Wigner function of relativistic spin-$\frac{1}{2}$ particles

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Using the recently developed relativistic Wigner formulation for the density matrix of spin-$\frac{1}{2}$ particles, we study the $J^\pi = \frac{1}{2}^+$ Coulomb-like and cavity relativistic states. One of our objectives is to understand the sharing of the total angular momentum of a quantum state between the spin and rotational degrees of freedom, arising due to the spin-orbit coupling. Another is to demonstrate that the 4×4-matrix Wigner function is the appropriate generalization from the 2×2 form of the nonrelativistic theory.

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Recently, Bialynicki-Birula, Górnicki, and Rafelski [1] have developed the relativistic Wigner phase-space formulation within the strong-field approximation to QED for the spin-$\frac{1}{2}$ Dirac-Heisenberg density matrix. In this note we would like to demonstrate that the 4×4-matrix Wigner function is the appropriate generalization of the nonrelativistic density-matrix Wigner function. Accordingly, the relativistic spin-$\frac{1}{2}$ particle in an external electromagnetic field can be described by the 16-component gauge-invariant Wigner function:

$$W_{\alpha\beta}(\mathbf{r},\mathbf{p},t) = \int d\mathbf{s}\, e^{-i\mathbf{p}\cdot\mathbf{s}}\Psi_\alpha(\mathbf{r}+\mathbf{s}/2,t)\Psi_\beta^\dagger(\mathbf{r}-\mathbf{s}/2,t)\exp\left(-i\int_{-1/2}^{+1/2} d\lambda \mathbf{s}\cdot\mathbf{A}(\mathbf{r}+\lambda\mathbf{s},t)\right). \tag{1}$$

Here $\mathbf{A}$ is the vector potential and the line integral assures that the expression we consider is manifestly gauge invariant. In the examples below, $\mathbf{A} = 0$. It is convenient to decompose this 4×4 matrix as follows:

$$\underline{W}(\mathbf{r},\mathbf{p},t) = \frac{1}{4}\left(f_0 + \sum_{i=1}^{3}\rho_i f_i + \boldsymbol{\sigma}\cdot\mathbf{g}_0 + \sum_{i=1}^{3}\rho_i\boldsymbol{\sigma}\cdot\mathbf{g}_i\right), \tag{2}$$

where f_0 is the phase-space charge and $\mathbf{g}_1$ is the phase-space current density; similarly $f_1, \mathbf{g}_0$ form the pseudo-four-vector phase-space density, with $\mathbf{g}_0$ being also the spin density, viz., $\mathbf{g}_0 = \frac{1}{2}\mathrm{Tr}\{\boldsymbol{\sigma}W\}$; f_2 is the pseudoscalar density, while f_3 is the scalar mass density. The structure functions $\mathbf{g}_2, \mathbf{g}_3$ are the electric and magnetic polarizibility phase-space distributions.

We will now obtain the explicit form of the relativistic Wigner function for a localized stationary spin-$\frac{1}{2}$ particle in a $J^+ = \frac{1}{2}^+$ state. The four-component Dirac wave function is

$$\Psi^{(\pm)}(\mathbf{r}) = \begin{pmatrix} G(r) \\ -i\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}F(r)\end{pmatrix}\chi^{(\pm)}, \tag{3}$$

where χ_s is the two-component spin function.

Phase-space densities obtained from Eqs. (1)–(3) can be written in the form

$$f_0 = A + \left((\mathbf{r})^2 + \frac{\hbar^2}{4}(\boldsymbol{\partial}_p)^2 - \hbar\mathbf{s}\cdot(\mathbf{r}\times\boldsymbol{\partial}_p)\right)D, \tag{4}$$

$$\mathbf{g}_0 = \mathbf{s}A + \left[\hbar\mathbf{r}\times\boldsymbol{\partial}_p + 2\mathbf{r}(\mathbf{r}\cdot\mathbf{s}) + \frac{\hbar^2}{2}\boldsymbol{\partial}_p(\boldsymbol{\partial}_p\cdot\mathbf{s}) - \mathbf{s}\left(\mathbf{r}^2 + \frac{\hbar^2}{4}\boldsymbol{\partial}_p^2\right)\right]D, \tag{5}$$

$$f_1 = [\hbar\mathbf{s}\cdot\boldsymbol{\partial}_p]B - [2\mathbf{s}\cdot\mathbf{r}]C, \tag{6}$$

$$\mathbf{g}_1 = [\hbar\boldsymbol{\partial}_p - 2\mathbf{s}\times\mathbf{r}]B - [2\mathbf{r} + \hbar\mathbf{s}\times\boldsymbol{\partial}_p]C, \tag{7}$$

$$f_2 = -[2\mathbf{s}\cdot\mathbf{r}]B - [\hbar\mathbf{s}\cdot\boldsymbol{\partial}_p]C, \tag{8}$$

$$\mathbf{g}_2 = -[2\mathbf{r} + \hbar\mathbf{s}\times\boldsymbol{\partial}_p]B - [\hbar\boldsymbol{\partial}_p - 2\mathbf{s}\times\mathbf{r}]C, \tag{9}$$

$$f_3 = A - \left((\mathbf{r})^2 + \frac{\hbar^2}{4}(\boldsymbol{\partial}_p)^2 - \hbar\mathbf{s}\cdot(\mathbf{r}\times\boldsymbol{\partial}_p)\right)D, \tag{10}$$

$$\mathbf{g}_3 = \mathbf{s}A - \left[\hbar\mathbf{r}\times\boldsymbol{\partial}_p + 2\mathbf{r}(\mathbf{r}\cdot\mathbf{s}) + \frac{\hbar^2}{2}\boldsymbol{\partial}_p(\boldsymbol{\partial}_p\cdot\mathbf{s}) - \mathbf{s}\left(\mathbf{r}^2 + \frac{\hbar^2}{4}\boldsymbol{\partial}_p^2\right)\right]D, \tag{11}$$

where the following auxiliary functions were used:

$$A(\mathbf{r},\mathbf{p}) = \int d\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}G(\mathbf{r}+\mathbf{x}/2)G(\mathbf{r}-\mathbf{x}/2), \tag{12}$$

$$B(\mathbf{r},\mathbf{p}) = \mathrm{Re}\left(\int d\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}G(\mathbf{r}+\mathbf{x}/2)\frac{F(\mathbf{r}-\mathbf{x}/2)}{|\mathbf{r}-\mathbf{x}/2|}\right), \tag{13}$$

$$C(\mathbf{r},\mathbf{p}) = \mathrm{Im}\left(\int d\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar} G(\mathbf{r}+\mathbf{x}/2)\frac{F(\mathbf{r}-\mathbf{x}/2)}{|\mathbf{r}-\mathbf{x}/2|}\right), \tag{14}$$

$$D(\mathbf{r},\mathbf{p}) = \int d\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar} \frac{F(\mathbf{r}+\mathbf{x}/2)}{|\mathbf{r}+\mathbf{x}/2|}\frac{F(\mathbf{r}-\mathbf{x}/2)}{|\mathbf{r}-\mathbf{x}/2|}, \tag{15}$$

where we introduced the spin direction vector, $\mathbf{s} = \chi_s^\dagger \boldsymbol{\sigma} \chi_s$, which for $\chi_s = \chi_\pm$ becomes $\mathbf{s} = (0,0,\pm 1)$. The spin-unpolarized distributions are averages of the spin-polarized quantities. The total energy and angular momentum contained in the state are the generators of the symmetry transformations and are given by [1]

$$E = \int d\Gamma (c\mathbf{p}\cdot\mathbf{g}_1 + mc^2 f_3), \tag{16}$$

$$\mathbf{J} = \int d\Gamma (\mathbf{r}\times\mathbf{p}) f_0 + \int d\Gamma \frac{\hbar}{2}\mathbf{g}_0, \tag{17}$$

where $d\Gamma = d\mathbf{r}\, d\mathbf{p}/(2\pi)^3$. Similarly, the total charge Q defined in units of e through the formula

$$Q = \int d\Gamma\, f_0, \tag{18}$$

is conserved by virtue of gauge invariance.

One easily finds that the charge Q and energy E have the same values as the spin-unpolarized quantities. The spin polarization of course has influence on the angular momentum $\mathbf{J}$ as we expect: the spin-unpolarized angular momentum gives us null results, but the spin-polarized angular momentum $\mathbf{J}$ gives us nonzero components:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \tag{19}$$

where

$$\mathbf{L} = \int d\Gamma (\mathbf{r}\times\mathbf{p}) f_0(\mathbf{r},\mathbf{p},t) = \mathbf{s}\frac{\hbar}{2}\frac{4}{3}\int d\mathbf{r}\, F(r)F(r) \tag{20}$$

and

$$\mathbf{S} = \int d\Gamma \left(\frac{\hbar}{2}\right) g_{0z}^{(\pm)}(\mathbf{r},\mathbf{p},t) = \mathbf{s}\frac{\hbar}{2}\int d\mathbf{r}[G(r)G(r) - \tfrac{1}{3}F(r)F(r)], \tag{21}$$

so that, as expected,

$$\mathbf{J} = \mathbf{s}\frac{\hbar}{2}\int d\mathbf{r}[G(r)G(r) + F(r)F(r)] = \mathbf{s}\frac{\hbar}{2}\frac{Q^{(\pm)}}{e} \tag{22}$$

and the last factor is unity for a normalized state with $Q = 1$.

Using the Coulomb wave function [2] for the hydrogen-like 1s atomic state, we find

$$\mathbf{L} = \mathbf{s}\frac{\hbar}{2}\frac{2-2\gamma}{3} \tag{23}$$

and

$$\mathbf{S} = \mathbf{s}\frac{\hbar}{2}\frac{1+2\gamma}{3}, \tag{24}$$

where $\gamma = \sqrt{1 - Z^2\alpha^2}$. In the nonrelativistic limit $\gamma \to 1 - Z^2\alpha^2/2$ and we find practically all angular momentum in the spin density of the state. For $Z\alpha \to 1$, $\gamma \to 0$ and we find that $\frac{2}{3}$ of the spin is in the rotational degree of freedom. More generally we note that when the integral of the scalar density vanishes, $\int d\mathbf{r}\, \bar{\Psi}^{(\pm)}\Psi^{(\pm)} = 0$, we have $\int d\mathbf{r}\, F^2 = \int d\mathbf{r}\, G^2 = \frac{1}{2}$, (with the last results holding by virtue of normalization of the wave function), which just leads to the last result.

For the bag-type cavity states [3], with a cavity of radius $R = 1$ we find

$$\mathbf{L} = \mathbf{s}\frac{\hbar}{2}\left(\frac{4}{3} - \frac{1}{3}\frac{\omega - \sin\omega\cos\omega}{(\omega-1)\sin^2\omega}\right) \simeq \mathbf{s}\frac{\hbar}{2}\,0.35, \tag{25}$$

$$\mathbf{S} = \mathbf{s}\frac{\hbar}{2}\left(\frac{1}{3}\frac{\omega - \sin\omega\cos\omega}{(\omega-1)\sin^2\omega} - \frac{1}{3}\right) \simeq \mathbf{s}\frac{\hbar}{2}\,0.65, \tag{26}$$

where the energy $\omega = 2.04$ is in natural units for the $J^\pi = \frac{1}{2}^+$, $\kappa = -1$ state. About a third of the spin is transferred in this case to the rotational degree of freedom. This interesting property of the relativistic cavity states was noted in the context of hadronic structure studies, as it reconciled the ratio of the weak interaction axial vector to vector coupling constants g_A/g_V with experiment [4]. Along this line we note the magnetic moment of a (confined) fermion

$$\boldsymbol{\mu} = \frac{e}{2}\int d\mathbf{r}(\mathbf{r}\times\mathbf{j}) \tag{27}$$

can be written as

$$\boldsymbol{\mu} = \frac{e}{2}\int d\Gamma(\mathbf{r}\times\mathbf{g}_1), \tag{28}$$

gives $\mu_z = 0.2R$ for $\mathbf{s} = (0,0,\pm 1)$ as in Ref.[4].

We believe that this study has conclusively demonstrated that the Wigner density-matrix formulation presented for the strong-field QED is the appropriate generalization of the Wigner density-matrix concept of nonrelativistic quantum mechanics. We have shown how the symmetries of the 16 Wigner functions of relativistic particles can be used to identify their physical meaning and used them in the study of angular momentum properties of the particularly interesting quantum states. In that way we have presented here a consistent discussion of polarized relativistic $J^\pi = \frac{1}{2}^+$ states, with emphasis on angular momentum sharing between rotational and spin degree of freedom. Our examples demonstrate that there is very appreciable sharing with the orbital rotation due to relativistic spin-orbit coupling.

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