but since the functions K_1 and K_2 decay rapidly with distance, it is quite insensitive to L as long as $L\gg H$. For $L\gg H\gg\Delta H$, the interactions in Eq. (2.18) are arranged in order of decreasing strength. The largest effect of randomness is to increase the Casimir attraction by an amount proportional to $(\Delta H/H)^2$. The last term in Eq. (2.18) decays as $1/H^{4-2\zeta_S}$ and in principle can be used to indirectly measure the roughness exponent ζ_S. In Eq. (2.18), if all lengths are measured in units of an atomic scale a_0 (e.g., the diameter of a surface atom), A_S becomes dimensionless. Using a reasonable set of parameters $\zeta_S\approx 0.35$, $a_0\approx 5$ Å, $A_S\approx 1$, and $L\approx 300$ Å, we estimate that for surfaces of 1 mm size and 100 Å apart, the forces generated by the three terms in Eq. (2.18) are 1.9×10^{-4}, 4.9×10^{-5}, and 3.7×10^{-6} N, respectively (using an appropriate lower cutoff of about 20 Å). (The corresponding pressures are 1.9×10^{-3}, 4.9×10^{-5}, and 3.7×10^{-6} atm.) The force generated by the last term is in fact measurable with current force apparatus [28], provided that the stronger forces generated by the first two terms can be properly subtracted.

III. GENERALIZATIONS

In Sec. II we obtained a general formula for fluctuation-induced forces between manifolds with type-I boundaries. We also explicitly worked out the fluctuation-induced forces between a flat and a deformed surface in an isotropic fluid governed by the simple Hamiltonian in Eq. (1.1). In this section we consider various generalizations.

A. Anisotropic fluids

First, consider a simple anisotropic fluid described by the Hamiltonian

$$\mathcal{H}_0=\tfrac{1}{2}\int d^dr\left[K_\parallel(\nabla_\parallel\phi)^2+K_\perp\left[\frac{\partial\phi}{\partial z}\right]^2\right]. \tag{3.1}$$

From the above Hamiltonian, we can compute the two-point correlation functions and the effective action using Eqs. (2.9) and (2.10). However, a simple rescaling $\mathbf{x}'=\mathbf{x}$, $z'=z\sqrt{K_\parallel/K_\perp}$ of coordinates allows us to arrive directly at the final results. As the Hamiltonian becomes isotropic in the new coordinates, we can directly use the results in Eqs. (2.15) and (2.16) with the substitution $H\to H'$, $h(\mathbf{x})\to h'(\mathbf{x})$. After rescaling back to the original coordinates, we get

$$\frac{\mathcal{H}_{\rm flat}}{kTA}=\int\frac{d^2p}{(2\pi)^2}\ln\left[\frac{1}{2K_\parallel p}\right]-\frac{\zeta(3)}{16\pi}\frac{1}{H^2}\beta \tag{3.2}$$

and

$$\frac{\mathcal{H}_{\rm corr}}{kT}=-\frac{3\zeta(3)}{16\pi H^4}\beta\int d^2\mathbf{x}\,h^2(\mathbf{x})+\tfrac{1}{4}\int d^2\mathbf{x}\,d^2\mathbf{y}[h(\mathbf{x})-h(\mathbf{y})]^2\left\{\frac{1}{8\pi^2|\mathbf{x}-\mathbf{y}|^6}\frac{1}{\beta}-\frac{1}{2\pi|\mathbf{x}-\mathbf{y}|^3H^3}\sqrt{\beta}K_1(t\sqrt{\beta})\right.$$
$$\left.+\frac{1}{H^6}\beta^2[K_1^2(t\sqrt{\beta})+K_2^2(t\sqrt{\beta})]\right\}, \tag{3.3}$$

where $t\equiv|\mathbf{x}-\mathbf{y}|/H$ and $\beta\equiv K_\perp/K_\parallel$. We see that the Casimir effect for two flat surfaces simply picks up a factor $K_\perp/K_\parallel$, and the same is true for the mass term. The effective action due to correlation of deformations depends on the anisotropy in a more complicated way through the kernel functions, which cannot be simply scaled out. Such anisotropy is inherent to nematic liquid crystals in which the long axis of the molecules has on average a preferred direction. The local preferred direction is described by a director field, subject to the Hamil-

tonian in Eq. (1.2). If the nematic director is on average perpendicular to the plates, we can write $\mathbf{n}\approx(n_x,n_y,1)$. Substituting $\mathbf{n}$ into Eq. (1.2), we get (keeping only quadratic terms)

$$\mathcal{H}_N=\int d^3r\frac{1}{2}\left\{\kappa_1\left[\frac{\partial n_x}{\partial x}+\frac{\partial n_y}{\partial y}\right]^2+\kappa_2\left[\frac{\partial n_x}{\partial y}-\frac{\partial n_y}{\partial x}\right]^2+\kappa_3\left[\left[\frac{\partial n_x}{\partial z}\right]^2+\left[\frac{\partial n_y}{\partial z}\right]^2\right]\right\}. \tag{3.4}$$

We now decompose the two-component field (n_x,n_y) into longitudinal and transverse components $\mathbf{n}_l$ and $\mathbf{n}_t$, satisfying $\nabla_\parallel\times\mathbf{n}_l=0$ and $\nabla_\parallel\cdot\mathbf{n}_t=0$, respectively. The deformation energy of Eq. (3.4) now separates into two independent terms,

$$\mathcal{H}_N=\int d^3r\left[\frac{\kappa_1}{2}(\nabla_\parallel\cdot\mathbf{n}_l)^2+\frac{\kappa_3}{2}\left[\frac{\partial\mathbf{n}_l}{\partial z}\right]^2\right]+\left[\frac{\kappa_2}{2}(\nabla_\parallel\times\mathbf{n}_t)^2+\frac{\kappa_3}{2}\left[\frac{\partial\mathbf{n}_t}{\partial z}\right]^2\right]. \tag{3.5}$$

Since $\mathbf{n}_t$ and $\mathbf{n}_l$ decouple, we can treat them as two independent fields, giving rise to additive contributions to the Casimir energy. For two flat surfaces, we find the H-dependent term

$$\frac{\mathcal{H}_{\rm flat}}{kTA}=-\frac{\zeta(3)}{16\pi}\frac{1}{H^2}\left[\frac{\kappa_3}{\kappa_1}+\frac{\kappa_3}{\kappa_2}\right], \tag{3.6}$$

which is exactly the result obtained by Ajdari, Peliti, and Prost [3]. The correction due to deformations also has two additive terms, with κ_3/κ_1 and κ_3/κ_2 replacing β in Eq. (3.3), respectively.

An extreme limit of anisotropy is exhibited by smectic liquid crystals where the molecules segregate into layers with fluidlike structure in each plane. The layer fluctuations $u(\mathbf{x},z)$ are then subject to the deformation energy given in Eq. (1.3). The Hamiltonian $\mathcal{H}_S$ is again quadratic, and we can directly apply Eqs. (2.9) and (2.10) to calculate the fluctuation-induced effective action. The Green's function $G^s(\mathbf{x},z)$ is obtained by inverting the quadratic action in Eq. (1.3) as

$$G^s(\mathbf{x},z)=\int\frac{d^3p}{(2\pi)^3}\frac{\exp[i(\mathbf{p}\cdot\mathbf{x}+p_z z)]}{Bp_z^2+\kappa p^4}. \tag{3.7}$$

Using the above expression, we obtain the two Fourier transforms defined in Eq. (2.8) as

$$\tilde{G}^s(\mathbf{p})=\frac{1}{2\sqrt{B\kappa}p^2},\qquad \tilde{G}^s(\mathbf{p},H)=\frac{1}{2\sqrt{B\kappa}p^2}\exp(-L\sqrt{\kappa/B}\,p^2). \tag{3.8}$$

Clearly, such anisotropy introduces a length scale $\lambda\equiv\sqrt{\kappa/B}$. Substituting the above expressions into Eqs. (2.9) and (2.10), after some lengthy but straightforward algebra, we find

$$\frac{\mathcal{H}_{\rm flat}}{kTA}=\int\frac{d^2\mathbf{p}}{(2\pi)^2}\ln\left[\frac{1}{2\sqrt{B\kappa}p^2}\right]-\frac{\zeta(2)}{16\pi}\frac{1}{H\lambda}. \tag{3.9}$$

The H-dependent term in the above equation was again previously obtained by Ajdari, Peliti, and Prost [3]. Comparing to the isotropic fluid, note that the fluctuation-induced free energy in a smectic liquid crystal decays as $1/H$, instead of $1/H^2$. The correction due to deformations is

$$\frac{\mathcal{H}_{\rm corr}}{kT}=-\frac{\zeta(2)}{16\pi}\frac{1}{H^3\lambda}\int h^2(\mathbf{x})d^2\mathbf{x}+\frac{1}{4}\frac{1}{H^4\lambda^2}\int d^2\mathbf{x}\,d^2\mathbf{y}[h(\mathbf{x})-h(\mathbf{y})]^2\times[K_3^2(t)+K_4^2(t)]. \tag{3.10}$$

Here, $t=|\mathbf{x}-\mathbf{y}|/\sqrt{\lambda H}$, and K_3 and K_4 are another two kernel functions defined as

$$K_3(t)=\int_0^\infty\frac{du}{2\pi}\frac{u^3}{\exp(2u^2)-1}J_0(tu),\qquad K_4(t)=\int_0^\infty\frac{du}{2\pi}\frac{u^3\exp(u^2)}{\exp(2u^2)-1}J_0(tu). \tag{3.11}$$

For quenched self-affine surfaces, the analog of Eq. (2.18) is

$$f(H)=-\frac{\zeta(2)}{16\pi}\frac{kT}{H\lambda}-\frac{\zeta(2)}{16\pi}\frac{kTA_S L^{2\zeta_S}}{\lambda H^3}+\frac{C_2}{4}\frac{kTA_S}{\lambda^{1-\zeta_S}H^{3-\zeta_S}}, \tag{3.12}$$

where C_2 is expressed in terms of the kernel functions K_3 and K_4 as

$$C_2=\int_{a/H}^{L/H}2\pi t^{2+2\zeta_S}[K_3^2(t)+K_4^2(t)]. \tag{3.13}$$

Again, we find characteristic corrections due to the self-affine structure of the surface. The decay of the last term is related to the roughness exponents of the surface. The presence of the additional length λ reduces the power to $3-\zeta_S$, compared to $4-2\zeta_S$ in an isotropic fluid. Clearly, these forces have the same magnitude as in the isotropic case of Eq. (2.18) for $H\approx\lambda$, but decay more slowly, and hence become comparatively stronger for $H\gg\lambda$.

B. Critical fluids

The fluid at criticality cannot be described by a Gaussian field. For example, a binary mixture at its demixing point belongs to the Ising universality class, and therefore is more appropriately described by a $\lambda\phi^4$ theory at T_C. In this case, the integration over the ϕ field in Eq. (2.1) cannot be performed easily, and in general multipoint correlations are needed. Nevertheless, we expect that the power laws we find in Eq. (2.18) remain valid since there is no additional length scale at T_C.

However, the amplitude should depend on the specific universality class. Exact results in two dimensions indicate that the Casimir interaction between two flat plates decays as $-c\pi kT/24H$ for like boundaries, where c is the central charge of the critical system [14]. For two flat surfaces, repeating our previous computation in two dimensions, and using Eq. (2.9), yields the H-dependent

term

$$\tfrac{1}{2}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\ln[1-\exp(-2pH)]=-\frac{\pi}{24H}\ ,$$

corresponding to $c=1$ as expected for a free scalar field. It is known that for the Ising universality class $c=\frac{1}{2}$ [31], so that the computation using a free scalar field gives an amplitude that is twice as big.

We can somewhat improve the computation of this amplitude by modifying the quadratic action $\mathcal{H}_0$ so that it gives the correct two-point correlation function. For a fluid at T_C, the two-point correlations take the general form

$$\langle\phi(r)\phi(0)\rangle\sim 1/r^{d-2+\eta}\ ,$$

where η is a critical exponent. To reproduce this two-point correlation, we modify $\mathcal{H}_0$ to (in Fourier space)

$$\mathcal{H}_0=\frac{K}{2}\int d^d p p^{2-\eta}|\tilde{\phi}(\mathbf{p})|^2\ . \tag{3.14}$$

Given this modified action, the Green's function is

$$G^d(\mathbf{x},z)=\int\frac{d^d p}{(2\pi)^d}\frac{\exp[i(\mathbf{p}\cdot\mathbf{x}+p_z z)]}{K(p^2+p_z^2)^{1-\eta/2}}\ , \tag{3.15}$$

and the analogs of Eqs. (2.14) are

$$\begin{aligned}\tilde{G}^d(\mathbf{p})&=\int_0^\infty\frac{dt}{\pi}\frac{1}{K(p^2+t^2)^{1-\eta/2}}\ ,\\ \tilde{G}^d(\mathbf{p},H)&=\int_0^\infty\frac{dt}{\pi}\frac{\cos(tH)}{K(p^2+t^2)^{1-\eta/2}}\ .\end{aligned} \tag{3.16}$$

The above expressions can be explicitly evaluated in terms of the gamma function $\Gamma(x)$ and the modified Bessel function $K_\nu(x)$, e.g.,

$$R\equiv\frac{\tilde{G}^d(\mathbf{p},H)}{\tilde{G}^d(\mathbf{p})}=\frac{(Hp)^{(1-\eta)/2}K_{(1-\eta)/2}(Hp)2^{(1+\eta)/2}}{\Gamma((1-\eta)/2)}\ .$$

Using Eq. (2.9), we then can compute the amplitude for the Casimir effect. For example, for the Ising model in two dimensions, with $\eta=\frac{1}{4}$, the result is

$$\mathcal{H}_{\rm flat}/kTA\approx-0.66\pi/24H\ ,$$

giving a much better estimate for the amplitude.

In general, for $\eta>0$, the amplitude computed using Eq. (3.14) is smaller than that computed from Eq. (1.1). We conjecture that the exact amplitude is even smaller due to multipoint correlations, as in the two-dimensional Ising model.

C. Like versus unlike boundary conditions

So far, we have calculated Casimir effects for manifolds with type-I boundary conditions. Manifolds with type-II or mixed boundary conditions can be handled similarly. The type-II constraint $\partial_\perp\phi=0$ is inserted into the functional integral in Eq. (2.1) via $\int\mathcal{D}\psi\exp(i\psi\partial_\perp\phi)$, thus representing a dipole source. After integrating over ϕ, we obtain a quadratic action for the auxiliary fields Ψ, as in Eq. (2.3). However, whereas the coupling between two type-I manifolds is $G^d(\mathbf{r}-\mathbf{r}')$, it is $\partial_\perp G^d(\mathbf{r}-\mathbf{r}')$ between type I and type II and $\partial_\perp\partial'_\perp G^d(\mathbf{r}-\mathbf{r}')$ between two type-II manifolds. The remaining computations proceed as before. As an illustration, again consider two plates, with type-I boundary for the deformed surface at $(\mathbf{x},h(\mathbf{x})+H)$ and type-II boundary at the flat surface $(\mathbf{x},0)$. After integrating out the ϕ field, we obtain the quadratic matrix

$$M=\begin{bmatrix}-\partial_z^2 G(\mathbf{x}-\mathbf{y},0) & -\partial_z G(\mathbf{x}-\mathbf{y},h(\mathbf{y})+H)\\ -\partial_z G(\mathbf{x}-\mathbf{y},h(\mathbf{x})+H) & G(\mathbf{x}-\mathbf{y},h(\mathbf{x})-h(\mathbf{y}))\end{bmatrix}. \tag{3.17}$$

As before, we can do an expansion for small deformations, $M\approx M_0+\delta M$, where M_0 is the matrix for two flat surfaces, i.e.,

$$M_0=\begin{bmatrix}-\partial_z^2 G(\mathbf{x}-\mathbf{y},0) & -\partial_z G(\mathbf{x}-\mathbf{y},H)\\ -\partial_z G(\mathbf{x}-\mathbf{y},H) & G(\mathbf{x}-\mathbf{y},0)\end{bmatrix}. \tag{3.18}$$

M_0 can be easily diagonalized by transforming to Fourier space, and we find Casimir effects between two flat surfaces with mixed type-I-II boundaries given by (keeping only the H-dependent term)

$$\begin{aligned}\frac{\mathcal{H}_{\rm flat}}{kTA}&=\tfrac{1}{2}\ln\mathrm{Det}M_0=\tfrac{1}{2}\int\frac{d^D\mathbf{p}}{(2\pi)^D}\ln[1+\exp(-2pH)]\\ &=(1-2^{1-d})\frac{\Gamma(d/2)\zeta(d)}{(4\pi)^{d/2}}\frac{1}{H^D}\ .\end{aligned} \tag{3.19}$$

Carrying out a similar calculation for two type-II boundaries, we find

$$\begin{aligned}\frac{\mathcal{H}_{\rm flat}}{kTA}&=\tfrac{1}{2}\ln\mathrm{Det}M_0=\tfrac{1}{2}\int\frac{d^D\mathbf{p}}{(2\pi)^D}\ln[1-\exp(-2pH)]\\ &=-\frac{\Gamma(d/2)\zeta(d)}{(4\pi)^{d/2}}\frac{1}{H^D}\ ,\end{aligned} \tag{3.20}$$

which is the same as for two type-I boundaries. Thus with both like boundaries, I-I or II-II, there is an attraction of the same amplitude, while unlike boundaries (I-II) lead to a repulsion which is $1-2^{1-d}$ times smaller. Repeating similar computations for smectic liquid-crystal layers, we find the ratio of the interactions between I-I, II-II, and I-II boundary conditions is given by $1:1:2^{(1-d)/2}-1$, respectively.

The corrections due to deformations can be computed perturbatively in terms of $h(\mathbf{x})$ as in the case of two type-I boundaries. The important thing to note is that due to the repulsive nature of the leading interaction, the mass term between unlike boundaries is positive, e.g., for $d=3$, the mass term is

$$\frac{9\zeta(3)}{64\pi}\frac{kT}{H^4}\int d^2\mathbf{x}\,h^2(\mathbf{x})\ . \tag{3.21}$$

D. Surface fluctuations of a thin liquid film

The $h(\mathbf{x})^2$ dependence of the Casimir energy in Eqs. (2.16), (3.3), (3.10), and (3.21) may have important consequences for the fluctuations of a fluid surface, or a film.

The fluctuations of a free surface are governed by a surface tension energy $\gamma \int d^2\mathbf{x}(\nabla h)^2/2$. For sufficiently long wavelengths, the first term in Eq. (2.16) dominates the surface tension and modifies the fluctuations. [The second term in Eq. (2.16) is equivalent to an increase in surface tension of roughly 10^{-3} dyn/cm.] For like boundaries (such as a free-standing film) there is an instability to deformations, while for unlike boundaries (e.g., a film on a solid substrate), there is an additional stabilizing force. In the absence of any other interactions, the crossover length is $\lambda_0^I \sim 24\sqrt{\gamma_{\rm eff}/kT}H^2$. For a film which is 100 Å thick, a typical fluid-air interfacial tension yields $\lambda_0^I \sim 4\ \mu$m. Of course, additional stabilizing forces may be present. For example, gravity produces an energy cost of $\rho g \int d^2\mathbf{x}\, h^2/2$ for deformations. This is larger than the Casimir deformation energy for thicknesses

$$H > 0.6(kT/\rho g)^{1/4} \approx 0.5\ \mu\text{m} .$$

Similarly, for a smectic liquid-crystal film, the crossover length is

$$\lambda_0^S \sim 35\sqrt{\gamma_{\rm eff}/kT}H^{3/2}\lambda^{1/2} .$$

We estimate that $\lambda_0^S \sim 6\ \mu$m for $H = 100$ Å, and gravity becomes important for thicknesses $H > 2\ \mu$m. There are indeed a number of experiments measuring the roughness of liquid surfaces [32]. It would be interesting if future experiments could probe the effect of Casimir forces on the surface roughness of thin liquid films.

IV. RELATED PROBLEMS

The general approach developed in Sec. II can also be used to study a number of other related problems.

A. Interaction between a line and a surface

This is an example of fluctuation-induced forces between manifolds with mixed dimensions. It could be relevant to adsorption of polymers on a surface within a correlated fluid. Consider a long, directed polymer parallel to a surface. Let's denote the position of the surface by $(\mathbf{x},0)=(x,y,0)$ and the location of the line as $(t,0,H)$. We can then read off the quadratic action $\mathcal{H}_1[\psi]$ from the general equation (2.3) as

$$\begin{aligned}\mathcal{H}_1[\Psi] = &\int dt\, dt' \psi_1(t) G^d(t-t',0,0)\psi_1(t') \\ &+ \int d\mathbf{x}\, d\mathbf{x}' \psi_2(\mathbf{x}) G^d(\mathbf{x}-\mathbf{x}',0)\psi_2(\mathbf{x}') \\ &+ 2\int d\mathbf{x}\, dt\, \psi_1(t) G^d(x-t,y,H)\psi_2(\mathbf{x}) , \end{aligned} \tag{4.1}$$

where $\psi_1(t)$ and $\psi_2(\mathbf{x})$ are auxiliary fields defined on the line and the surface, respectively. After transforming to Fourier space, we have

$$\begin{aligned}\mathcal{H}_1[\Psi] = &\int \frac{dq_x}{2\pi} \tilde{G}_1^d(q_x)|\psi_1(q_x)|^2 \\ &+ \int \frac{d^2q}{(2\pi)^2} \tilde{G}^d(\mathbf{q})|\psi_2(\mathbf{q})|^2 \\ &+ 2\int \frac{d^2q}{(2\pi)^2} \tilde{G}^d(\mathbf{q},H)\psi_1(-q_x)\psi_2(\mathbf{q}) , \end{aligned} \tag{4.2}$$

with a new transformed function defined by

$$\tilde{G}_1^d(q_x) \equiv \int dx\, \exp(iq_x x)\tilde{G}^d(x,0,0) . \tag{4.3}$$

The matrix M in Eq. (2.3) then takes block diagonal form with respect to q_x. Therefore,

$$\mathrm{Det}\tilde{M} = \prod_{q_x} \mathrm{Det}\tilde{M}_s(q_x) , \tag{4.4}$$

where the submatrix $\tilde{M}_s(q_x)$ has the form

$$\tilde{M}_s(q_x) = \begin{bmatrix} \tilde{G}_1^d(q_x) & V(q_x) \\ V^T(q_x) & U(q_x) \end{bmatrix} . \tag{4.5}$$

Here, $V(q_x)$ is a vector indexed by q_y with elements $\tilde{G}^d(q_x,q_y,H)$, and $U(q_x)$ is a diagonal matrix with respect to q_y with elements $\tilde{G}^d(q_x,q_y)$. It is straightforward to work out $\mathrm{Det}\tilde{M}_s(q_x)$, with the result (in discrete notation)

$$\begin{aligned}\mathrm{Det}\tilde{M}_s(q_x) = &\tilde{G}_1^d(q_x) \prod_{q_y} \tilde{G}^d(q_x,q_y) \\ &- \sum_{q_y^0} [\tilde{G}^d(q_x,q_y^0,H)]^2 \prod_{q_y \neq q_y^0} \tilde{G}^d(q_x,q_y) . \end{aligned} \tag{4.6}$$

Using the above result, we find the H-dependent free energy

$$\frac{\mathcal{H}_{\rm eff}}{kTS} = \tfrac{1}{2}\int \frac{dq_x}{2\pi} \ln\left[1 - \int \frac{dq_y}{2\pi} \frac{[\tilde{G}^d(q_x,q_y,H)]^2}{\tilde{G}_1^d(q_x)\tilde{G}^d(q_x,q_y)} \right] , \tag{4.7}$$

where S is the total length of the line. The Fourier transform $\tilde{G}_1^d(q_x)$ defined in Eq. (4.3) can be explicitly evaluated; the result depends on both upper and lower cutoffs Λ_L and Λ_s in momentum space, and is given by

$$\tilde{G}_1^d(q_x) = \frac{1}{4\pi K} \ln\left[\frac{q_x^2+\Lambda_L^2}{q_x^2+\Lambda_s^2} \right] . \tag{4.8}$$

After carrying out the integration over q_y, we end up with the final result

$$\frac{\mathcal{H}_{\rm eff}}{kTS} = \frac{1}{H}\int_{H\Lambda_s}^{H\Lambda_L} \frac{dt}{2\pi} \ln\left\{ 1 - f_0(t) \Big/ \ln\left[\frac{t^2+(H\Lambda_L)^2}{t^2+(H\Lambda_s)^2} \right] \right\} , \tag{4.9}$$

where $f_0(t)$ is defined as

$$f_0(t) \equiv 2\int_{H\Lambda_s}^{H\Lambda_L} du \frac{\exp[-2(u^2+t^2)^{1/2}]}{\sqrt{u^2+t^2}} . \tag{4.10}$$

Equation (4.9) indicates that there is a $1/H$ attraction between a line and a surface due to correlated fluctuations of the fluid. Its magnitude, however, depends on the explicit cutoffs in momentum space.

B. Fluctuation-induced many-body interactions

In general, if more than two manifolds are immersed in the correlated fluid, the total induced interactions cannot

be obtained by summing of the pairwise interactions. The presence of other manifolds modifies the fluctuations of the fluid, giving rise to many-body interactions. Such interactions are important in a number of physical systems such as in colloidal suspensions in polymer solutions [33], where the effect of fluctuation- (in conformation of polymers) induced many-body forces has been investigated. Here we demonstrate how the many-body interactions for manifolds in a correlated fluid can be obtained using the general approach of Sec. II.

Consider a simple example where three parallel lines are immersed in a three-dimensional fluid governed by the simple action in Eq. (1.1), separated by a distance H between each pair. Following the general procedure in Sec. II, we find the quadratic matrix $\tilde{M}$ (diagonal in Fourier space) as

$$\tilde{M}=\begin{bmatrix} \tilde{G}_1^d(p) & \tilde{G}_1^d(p,H) & \tilde{G}_1^d(p,H) \\ \tilde{G}_1^d(p,H) & \tilde{G}_1^d(p) & \tilde{G}_1^d(p,H) \\ \tilde{G}_1^d(p,H) & \tilde{G}_1^d(p,H) & \tilde{G}_1^d(p) \end{bmatrix} , \tag{4.11}$$

where $\tilde{G}_1^d(p,H)$ is defined as

$$\tilde{G}_1^d(p,H)\equiv\int G^d(x,0,H)\exp(ipx)dx=\frac{1}{4\pi K}g_0(pH) , \tag{4.12}$$

with $g_0(t)$ given by

$$g_0(t)=2\int_{H\Lambda_s}^{H\Lambda_L}du\frac{\exp[-(u^2+t^2)^{1/2}]}{(u^2+t^2)^{1/2}} . \tag{4.13}$$

The determinant of $\tilde{M}$ is easily evaluated, leading to the H-dependent free energy

$$\frac{\mathcal{H}_{\rm eff}}{kTS}=\tfrac{1}{2}\int\frac{dp}{2\pi}\ln\left\{1+2\left[\frac{\tilde{G}_1^d(p,H)}{\tilde{G}_1^d(p)}\right]^3-3\left[\frac{\tilde{G}_1^d(p,H)}{\tilde{G}_1^d(p)}\right]^2\right\} . \tag{4.14}$$

The three-body interaction is formally defined as the difference between the total free energy and the sum of pairwise interactions, and is given by

$$\frac{\mathcal{H}_3}{kTS}=\tfrac{1}{2}\int\frac{dp}{2\pi}\left\{\ln\left[1+2\frac{\tilde{G}_1^d(p,H)}{\tilde{G}_1^d(p)}\right]-\ln\left[1-\frac{\tilde{G}_1^d(p,H)}{\tilde{G}_1^d(p)}\right]-3\ln\left[1+\frac{\tilde{G}_1^d(p,H)}{\tilde{G}_1^d(p)}\right]\right\} . \tag{4.15}$$

It can be shown that the integrand in the above equation is positive, indicating a repulsive three-body interaction. After the H dependence is scaled out, we obtain $\mathcal{H}_3/(kTS)=\Delta_3/H$, with amplitude Δ_3 given by

$$\Delta_3=\int_{H\lambda_s}^{H\lambda_L}\frac{dt}{2\pi}\{\ln[1+2R(t)]-\ln[1-R(t)]-3\ln[1+R(t)]\} , \tag{4.16}$$

where

$$R(t)\equiv g_0(t)\Big/\ln\left[\frac{t^2+(H\lambda_L)^2}{t^2+(H\lambda_s)^2}\right] .$$

For more than three lines in the fluid, the many-body interactions can be obtained similarly. Although the above computation is for an isotropic fluid, it can be easily extended to directed polymers in nematic liquid crystals where the relevance of the many-body interactions to the phase behavior remains to be investigated.

V. CONCLUSIONS AND FUTURE APPLICATIONS

In this paper we developed a general approach for computing fluctuation-induced forces between manifolds immersed in a correlated fluid. This approach allows us to deal with manifolds with arbitrary intrinsic and embedding dimensions, for both like and unlike boundaries. It also enables us to perturbatively calculate fluctuation-induced forces for slightly deformed manifolds. As an explicit example, we computed the fluctuation-induced forces between two parallel plates, with one deformed, for a variety of correlated fluids. We find that the fluctuation-induced force between a flat and a self-affine surface has a characteristic correction which decays with the average separation of the plates through an exponent related to the roughness of the surface. We also find that fluctuation-induced forces due to deformations can either stabilize or destabilize the surface of a fluid film, depending on its boundary conditions. Other examples presented in this paper are fluctuation-induced forces between a surface and a line, which may be applicable for adsorption of polymers on a surface, and fluctuation-induced many-body interactions which may be relevant to phase behavior in mixtures (e.g., polymer nematic liquid-crystal systems).

There are a number of interesting extensions and possible further applications. The method we used in this paper can be extended to other problems related to rough boundaries. For example, it can be used to study the capacitance between flat and self-affine conducting surfaces. We find that this problem can be formulated in a path-integral form, with boundary conditions implemented by inserting δ functions. Thus we avoid the problem of solving Laplace's equation with complicated boundary conditions (equipotentials at the flat and rough electrodes). Another problem that may be amenable to similar treatment is that of classical waves scattered from a rough surface, which has many practical applications.

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APPENDIX

In this appendix we give the details of the derivations that lead to Eq. (2.10). Expanding the effective action around the flat geometry, the correction due to deformations is

$$\mathcal{H}_{\rm corr}=\tfrac{1}{2}\ln \mathrm{Det}[1+\tilde{M}_0^{-1}\delta\tilde{M}]\ , \tag{A1}$$

where $\tilde{M}_0$ is given by Eq. (2.7). The matrix δM is obtained by expanding the Green's functions in Eq. (2.5) up to quadratic order in the deformation

$$G^d(\mathbf{x}-\mathbf{y},H+h(\mathbf{x}))=G^d(\mathbf{x}-\mathbf{y},H)+\frac{\partial G^d}{\partial z}(\mathbf{x}-\mathbf{y},H)h(\mathbf{x}) +\frac{1}{2}\frac{\partial^2 G^d}{\partial z^2}(\mathbf{x}-\mathbf{y},H)h^2(\mathbf{x})\ , \tag{A2}$$

$$G^d(\mathbf{x}-\mathbf{y},h(\mathbf{x})-h(\mathbf{y})) = G^d(\mathbf{x}-\mathbf{y},0)+\frac{1}{2}\frac{\partial^2 G^d}{\partial z^2}(\mathbf{x}-\mathbf{y},0)[h(\mathbf{x})-h(\mathbf{y})]^2\ . \tag{A3}$$

By transforming to Fourier space,

$$\delta\tilde{M}=\begin{bmatrix} 0 & A(\mathbf{p},\mathbf{q}) \\ A(\mathbf{q},\mathbf{p}) & B(\mathbf{p},\mathbf{q}) \end{bmatrix}\ , \tag{A4}$$

where

$$A(\mathbf{p},\mathbf{q})=\int d^D\mathbf{x}\,d^D\mathbf{y}\exp(i\mathbf{p}\cdot\mathbf{x}+i\mathbf{q}\cdot\mathbf{y}) \times\left[\frac{\partial G^d}{\partial z}(\mathbf{x}-\mathbf{y},H)h(\mathbf{y}) +\frac{1}{2}\frac{\partial^2 G^d}{\partial z^2}(\mathbf{x}-\mathbf{y},H)h^2(\mathbf{y})\right]\ ,$$
$$B(\mathbf{p},\mathbf{q})=\int d^D\mathbf{x}\,d^D\mathbf{y}\exp(i\mathbf{p}\cdot\mathbf{x}+i\mathbf{q}\cdot\mathbf{y}) \times\frac{1}{2}\frac{\partial^2 G^d}{\partial z^2}(\mathbf{x}-\mathbf{y},0)[h(\mathbf{x})-h(\mathbf{y})]^2\ . \tag{A5}$$

The matrix $\tilde{M}_0$ can be easily inverted to give

$$\tilde{M}_0^{-1}=\frac{1}{\mathcal{N}}\begin{bmatrix} \tilde{G}^d(\mathbf{p}) & -\tilde{G}^d(\mathbf{p},H) \\ -\tilde{G}^d(\mathbf{p},H) & \tilde{G}^d(\mathbf{p}) \end{bmatrix}\ , \tag{A6}$$

where

$$\mathcal{N}=[\tilde{G}^d(\mathbf{p})]^2-[\tilde{G}^d(\mathbf{p},H)]^2\ .$$

The matrix $Q\equiv 1+\tilde{M}_0^{-1}\delta M$ then has the following form:

$$Q=\begin{bmatrix} 1-\dfrac{\tilde{G}^d(\mathbf{p},H)A(\mathbf{p},\mathbf{q})}{\mathcal{N}(\mathbf{p})} & \dfrac{\tilde{G}^d(\mathbf{p})A(\mathbf{p},\mathbf{q})-\tilde{G}^d(\mathbf{p},H)B(\mathbf{p},\mathbf{q})}{\mathcal{N}(\mathbf{p})} \\ \dfrac{\tilde{G}^d(\mathbf{p})A(\mathbf{q},\mathbf{p})}{\mathcal{N}(\mathbf{p})} & 1+\dfrac{\tilde{G}^d(\mathbf{p})B(\mathbf{p},\mathbf{q})-\tilde{G}^d(\mathbf{p},H)A(\mathbf{p},\mathbf{q})}{\mathcal{N}(\mathbf{p})} \end{bmatrix}\ . \tag{A7}$$

Using Eq. (A7), we can expand $\mathrm{Det}Q$ to quadratic order in the deformation. The result is (in discrete notation)

$$\mathrm{Det}Q=1+\sum_{\mathbf{p}}\left\{\frac{\tilde{G}^d(\mathbf{p})B(\mathbf{p},-\mathbf{p})}{\mathcal{N}(\mathbf{p})}-2\frac{\tilde{G}^d(\mathbf{p},H)A(\mathbf{p},-\mathbf{p})}{\mathcal{N}(\mathbf{p})}\right\}-\sum_{\substack{\mathbf{q},\mathbf{p}\\ \mathbf{q}\neq-\mathbf{p}}}\frac{\tilde{G}^d(\mathbf{p},H)A(\mathbf{p},\mathbf{q})\tilde{G}^d(\mathbf{q},H)A(\mathbf{q},\mathbf{p})}{\mathcal{N}(\mathbf{p})\mathcal{N}(\mathbf{q})} -\sum_{\mathbf{p},\mathbf{q}}\frac{\tilde{G}^d(\mathbf{p})A(\mathbf{p},\mathbf{q})\tilde{G}^d(\mathbf{q})A(\mathbf{q},\mathbf{p})}{\mathcal{N}(\mathbf{p})\mathcal{N}(\mathbf{q})}\ , \tag{A8}$$

where the first correction is from the contributions of diagonal elements in block Q_{11} and Q_{22}, the second correction is from the off-diagonal elements in Q_{11} and Q_{22}, and the last term has the off-diagonal contributions from blocks Q_{12} and Q_{21}. Notice that by choosing $\int d^D\mathbf{x}\,h(\mathbf{x})=0$, the term linear in $h(\mathbf{x})$ in $A(\mathbf{p},-\mathbf{p})$ vanishes. Using $A(\mathbf{p},\mathbf{q})$ and $B(\mathbf{p},\mathbf{q})$ in Eq. (A5), we find

$$\frac{\mathcal{H}_{\rm corr}}{kTA}=\tfrac{1}{2}\int\frac{d^D\mathbf{p}}{(2\pi)^D}d^D\mathbf{x}\,d^D\mathbf{y}\left\{\frac{\tilde{G}^d(\mathbf{p})}{\mathcal{N}(\mathbf{p})}\frac{1}{2}\frac{\partial^2 G^d}{\partial z^2}(\mathbf{x}-\mathbf{y},0)[h(\mathbf{x})-h(\mathbf{y})]^2\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})] -\frac{\tilde{G}^d(\mathbf{p},H)}{\mathcal{N}(\mathbf{p})}\frac{\partial^2 G^d}{\partial z^2}(\mathbf{x}-\mathbf{y},H)h^2(\mathbf{y})\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})]\right\}$$
$$-\tfrac{1}{2}\int\frac{d^D\mathbf{p}}{(2\pi)^D}\frac{d^D\mathbf{q}}{(2\pi)^D}\frac{\tilde{G}^d(\mathbf{p},H)\tilde{G}^d(\mathbf{q},H)+\tilde{G}^d(\mathbf{p})\tilde{G}^d(\mathbf{q})}{\mathcal{N}(\mathbf{p})\mathcal{N}(\mathbf{q})} \times\int d^D\mathbf{x}\,d^D\mathbf{v}\exp[i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{v})]\frac{\partial G^d}{\partial z}(\mathbf{v}-\mathbf{x},H)h(\mathbf{x})$$
$$\times\int d^D\mathbf{u}\,d^D\mathbf{y}\exp[i(\mathbf{q}\cdot\mathbf{u}-\mathbf{p}\cdot\mathbf{y})]\frac{\partial G^d}{\partial z}(\mathbf{u}-\mathbf{y},H)h(\mathbf{y})\ . \tag{A9}$$

In the above equation, the integrations over the relative coordinates $\mathbf{v}-\mathbf{x}$ and $\mathbf{u}-\mathbf{y}$ can be first performed, followed by integrations over $\mathbf{p}$ and $\mathbf{q}$. With some rearrangement of terms, we then end up with Eq. (2.10).

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