Finite model of two-dimensional ideal hydrodynamics

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A finite-dimensional su(N) Lie algebra equation is discussed that in the infinite N limit tends to the two-dimensional, inviscid vorticity equation on the torus. The equation is numerically integrated, for various values of N, and the time evolution of an (interpolated) stream function is compared with that obtained from a simple mode truncation of the continuum equation. The time-averaged vorticity moments and correlation functions are compared with canonical ensemble averages.

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I. INTRODUCTION

The vorticity equation for an ideal fluid on a two-dimensional manifold $\mathcal{M}$ is

$$\dot{\zeta}+\{\zeta,\psi\}=0\ , \tag{1}$$

where ζ is the vorticity and ψ the stream function related to ζ by

$$\zeta=-\Delta_2\psi\ . \tag{2}$$

Δ_2 is the Laplace operator on $\mathcal{M}$ and $\{f,g\}$ is the Poisson bracket of f and g.

This equation has, of course, been the subject of numerous studies over the years. It will be enough to mention the analysis of atmospheric motion (in the zero height approximation) and the theory of turbulence.

A standard approach is to expand ψ in modes of Δ_2 so that (1) becomes a coupled-mode equation, the coupling coefficients being the structure constants of the Poisson algebra, with factors involving the eigenvalues.

Precisely, we define modes Y_α and eigenvalues λ_α by

$$\Delta_2 Y_\alpha=-\lambda_\alpha Y_\alpha$$

and expand ζ and ψ,

$$\zeta=\sum_\alpha \zeta^\alpha Y_\alpha=\sum_\alpha \lambda_\alpha\psi^\alpha Y_\alpha,\quad \psi=\sum_\alpha \psi^\alpha Y_\alpha\ , \tag{3}$$

so that (1) reads

$$\lambda_\alpha\dot{\psi}^\alpha+\lambda_\beta C^\alpha{}_{\beta\gamma}\psi^\beta\psi^\gamma=0\ , \tag{4}$$

where the structure coefficients are defined by

$$\{Y_\beta,Y_\gamma\}=Y_\alpha C^\alpha{}_{\beta\gamma}$$

and are given, by orthogonality, as an integral over three harmonics. On the two-sphere, Elsasser [1] appears to have been the first to write down this integral, although he does not refer to the Poisson algebra. We will not enter into a detailed history of these sphere coefficients. They occur in the work of Silberman [2] and of Baer and Platzman [3] in early studies of atmospheric vorticity. It was noted [4,5] sometime later that the coefficients were proportional to Clebsch-Gordan coefficients, although the calculation of the reduced matrix element was cumbersome.

The two-torus T^2 presents, in some aspects, a simpler situation and its Poisson algebra was first discussed by Lorenz [6], who was concerned to truncate an infinite coupled-mode system (in the atmosphere) to the simplest nontrivial finite one. The same algebra was later investigated by Arnol'd [7], also in connection with hydrodynamics.

The modes on the torus are plane waves $\exp(i\mathbf{n}\cdot\mathbf{r})$, $(-\pi<x\le\pi, -\pi<y\le\pi)$, and the eigenvalues are $\lambda_{\mathbf{n}}=\mathbf{n}^2$, $\mathbf{n}\in\mathbb{Z}^2$. The expansions of the stream function and vorticity are

$$\psi(\mathbf{r})=\sum_{\mathbf{n}}\psi^{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{r}},\quad \zeta(\mathbf{r})=\sum_{\mathbf{n}}\zeta^{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{r}} \tag{5}$$

with $\zeta^{\mathbf{n}}=\lambda_{\mathbf{n}}\psi^{\mathbf{n}}$. The structure constants are

$$C^{\mathbf{n}}{}_{\mathbf{n}',\mathbf{n}''}=\mathbf{n}'\wedge\mathbf{n}''\delta^{\mathbf{n}}_{\mathbf{n}'+\mathbf{n}''}\ . \tag{6}$$

The problem of mode truncation is a vital one in numerical weather prediction and there seem to be no theoretical criteria for its optimum solution. One point is that any truncation does violence to the infinite set of conserved quantities (which may be taken to be the integrated powers of the vorticity) for Eq. (1). In Lorenz's truncation, for example, only the energy and the enstrophy were conserved, and this was considered to be remarkable.

It is therefore of some interest to develop finite-mode approximations that preserve more conserved quantities. Such models are suggested by the fact [8,9] that the Poisson algebra structure constants [or equivalently, the structure constants of the area-preserving diffeomorphism group SDiff($\mathcal{M}$)], are the limits of the structure constants of SU(N) as N tends to infinity, after a simple change of normalization. We can say that the commutator of two elements of the Lie algebra of SU(N) "corresponds," in the limit, to the Poisson bracket of two functions on $\mathcal{M}$. It can be seen that there are at least N constants of the motion, corresponding to the energy and the $N-1$ Casimir operators.

Zeitlin [10] has also suggested and investigated these models in works that appeared after our analysis was undertaken. There are certain differences of detail and emphasis. A reference to finite models is also made by Zakharov [11].

II. MATRIX ANALOG OF THE VORTICITY EQUATION ON T^2

Since we are interested in SU(N) we first present some standard algebraic material regarding its generators [12,8,9], which we write in the Weyl [13] form (see also Schwinger [14])

$$J_{\mathbf{n}}=\omega^{n_1 n_2/2} g^{n_1} h^{n_2} , \tag{7}$$

where the unitary $N\times N$ matrices h and g satisfy $hg=\omega gh$ and $g^N=1=h^N$. We choose N odd and $\omega=\exp(2ik\pi/N)$, where k and N are coprime. The periodicities

$$J_{\mathbf{n}+N_{\mathbf{p}}}=(-1)^{\mathbf{p}\wedge\mathbf{n}+p_1p_2}J_{\mathbf{n}},\quad k \text{ odd}$$

and

$$J_{\mathbf{n}+N_{\mathbf{p}}}=J_{\mathbf{n}},\quad k \text{ even} \tag{8}$$

can be used to bring any $\mathbf{n}$ onto the $N\times N$ lattice $\mathcal{C}_N$ (the unit cell), defined by $-(N-1)/2\le n_i\le(N-1)/2$. We occasionally use $\mathcal{C}_\infty$ to denote the entire square lattice $\mathbb{Z}^2$.

The most popular choice appears to be $\omega=\exp(4i\pi/N)$ because of the simple periodicity (8).

The $J_{\mathbf{n}}$ satisfy the relations

$$\begin{aligned}
&J_{\mathbf{n}}^\dagger=J_{-\mathbf{n}}=(J_{\mathbf{n}})^{-1} ,\\
&J_{\mathbf{n}'}J_{\mathbf{n}''}=\omega^{\mathbf{n}''\wedge\mathbf{n}'/2}J_{\mathbf{n}'+\mathbf{n}''} ,\\
&\frac{1}{N}\mathrm{Tr}(J_{\mathbf{n}'}^\dagger J_{\mathbf{n}''})=\delta_{\mathbf{n}''}^{\mathbf{n}'},\quad (\mathbf{n}',\mathbf{n}''\in\mathcal{C}_N) .
\end{aligned} \tag{9}$$

Splitting up (9) gives the commutation and anticommutation rules

$$[J_{\mathbf{n}'},J_{\mathbf{n}''}]=i\frac{2}{k}\sin\left[\frac{k\pi}{N}\mathbf{n}''\wedge\mathbf{n}'\right]J_{\mathbf{n}'+\mathbf{n}''} \tag{10}$$

and

$$[J_{\mathbf{n}'},J_{\mathbf{n}''}]_+=\frac{2}{k}\cos\left[\frac{2\pi}{N}\mathbf{n}''\wedge\mathbf{n}'\right]J_{\mathbf{n}'+\mathbf{n}''} . \tag{11}$$

Another way of writing (9) is

$$J_{\mathbf{n}_1}J_{\mathbf{n}_2}=\sum_{\mathbf{n}_3}e^{-i(2\pi k/N)A_{123}}J_{\mathbf{n}_3}\delta_{\mathbf{n}_1+\mathbf{n}_2}^{\mathbf{n}_3} , \tag{12}$$

where A_{123} is the area of the triangle formed by the vectors $\mathbf{n}_1,\mathbf{n}_2,-(\mathbf{n}_1+\mathbf{n}_2)$.

If $\mathbf{n}$ includes the origin, with $J_0=1$, the $J_{\mathbf{n}},\mathbf{n}\in\mathcal{C}_N$, form a complete operator basis,

$$\frac{1}{N}\sum_{\mathbf{n}}J_{\mathbf{n}}^\dagger T J_{\mathbf{n}}=1\,\mathrm{Tr}T .$$

(Incidentally, Schwinger [14] chooses $k=1$.)

We note [8,9] that as $N\to\infty$, the structure constants in (10) for finite $\mathbf{n}'$ and $\mathbf{n}''$, tend to those of the torus Poisson algebra, up to a normalization constant.

Let v and w be two time-dependent elements of su(N). They can be expanded in the Lie algebra generators $J_{\mathbf{n}}$,

$$v(t)=\sum_{\mathbf{n}}v^{\mathbf{n}}(t)J_{\mathbf{n}},\quad w(t)=\sum_{\mathbf{n}}w^{n}(t)J_{\mathbf{n}} . \tag{13}$$

The summation over $\mathbf{n}$ is restricted to the lattice $\mathcal{C}_N-\{\mathbf{O}\}$ where $\mathbf{O}$ is the origin. If we wish to extend the summation to $\mathcal{C}_N$, as we do, v^0 is set equal to zero for traceless v. Hermiticity is equivalent to the conditions on the coefficients

$$v^{\mathbf{n}*}=v^{-\mathbf{n}},\quad w^{\mathbf{n}*}=w^{-\mathbf{n}} . \tag{14}$$

Consider the equation

$$\dot v+iN\beta[v,w]=0 , \tag{15}$$

which we wish to compare to the hydrodynamic equation (1) with v the vorticity and w the stream matrices. β is a constant that will be specified later. There is no real significance to its value since the overall normalization is actually arbitrary and could be absorbed into a redefined time.

In order to correspond with (2), we require that as $N\to\infty$, $v^{\mathbf{n}}\to\lambda_{\mathbf{n}}w^{\mathbf{n}}$. Assuming that this has been achieved, the statement about the structure constants is that as N tends to infinity, the equation that the coefficients $w^{\mathbf{n}}$ satisfy tends to the same equation that the coefficients of the expansion of ψ in torus modes satisfy. Then, in the limit, we might hope to identify $w^{\mathbf{n}}$ with $\psi^{\mathbf{n}}$. This will be made more precise later. Our intention is to look upon these finite-dimensional models as playing the role of consistent Lorenz-type truncations although it is not clear *a priori* whether they will prove to be of practical interest.

We turn first to the relation between v and w, and it is here that we differ from Zeitlin [10]. He simply sets $v^{\mathbf{n}}$ equal to $\lambda_{\mathbf{n}}w^{\mathbf{n}}$. We feel that the relation should be expressible directly in terms of the Lie algebra elements themselves and it is not clear whether this is true for Zeitlin's relation.

Looking at (2), we require the Lie algebra analog of the Laplacian. To find this we recall the significance of the operators g and h in (7) as stepping operators in the quantum mechanics on the discretized circle [13–15].

It is easy to verify that

$$\mathcal{L}J_{\mathbf{n}}\equiv\left[\frac{N}{2\pi}\right]^2([[h,J_{\mathbf{n}}],h^{-1}]+[[g,J_{\mathbf{n}}],g^{-1}])=-\Lambda_{\mathbf{n}}J_{\mathbf{n}} ,$$

where

$$\Lambda_{\mathbf{n}}(k)=\left[\frac{N}{\pi}\right]^2\left[\sin^2\left[\frac{k\pi n_1}{N}\right]+\sin^2\left[\frac{k\pi n_2}{N}\right]\right] , \tag{16}$$

which we recognize as proportional to the eigenvalues of the difference Laplacian on the discretized torus $(2k\pi/N)\mathbb{Z}_N\otimes(2k\pi/N)\mathbb{Z}_N$. The normalization factor is chosen to give the correct continuum limit. If $k=1$, $\Lambda_{\mathbf{n}}\to\lambda_{\mathbf{n}}=\mathbf{n}^2$ as $N\to\infty$ for fixed $\mathbf{n}$. As a set, the $\Lambda_{\mathbf{n}}$ are independent of k. In fact $\Lambda_{\mathbf{n}}(1)=\Lambda_{\mathbf{n}_p}(k)$, where the components of $\mathbf{n}_p$ are cyclic permutations of those of $\mathbf{n}$ according to

$$\mathbf{n}_P=\bar{k}\mathbf{n}\bmod(N,N) \ , \quad (17)$$

where $\bar{k}$ is the mod inverse of k, i.e., $k\bar{k}=1 \bmod N$. (See, e.g., Čizek [16].)

Another way of expressing this is to say that the discrete ζ function

$$\sum_{\mathbf{n}\in\mathcal{C}_N}{}' \frac{\exp(2ik\pi\mathbf{n}\cdot\mathbf{m}/N)}{[\Lambda_\mathbf{n}(k)]^8}$$

is independent of k. The prime means that the $\mathbf{n}=\mathbf{0}$ term is to be omitted. We note that J_0, the unit matrix, is the zero mode $\mathcal{L}J_0=0$.

It might be helpful to remark that the continuum ∇^2 can be written in terms of repeated Poisson brackets

$$\nabla^2\psi=\{\{e^{ix},\psi\},e^{-ix}\}+\{\{e^{iy},\psi\},e^{-iy}\} \ .$$

We take the operator $\mathcal{L}$ to be the discrete Lie algebra analog of $\Delta_2=\nabla^2$ so that the generators $J_\mathbf{n}$ are the analogs of the modes Y_α (plane waves on T^2), as befits a complete set. The relation between v and w is thus written neatly as $v=\mathcal{L}w$ and (15) becomes

$$\mathcal{L}\dot{w}+iN\beta[\mathcal{L}w,w]=0 \ . \quad (18)$$

At this point it is convenient to discuss the conservation properties of (5). We first need the fact that $\mathcal{L}$ is Hermitian, i.e.,

$$\mathrm{Tr}(a^\dagger\mathcal{L}b)=\mathrm{Tr}(\mathcal{L}a^\dagger b) \ ,$$

where a and b are elements of su(N). (Of course $a^\dagger=a$). The trace is the finite analog of integration over $\mathcal{M}$. It is then easy to show that the quantity

$$E\equiv\frac{1}{2N}\mathrm{Tr}(vw)=\tfrac{1}{2}\sum_\mathbf{n}\Lambda_\mathbf{n}w^\mathbf{n}w^{-\mathbf{n}} \ , \quad (19)$$

which we refer to as the energy, is time independent.

Also, quite trivially and independently of the relation between v and w, the traces of powers of v are conserved

$$S_l=\frac{1}{N}\mathrm{Tr}(v^l)=\sum_{\mathbf{n}_1,\ldots,\mathbf{n}_l}v^{\mathbf{n}_1}\cdots v^{\mathbf{n}_l}\cos\left[\frac{2k\pi}{N}A_{1,\ldots,l}\right] , \quad (20)$$

where $A_{1\ldots l}$ is the area of the $(l+1)$-gon in $\mathcal{C}_\infty$ with edges $\mathbf{n}_1,\ldots,\mathbf{n}_l,\mathbf{n}_{l+1}$ subject to the restriction $\mathbf{n}_{l+1}\equiv-\sum_{i=1}^{l}\mathbf{n}_i=(0 \bmod N, 0 \bmod N)$.

If k is even, the periodicity of the cosine allows one to replace $A_{1,\ldots,l}$ by the area of the l-gon, $\mathbf{n}_1,\ldots,-\sum_1^{l-1}\mathbf{n}_i$. (In fact the whole polygon can be pulled back to fit into the unit cell.)

There are $N-1$ independent $S_l, l=1,\ldots,N-1$, corresponding to the anticommutator (11), i.e., the Casimir invariants constructed from the symmetric d^i_{jk} SU(N) invariant tensors [17–19]. S_1 always vanishes. S_2 is the enstrophy Ω.

These invariants also arise in the analyses of the generalized Euler equations of rigid body motion [9,20–22], except that the group there is taken to be SO(N) so that only the even powers remain. As N tends to infinity, (20) should become the continuum expression, assuming that the $v^\mathbf{n}$ tend to the $\zeta^\mathbf{n}$ of (5).

The dynamical equations for the stream element coefficients are

$$\dot{v}^\mathbf{n}(t)=\Lambda_\mathbf{n}\dot{w}^\mathbf{n}(t)=-\sum_{\mathbf{n}',\mathbf{n}''}G^\mathbf{n}_{\mathbf{n}',\mathbf{n}''}\Lambda_{\mathbf{n}'}w^{\mathbf{n}'}(t)w^{\mathbf{n}''}(t) \ , \quad (21)$$

where the summations are restricted to lie on the lattice $\mathcal{C}_N-\{\mathbf{O}\}$ and where the coupling coefficients are given by

$$G^\mathbf{n}_{\mathbf{n}',\mathbf{n}''}=2N\beta\sin\left[\frac{k\pi}{N}\mathbf{n}'\wedge\mathbf{n}''\right]\bar{\delta}^{\,\mathbf{n}}_{\mathbf{n}'+\mathbf{n}''} \ . \quad (22)$$

The periodicity (8) has been incorporated by defining the (quasi-)periodic delta $\bar{\delta}$ with

$$\bar{\delta}^{\,n}_l=\begin{cases}\displaystyle\sum_{\mathbf{p}\in\mathcal{C}_\infty}\delta_l^{\mathbf{n}+N_\mathbf{p}}, & k \text{ even}\\ \displaystyle\sum_{\mathbf{p}\in\mathcal{C}_\infty}(-1)^{\mathbf{n}\wedge\mathbf{p}+p_1p_2}\delta_l^{\mathbf{n}+N_\mathbf{p}}, & k \text{ odd}\end{cases} \quad (23)$$

so that $\mathbf{n}$ can be restricted to the unit cell. In (23), $l=\mathbf{n}'+\mathbf{n}''$ and the sums are actually restricted to $\mathbf{p}\in\mathcal{C}_3$ because adding two elements of $\mathcal{C}_N$ can take us only to the "nearest-neighbor" unit cells.

As $N\to\infty$, the $G^\mathbf{n}_{\mathbf{n}',\mathbf{n}''}$ tend to the Poisson algebra structure constants (for fixed $\mathbf{n}'$ and $\mathbf{n}''$) and we expect (21) to turn into Lorenz's torus equation. However it is necessary to be careful when taking the $N\to\infty$ limit. One cannot simply substitute the limiting form of (22) directly into (21) because of the behavior of terms for which $k\mathbf{n}'$ is of order N, for example. To elucidate this limit we shall rewrite the equation in coordinate representation, but first another motivation for this particular step will be given.

The aim is to solve Eq. (21) numerically for given initial conditions, and then to compare with the corresponding discussion of Lorenz, i.e., with a simple truncation. Hence there arises a question concerning the appropriate quantity to construct once the coefficients have been computed. It is possible to compare the coefficients directly but this is sensible only for N large, so making the comparison impractical. A stream "function" is needed for a global picture, that is, a continuous quantity constructed from $w(t)$, for any finite N, that can be compared with the conventional stream function $\psi(\mathbf{r},t)$ after a numerical integration.

It has been noted [23,24] that the limit $N\to\infty$ is akin to the transition from quantum to classical mechanics with, in these references, $4\pi/N$ playing the role of Planck's constant. This suggests that we regard (18) as a Heisenberg equation of motion and derive the corresponding classical equation in the standard fashion using, say, coherent states [25]. This would give a concrete connection between w and the "classical" stream function and will be pursued elsewhere. We have not seen a discussion of coherent states in Weyl's finite formulation of quantum mechanics although there are several applications of the Wigner phase-space technique [26–28], to which we now turn.

This more formal point of view is provided by the representation of quantum mechanics (called "treacherous" by Groenewold [29]) introduced by Groenewold [30], Moyal [31], and others, based on the Wigner [32] phase-space distribution, and much studied since. The quantum equations are replaced, exactly, by a classical-looking equation but with the Moyal bracket (actually due to Groenewold) instead of the Poisson bracket.

In this approach, which *a priori* is distinct from the coherent-state method, one constructs the Weyl-Wigner distributions $\mathrm{Tr}(aJ_{\mathbf{n}})$, which are then interpreted as the Fourier components of the classical quantity corresponding to the operator a. Usually one starts from a classical quantity and then asks for the corresponding quantum operator. This is the well-known ordering problem.

A more general ordering [30,33,34] is provided by setting

$$a(\mathbf{m})=\frac{1}{N}\sum_{\mathbf{n}}\mathrm{Tr}(aJ_{\mathbf{n}}^{\dagger})\Omega(\mathbf{n})e^{i(2\pi/N)k\mathbf{n}\cdot\tilde{\mathbf{m}}} \tag{24}$$

(usually $k=1$). The Weyl ordering corresponds to $\Omega=1$ and then $a(\mathbf{m})$ is called the Wigner function.

If $a(\mathbf{r})$ is to be real when a is Hermitian, the function Ω must satisfy $\Omega^*(\mathbf{n})=\Omega(-\mathbf{n})$ and we also want $\Omega\to 1$ as $N\to\infty$. Typically, $\Omega(\mathbf{n})$ is a trigonometric function of the product n_1n_2. A Gaussian form for Ω is associated with normal or antinormal ordering.

The quantity that corresponds to the commutator $[a,b]$ is the Moyal bracket (if $\Omega=1$) and, in this case, the coordinate-space representation of the vorticity equation (15) is (the proof is given shortly)

$$\dot{v}(\mathbf{m},t)+\{v,w\}_M(\mathbf{m},t)=0\ , \tag{25}$$

where the *finite* Moyal bracket $\{a,b\}_M$ is defined by

$$\{a,b\}_M(\mathbf{m})=\frac{k^3}{N\pi}\sum_{\mathbf{m}',\mathbf{m}''}a(\mathbf{m}')b(\mathbf{m}'')\sin\left[\frac{8k\pi}{N}A\right]\ , \tag{26}$$

with

$$A=\tfrac{1}{2}(\mathbf{m}\wedge\mathbf{m}'+\mathbf{m}'\wedge\mathbf{m}''+\mathbf{m}''\wedge\mathbf{m})\ ,$$

the area of the dual triangle with vertices at $\mathbf{m}$, $\mathbf{m}'$, and $\mathbf{m}''$.

It is clearly possible [35] to generalize the Moyal bracket to allow for the more general ordering (24) involving Ω, but we will not pursue this point here except to say that (7) and (24) show that the different choices for k are related to the ordering question.

As mentioned before, one reason for introducing the coordinate-space representation is that the infinite N limit appears to be more transparent than in the mode representation (21), which always remains discrete. We will deduce a value for the constant β.

Equations (25) and (26) are now derived. The finite Fourier relation we require reads

$$\frac{1}{N^2}\sum_{\mathbf{n}\in\mathcal{C}_N}e^{i(2\pi/N)k\mathbf{n}\cdot\mathbf{m}}=\sum_{\mathbf{p}\in\mathcal{C}_\infty}\delta^{m}_{N\mathbf{p}}\ , \tag{27}$$

and the transform is defined by

$$\begin{aligned}a(\mathbf{m})&=\sum_{\mathbf{n}\in\mathcal{C}_N}a^{\mathbf{n}}e^{i(2\pi/N)k\mathbf{n}\cdot\tilde{\mathbf{m}}}=\sum_{\mathbf{n}\in\mathcal{C}_N}a^{\mathbf{n}_{\mathbf{p}}}e^{i(2\pi/N)\mathbf{n}\cdot\tilde{\mathbf{m}}}\ ,\\ a^{\mathbf{n}}&=\frac{1}{N^2}\sum_{\mathbf{m}\in\mathcal{C}_N^*}a(\mathbf{m})e^{-i(2\pi/N)k\mathbf{n}\cdot\tilde{\mathbf{m}}}\ .\end{aligned} \tag{28}$$

$\mathcal{C}_N^*$ is the dual lattice. We often identify $\mathcal{C}_N$ and $\mathcal{C}_N^*$. The expressions provide periodic extensions of $a(\mathbf{m})$ and $a^{\mathbf{n}}$ off the corresponding unit cells.

As a technical point of some interest, the appearance of the factor of k in (28) is related to the use of the eigenvalues $\Lambda_{\mathbf{n}}(k)$ of (16). If we had simply chosen to set $k=1$ in (16) (but *not* in the definition of ω), then the k in (28) must be unity too. There is nothing wrong in this, but it would not then be possible to write the dynamical equations in purely Lie algebra terms, as we have done in (18). We believe this is of more than aesthetic importance.

As $N\to\infty$, with $\mathbf{r}=(2\pi/N)\tilde{\mathbf{m}}$ and $a(\mathbf{m})\to a(\mathbf{r})$, the formulas (28) turn into a standard Fourier series,

$$\begin{aligned}a(\mathbf{r})&=\sum_{\mathbf{n}}\bar{a}^{\,\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{r}}\ ,\\ \bar{a}^{\,n}&=\frac{1}{(2\pi)^2}\int_{T^2}d\mathbf{r}\,a(\mathbf{r})e^{-i\mathbf{n}\cdot\mathbf{r}}\ ,\end{aligned} \tag{29}$$

and $\bar{a}^{\,\mathbf{n}}=a^{\bar{\mathbf{n}}}$, where $\bar{\mathbf{n}}$ stands for a pair of reordered sets of *all* the integers. We have used the continuum replacement

$$\sum_{\mathbf{m}\in\mathcal{C}_N^*}\to\left[\frac{N}{2\pi}\right]^2\int_{T^2}d\mathbf{r}\ .$$

It is formally attractive to define the transformed generators, $J(\mathbf{m})$, by

$$\begin{aligned}J(\mathbf{m})&=\sum_{\mathbf{n}\in\mathcal{C}_N}J_{\mathbf{n}}e^{-i(2\pi/N)k\mathbf{n}\cdot\tilde{\mathbf{m}}}\ ,\\ J_{\mathbf{n}}&=\frac{1}{N^2}\sum_{\mathbf{m}\in\mathcal{C}_N^*}J(\mathbf{m})e^{i(2\pi/N)k\mathbf{n}\cdot\tilde{\mathbf{m}}}\ ,\end{aligned} \tag{30}$$

so that

$$v=\sum_{\mathbf{n}}v^{\mathbf{n}}J_{\mathbf{n}}=\frac{1}{N^2}\sum_{\mathbf{m}}v(\mathbf{m})J(\mathbf{m})\ .$$

The $J(\mathbf{m})$ are Hermitian, and as we have defined them, satisfy the relations dual to (9)

$$\begin{aligned}&\mathrm{Tr}J(\mathbf{m})=N^2\quad\text{for all }\mathbf{m}\ ,\\ &J(\mathbf{m}')J(\mathbf{m}'')=\sum_{\mathbf{m}}f(\mathbf{m}',\mathbf{m}'';\mathbf{m})J(\mathbf{m})\ ,\\ &\frac{1}{N^2}\mathrm{Tr}[J(\mathbf{m})J(\mathbf{m}')]=N\bar{\delta}_{\mathbf{m}-\mathbf{m}'}\ ,\\ &J_{-\mathbf{m}}J(\mathbf{m}')J_{\mathbf{m}}=J(\mathbf{m}'-\mathbf{m})\ .\end{aligned} \tag{31}$$

These relations hold for all k but an even value would be preferred because of the implied simple periodicity of the $J_{\mathbf{n}}$.

If k is even, a short calculation using (27) shows that the composition constants are given by

$$f(\mathbf{m}',\mathbf{m}'';\mathbf{m})=\exp\left[i\frac{8k\pi}{N}A(\mathbf{m}',\mathbf{m}'',\mathbf{m})\right], \qquad (32)$$

where $A(\mathbf{m}',\mathbf{m}'',\mathbf{m}),=-A(\tilde{\mathbf{m}}',\tilde{\mathbf{m}}'',\tilde{\mathbf{m}})$, is the area of the triangle $(\mathbf{m}',\mathbf{m}'',\mathbf{m})$ on the dual lattice, given before.

Consider the product of two operators (i.e., Lie algebra elements)

$$ab=\frac{1}{N^4}\sum_{\mathbf{m}'}\sum_{\mathbf{m}''}a(\mathbf{m}')b(\mathbf{m}'')J(\mathbf{m}')J(\mathbf{m}'')$$

$$\equiv\frac{1}{N^2}\sum_{\mathbf{m}}(a*b)(\mathbf{m})J(\mathbf{m}),$$

all sums being over $\mathcal{C}_N$. This defines the $*$ or Moyal product. Therefore from (31)

$$(a*b)(\mathbf{m})=\frac{1}{N^2}\sum_{\mathbf{m}',\mathbf{m}''}a(\mathbf{m}')b(\mathbf{m}'')f(\mathbf{m}',\mathbf{m}'';\mathbf{m}). \qquad (33)$$

Taking the commutator gives

$$[a,b]=\frac{1}{N^2}\sum_{\mathbf{m}}[a,b](\mathbf{m})J(\mathbf{m}),$$

where

$$[a,b](\mathbf{m})=(a*b-b*a)(\mathbf{m})$$

$$=\frac{2i}{N^2}\sum_{\mathbf{m}',\mathbf{m}''}a(\mathbf{m}')b(\mathbf{m}'')\sin\left[\frac{8k\pi}{N}A(\mathbf{m}',\mathbf{m}'',\mathbf{m})\right]$$

$$=\frac{2i\pi}{k^3N}\{a,b\}_M(\mathbf{m}). \qquad (34)$$

Equation (15) can then be written as (25), with (26), as promised, if $\beta=k^3/2\pi$.

The continuum limit of (25) can be checked by replacing $\mathbf{m},\mathbf{m}',\mathbf{m}''$ by $\mathbf{r},\mathbf{r}',\mathbf{r}''$, respectively, where $\mathbf{r}=(2\pi/N)\tilde{\mathbf{m}}$, etc. (for all $\mathbf{k}$). Then, in the infinite N limit $\mathbf{m}\to\infty$ the sums turn into integrals and just as (28) becomes (29), (26) goes over into

$$\{a,b\}_M(\mathbf{m})\to\lim_{N\to\infty}\frac{k^3N^3}{4\pi^5}\times\int_{T^2\times T^2}d\mathbf{r}'d\mathbf{r}''a(\mathbf{r}')b(\mathbf{r}'')\times\sin\left[\frac{2kN}{\pi}A(\mathbf{r},\mathbf{r}',\mathbf{r}'')\right]=\{a,b\}(\mathbf{r}), \qquad (35)$$

the Poisson bracket, as required. k is assumed to remain fixed and $\mathbf{r}\in T^2$. (The finite integration ranges could be replaced by infinite ones, in the limit, to give precisely the same integrals as in Baker [36]. $A(\mathbf{r},\mathbf{r}',\mathbf{r}'')=\frac{1}{2}(\mathbf{r}\wedge\mathbf{r}'+\mathbf{r}'\wedge\mathbf{r}''+\mathbf{r}''\wedge\mathbf{r})$ is the area of the coordinate-space triangle $(\mathbf{r},\mathbf{r}',\mathbf{r}'')$. The conclusion is that Eq. (25) becomes the continuum Euler equation (1).

The coordinate-space representation also allows us to confirm the form of the discrete Laplacian $\mathcal{L}$. The standard expression for the finite-difference Laplacian on $\mathbf{T}^2$, scaled to give the correct continuum limit, is

$$\Delta^2J(\mathbf{m})=\left[\frac{N}{2\pi}\right]^2\sum_{\langle\mathbf{m}\rangle}[J(\langle\mathbf{m}\rangle)-J(\mathbf{m})],$$

where $\langle\mathbf{m}\rangle$ are the nearest neighbors to $\mathbf{m}$. Thus

$$\Delta^2J(\mathbf{m})=\left[\frac{N}{2\pi}\right]^2[J(m_1,m_2-1)+J(m_1,m_2+1)-2J(m_1,m_2)+J(m_1+1,m_2)+J(m_1-1,m_2)-2J(m_1,m_2)].$$

It is easily shown that

$$\Delta^2J(\mathbf{m})=\left[\frac{N}{2\pi}\right]^2([[h,J(\mathbf{m})],h^{-1}]+[[g,J(\mathbf{m})],g^{-1}]).$$

Thus one can write

$$\mathcal{L}v=\frac{1}{N^2}\sum_{\mathbf{m}\in\mathcal{C}_N^*}v(\mathbf{m})\Delta^2J(\mathbf{m})=\frac{1}{N^2}\sum_{\mathbf{m}\in\mathcal{C}_N^*}\Delta^2v(\mathbf{m})J(\mathbf{m}),$$

with $\Delta^2v(\mathbf{m})\to\nabla^2v(\mathbf{r})$, as required.

The invariants S_l too can be recast in terms of $v(\mathbf{m})$. We write

$$S_l=\frac{1}{(N)^{2l}}\sum_{\mathbf{m}_1,\ldots,\mathbf{m}_l}v(\mathbf{m}_1)v(\mathbf{m}_2)\cdots v(\mathbf{m}_l)G(\mathbf{m}_1,\ldots,\mathbf{m}_l). \qquad (36)$$

G is related to a finite Fourier transform (a Gaussian sum) by

$$G(\mathbf{m}_1,\ldots,\mathbf{m}_l)=\frac{1}{N}\mathrm{SymTr}[J(\mathbf{m}_1)\ldots J(\mathbf{m}_l)], \qquad (37)$$

where from (30) and (12)

$$\frac{1}{N}\mathrm{Tr}[J(\mathbf{m}_1)\cdots J(\mathbf{m}_l)]=\sum_{\mathbf{n}_1,\ldots,\mathbf{n}_l}\exp\left[-i(2k\pi/N)\left[\sum_{l=1}\mathbf{n}_i\cdot\tilde{\mathbf{m}}_i-A_{1\ldots l}\right]\right].$$

The symmetrization on the $\mathbf{m}_i$ can be performed in various ways. Simply reversing their order gives

$$G(\mathbf{m}_1,\ldots,\mathbf{m}_l)=\sum_{\mathbf{n}_1,\ldots,n_l}\exp\left[-i(2k\pi/N)\sum_{i=1}^{l}\mathbf{n}_i\cdot\tilde{\mathbf{m}}_i\right]\times\cos\left[\frac{2k\pi}{N}A_{1,\ldots,l}\right] \qquad (38)$$

with the mod$N$ condition on $\sum\mathbf{n}_i$.

Another formula results on combining the $J(\mathbf{m})$ in (37) using the composition law (31) to give

$$\frac{1}{N}\mathrm{Tr}[J(\mathbf{m}_1)\cdots J(\mathbf{m}_l)]=\sum_{\{\mathbf{m}_{i'}\}}f(\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_{3'})\times f(\mathbf{m}_{3'},\mathbf{m}_3,\mathbf{m}_4)\cdots f(\mathbf{m}_{(l-1)'},\mathbf{m}_{l-1},\mathbf{m}_l). \qquad (39)$$

It is interesting to check that, in the infinite N limit, S_l becomes the integrated power of the vorticity function, that is, up to a renormalization factor,

$$S_l \to \int_{T^2} d\mathbf{r}\, v(\mathbf{r})^l \tag{40}$$

and it is instructive to carry the limit through completely in coordinate space *after* the summations over the $\mathbf{n}_i$ have been done.

The behavior of the function $G(\mathbf{m}_1, \ldots, \mathbf{m}_l)$ as N tends to infinity is required. As usual we set $\mathbf{r}_i = (2\pi/N)\tilde{\mathbf{m}}_i$ and write $G(\mathbf{r}_1, \ldots, \mathbf{r}_l)$. Taking the expression (36) for the invariant S_l, rescaling the $\tilde{\mathbf{m}}_i$ to the $\mathbf{r}_i$ and changing the summation into integrations produces

$$S_l \to \frac{1}{(2\pi)^{2l}} \int_{T^2 \times \cdots \times T^2} d\mathbf{r}_1 \cdots d\mathbf{r}_l v(\mathbf{r}_1) \cdots v(\mathbf{r}_l) \times G(\mathbf{r}_1, \ldots, \mathbf{r}_l) \,. \tag{41}$$

The simplest nontrivial example is l equal to 3, when the polygons are triangles. Then, immediately, from (31) and (32)

$$G(\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3) = N^2 \cos\left[\frac{8k\pi}{N} A(\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3)\right].$$

On rescaling, $A(\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3)$ becomes $(N/2\pi)^2 A(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ and we can now apply another formula given in Baker [36],

$$\lim_{N\to\infty} \left[\frac{kN}{\pi}\right]^{2v} \int_{-\infty}^{\infty} d\mathbf{r}_1 \cdots d\mathbf{r}_{2v+1} \cos\left[\frac{2kN}{\pi}[A(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) + A(\mathbf{r}_1,\mathbf{r}_4,\mathbf{r}_5) + \cdots + A(\mathbf{r}_1,\mathbf{r}_{2v},\mathbf{r}_{2v+1})]\right] F(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2v+1}) = \int_{-\infty}^{\infty} d\mathbf{r}_1 F(\mathbf{r}_1,\ldots,\mathbf{r}_1) \,, \tag{42}$$

for $v=1$ to give the desired continuum integral of $v(\mathbf{r})^3$ in the infinite N limit. We note that in (42) F must be a reasonable function.

The case of any l will now be discussed. Although (39) is completely in coordinate space it is not in a convenient form for the application of (42). In fact, the general form of $G(\mathbf{m}_1, \ldots, \mathbf{m}_l)$ can be given with no summations. The expression depends on whether l is even or odd and in fact, we shall restrict the discussion to odd l for brevity.

We start from the form (38) and begin by replacing $\mathbf{n}_l$ by $\bar{\mathbf{n}}_l \equiv -\sum_1^{l-1} \mathbf{n}_i$. This is allowed because of the periodicity of the exponential and the cosine. We might then just as well rename $\bar{\mathbf{n}}_l$ as $\mathbf{n}_l$ and restrict the sum (38) to *closed* l-gons, as mentioned earlier.

The evaluation proceeds by alternate application of (27) and imposition of the resulting $\bar{\delta}$. The choice of which $\mathbf{n}_i$ to sum over is crucial to obtaining a simple symmetrical result. It is convenient to first perform a cyclic permutation of the $\mathbf{n}_i$ (under which $A_{1\ldots l}$ is invariant) so that $\mathbf{n}_b$ becomes $\mathbf{n}_l$ where b is the next integer after $l/4$. Then, performing the sums in the order $\mathbf{n}_l$ downwards, we find

$$G(\mathbf{m}_1, \ldots, \mathbf{m}_t) = N^{l-1} \cos\left[\frac{8k\pi}{N} E(\{\mathbf{m}_i\})\right], \tag{43}$$

where $E(\{\mathbf{m}_i\})$ is given by

$$\begin{aligned} E(\{\mathbf{m}_i\}) &= \sum_{i=2,4,\ldots,l} A(\mathbf{m}_1,\mathbf{m}_i,\mathbf{m}_{i+1}) + \sum_{\substack{i\neq j \\ i=2,4,\ldots \\ j=2,4,\ldots}}^{l-1} (\mathbf{m}_i - \mathbf{m}_{i+1}) \wedge (\mathbf{m}_j - \mathbf{m}_{j+1}) \\ &\equiv A(\{\mathbf{m}_i\}) + B(\{\mathbf{m}_i\}) \,. \end{aligned} \tag{44}$$

This is a closed form for the Gauss sum. The first summation is the area of $(l-1)/2$ triangles connected at the vertex $\mathbf{m}_1$ in a windmill sail pattern. The second sum is that of the cross products of all pairs of vane ends.

Equation (43) with (44) yields (41) with G given by

$$\begin{aligned} G(\mathbf{r}_1, \ldots, \mathbf{r}_l) &= N^{l-1} \cos\left[\frac{2kN}{\pi}[A(\{\mathbf{r}_i\}) + B(\{\mathbf{r}_i\})]\right] \\ &= N^{l-1}\left[\cos\left[\frac{2kNA}{\pi}\right]\cos\left[\frac{2kNB}{\pi}\right] - \sin\left[\frac{2kNA}{\pi}\right]\sin\left[\frac{2kNB}{\pi}\right]\right] \end{aligned} \tag{45}$$

in terms of rescaled quantities.

A completely immediate application of (42) to (41) is not possible because the function F now contains N. However we note that the effect of the $\cos(2kNA/\pi)$ terms in (42) in the $N \to \infty$ limit is to force $\mathbf{r}_i$ to equal $\mathbf{r}_{i+1}, (i=2,4,\ldots,2l)$ and also both to equal $\mathbf{r}_1$. Since B has a product structure [see (44)], the $\cos(2kNB/\pi)$ factor can be replaced by unity in the limit. We also note that the same equation as (42), but with a sine [as used in (35) with an extra factor of N] gives a result of the order of $1/N$ and so the second term in (45) goes away. Equation (42) can now be applied directly with $F(\mathbf{r}_1, \ldots, \mathbf{r}_1) = v(\mathbf{r}_1) \cdots v(\mathbf{r}_l)$ to give the required continuum expression (40). Our discussion of the coordinate representation of the invariants ends at this point.

We can now give a reasonable answer to a question posed earlier regarding the appropriate quantity to construct from the matrix $w(t)$ that can be compared with the continuum steam function $\psi(\mathbf{r},t)$ resulting from a standard mode truncation of Euler's equations (1). The Fourier transform (28) suggests the simple interpolation

$$w(\mathbf{r},t)=\sum_{\mathbf{n}\in\mathscr{C}_N} w^{\mathbf{n}_P}(t)e^{i\mathbf{n}\cdot\mathbf{r}} \quad (46)$$

as a possible analog of $\psi(\mathbf{r},t)$. For convenience, it is this quantity that is plotted, but it is clear of course that there cannot be a unique quantity corresponding to ψ.

We note that the Fourier coefficients in (46) are evaluated at the permuted points $\mathbf{n}_P$. This means that for any k, the plane-wave modes with the smaller $|n_1|,|n_2|$ are associated with the smaller eigenvalues $\Lambda_{\mathbf{n}_P}(k)=\Lambda_{\mathbf{n}}(1)$, as occurs in the continuous case. A naive application of the $N\to\infty$ limit to the momentum-space equations (21) and (22) does not give the correct result.

There is a peculiarity in that the coordinate-space treatment of the $N\to\infty$ limit is not easily available to us for what appears to be the simplest value of k (from the Fourier transform point of view), namely, unity. For finite N, the models for different values of the parameter k seem to be distinct. Our treatment shows, however, that they will all yield the same continuum limit but we can vouch for our coefficients only when k is even.

III. MAXIMUM SIMPLIFICATION

We now turn to the practical solution of Eqs. (21) along the lines of Lorenz's calculation [6]. Since ψ is real its Fourier coefficients satisfy the condition $\psi^{\mathbf{n}*}=\psi^{-\mathbf{n}}$. The obvious finite equivalent is the hermiticity of w (14).

Lorenz [6] notices that if the coefficients are chosen to be real at some initial time, they will remain real throughout the time development. The reality condition means that $w^{\mathbf{n}}=w^{-\mathbf{n}}$ and using the symmetries $\Lambda_{-\mathbf{n}}=\Lambda_{\mathbf{n}}$ and $G^{\mathbf{n}}_{\mathbf{n}',\mathbf{n}''}=G^{-\mathbf{n}}_{-\mathbf{n}',-\mathbf{n}''}=-G^{\mathbf{n}}_{\mathbf{n}'',\mathbf{n}'}$ we can deduce from (21) that

$$\Lambda_{\mathbf{n}}\frac{d}{dt}(w^{\mathbf{n}}-w^{-\mathbf{n}})=\tfrac{1}{2}\sum_{\mathbf{n}',\mathbf{n}''}(\Lambda_{\mathbf{n}'}-\Lambda_{\mathbf{n}''})G^{\mathbf{n}}_{\mathbf{n}',\mathbf{n}''}(w^{\mathbf{n}'}+w^{-\mathbf{n}'})\times(w^{\mathbf{n}''}-w^{-\mathbf{n}''}) \quad (47)$$

showing that if the condition $w^{\mathbf{n}}=w^{-\mathbf{n}}$ is valid for all $\mathbf{n}$ at some time, it remains valid.

For $N=3$ the number of independent real coefficients is four. This is the smallest number that we can take for a consistent group-theoretical structure. Lorenz makes a further identification, that is also propagated in time, and this reduces his number to three. In general, the number of coefficients in our maximum simplification is $(N^2-1)/2$.

For real w^n, w can be rearranged into

$$w=\tfrac{1}{2}\sum_{\mathbf{n}} w^{\mathbf{n}}(J_{\mathbf{n}}+J_{-\mathbf{n}}) .$$

The combination of generators that occurs on the right-hand side gives just the generators of the subgroup $U(SU((N+1)/2)\otimes SU((N-1)/2))$, which means that the torus has actually been turned into a tetrahedron [37], which might not be unreasonable for discussing atmospheric motion on the whole earth.

In all our calculations k was set equal to 2. The numerical procedure consisted of choosing an initial distribution of the coefficients $w^{\mathbf{n}}$ and then integrating (21) by standard routines. The results for the stream "function" were displayed in coordinate space using the Fourier interpolation (46). For each odd value of N from 3 to 31 the results were compared with those for a simple truncation method using the same number of modes.

The conservation of E and of the S_l was used as a check of the algorithms and algebra. The initial configurations are discussed in the next section.

IV. VORTICES ON A LATTICE

For the sake of having something definite, it is interesting to propagate a system of lattice vortices. A suitable set of initial stream coefficients for a single vortex situated at $\mathbf{m}=\mathbf{m}_i$ would be

$$w_i(\mathbf{m})=\sum_{\mathbf{n}\in\mathscr{C}_N}{}' \frac{\exp(2ik\pi\mathbf{n}\cdot(\tilde{\mathbf{m}}-\tilde{\mathbf{m}}_i)/N)}{\Lambda_{\mathbf{n}}(k)} . \quad (48)$$

They are independent of k. Further, $\sum_{\mathbf{m}} w_i(\mathbf{m})=0$. The mode coefficients are $w_i^n=\exp(2ik\pi\mathbf{n}\cdot\mathbf{m}_i/N)/\Lambda_{\mathbf{n}}(k)$ if $\mathbf{n}\neq 0$ and $w_i^0=0$. For the vorticity, $v_i^{\mathbf{n}}=\exp(2ik\pi\mathbf{n}\cdot\mathbf{m}_i/N)$ if $\mathbf{n}\neq\mathbf{0}$ with $v_i^0=0$. In coordinate space, $v_i(\mathbf{m})=N^2\delta^m_{\mathbf{m}_i}-1$ showing that the vorticity is mostly concentrated at the point $\mathbf{m}_i$, justifying the term "vortex." The smaller, negative value of -1 makes the total "integrated" vorticity $\sum_{\mathbf{m}} v_i(\mathbf{m})$ zero as necessitated by the compactness of the domain. However, it is not possible to conserve the vorticity located (in some sense) at $\mathbf{m}_i$, which can leak away.

As N and $\mathbf{m}$ tend to infinity, the discrete stream function $w_i(\mathbf{m})$ of (48) turns into the Epstein ζ function, except at $x=0, y=0$,

$$w_i(\mathbf{m})\to\sum_{\mathbf{n}\in\mathscr{C}_\infty}{}' \frac{\exp(i\mathbf{n}\cdot\mathbf{r})}{\mathbf{n}^2}=Z\begin{vmatrix}0 & 0\\ x/2\pi & y/2\pi\end{vmatrix}(2) , \quad (49)$$

in Epstein's notation [38]. For simplicity, we have set $k=1$ and $\mathbf{m}_i=\mathbf{0}$. x and y are the coordinates of $\mathbf{r}$ with $x=-2\pi m_2/N$ and $y=2\pi m_1/N$. In the limit we would regard x and y as being continuous and restricted to the range $-\pi$ to π. It should be remarked that for any fixed value of $\mathbf{m}$, the difference between $w_i(\mathbf{m})$ and the ζ function evaluated at $\mathbf{r}=2\pi\tilde{\mathbf{m}}/N$ will be a zero constant that decreases with increasing $\mathbf{m}$.

We can now compare (49) with some results for the stream function on the torus derived in earlier times. Greenhill [39] and Hicks [40] give the stream function in a rectangle. The expression on the torus can be found in the intermediate calculation, but perhaps the easiest method of proceeding is the following.

The stream function on the torus for a vortex at the origin is given as the image expression (cf. [40])

$$\psi(x,y)=\frac{\kappa}{4\pi}\sum_{\mathbf{M}}\ln\frac{(x/2\pi+M_1)^2+(y/2\pi+M_2)^2}{(M_1^2+M_2^2)} ,$$

where the sums run over all the integers and x and y are restricted to the range $-\pi$ to π. Up to an additional constant,

$$\psi(x,y)=-\frac{\kappa}{2\pi}\left[\frac{\partial}{\partial s}\sum_{\mathbf{M}}\frac{1}{[(x/2\pi+M_1)^2+(y/2\pi+M_2)^2]^{s/2}}\right]_{s=0}$$
$$=-\frac{\kappa}{2\pi}\left[\frac{\partial}{\partial s}Z\begin{vmatrix}x/2\pi & y/2\pi\\ 0 & 0\end{vmatrix}(s)\right]_{s=0}\equiv-\frac{\kappa}{2\pi}Z'\begin{vmatrix}x/2\pi & y/2\pi\\ 0 & 0\end{vmatrix}(0)\ .$$

Evaluation of the transformation formula [38]

$$\pi^{-s/2}\Gamma(s/2)Z\begin{vmatrix}h_1 & h_2\\ 0 & 0\end{vmatrix}(s)$$
$$=\pi^{s/2-1}\Gamma(1-s/2)Z\begin{vmatrix}0 & 0\\ -h_1 & -h_2\end{vmatrix}(2-s)$$

at $s=0$ yields

$$Z'\begin{vmatrix}h_1 & h_2\\ 0 & 0\end{vmatrix}(0)=\frac{1}{2\pi}Z\begin{vmatrix}0 & 0\\ -h_1 & -h_2\end{vmatrix}(2)$$

since

$$Z\begin{vmatrix}h_1 & h_2\\ 0 & 0\end{vmatrix}(0)=0$$

if h_1 and h_2 are not integers. Hence

$$\psi(x,y)=-\frac{\kappa}{(2\pi)^2}Z\begin{vmatrix}0 & 0\\ x/2\pi & y/2\pi\end{vmatrix}(2)\ . \tag{50}$$

Looking at (49), it can be seen that the discrete stream function $w_i(\mathbf{m})$ tends to the torus stream function $\psi(x,y)$ as $N\to\infty$ for strength $\kappa=-4\pi^2$.

Hicks [40] gives an expression for $\psi(x,y)$ in terms of simpler θ functions. This is, up to the usual additive constant,

$$\psi(x,y)=\frac{\kappa}{4\pi}\left[\frac{1}{4\pi}(x^2-y^2)+\ln[\theta_1(z/2\pi,i)\theta_1(z^*/2\pi,i)]\right]\ , \tag{51}$$

where $z=x+iy$ and $\theta_1(u)$ is denoted by $H(2Ku)$ in Hicks and Greenhill. Using the Jacobi transformation formula for θ functions, it is easily checked that $\psi(x,y)$ is symmetrical in x and y. Our definitions of θ functions are those of Oberhettinger and Magnus [41], where a brief discussion of the motion of vortex systems can also be found in Chap. 4.

Incidentally, Kronecker [41] reduced the Epstein ζ function (49) to a form in θ functions,

$$Z\begin{vmatrix}0 & 0\\ h_1 & h_2\end{vmatrix}(2)=2\pi^2h_1^2-\pi\ln\frac{\theta_1(u_1,\omega_1)\theta_1(u_2,-\omega_2)}{\eta(\omega_1)\eta(-\omega_2)}\ , \tag{52}$$

where η is the Dedekind function $\eta(\omega)=q^{1/12}\prod_{n=1}^{\infty}(1-q^{2n})$. The variables h_1 and h_2 are to be taken in the range from 0 to 1. For the square torus $\omega_1=\omega_2^*=i$, $q=e^{-\pi}$, and $u_1=u_2^*=(x+iy)/2\pi$. Comparing (52) with (50) and (51), we see that Hicks and Greenhill had earlier obtained an equivalent reduction. Kronecker's formula has been rediscovered a number of times since. These formulas can be used for numerical evaluation of ψ (50), although there exists a form in terms of incomplete Γ functions that is generally more efficient, except for small $x+iy$.

For a single vortex, a glance at Eq. (21) reveals that $\dot{w}^{\mathbf{n}}$ vanishes for all $\mathbf{n}$ and the vortex is thus stationary. The energy is given by $E=w(0)=\sum[1/\Lambda_{\mathbf{n}}(k)]$ and the enstrophy is $S_2=N^2-1$. Both these quantities diverge as $N\to\infty$.

The stream matrix coefficients for a set of vortices is

$$w(\mathbf{m})=\sum_i\frac{\kappa_i}{4\pi^2}w_i(\mathbf{m})\ ,$$

where the κ_i are the vortex strengths. For two equal vortices setting $\mathbf{m}_2=-\mathbf{m}_1$ and $\kappa_2=\kappa_1=-4\pi^2$, we get real coefficients $w^{\mathbf{n}}=2\cos(2k\pi\mathbf{n}\cdot\mathbf{m}_1/N)/\Lambda_{\mathbf{n}}(k)$, $w^0=0$ so that we can place this configuration on the tetrahedron. The enstrophy equals $2(N^2-2)$, while the next invariant is $4(16-N^2)$.

The systems with N from 3 to 31 were evolved in time. We present the results in Fig. 1 for SU(9) as being typical. The initial position of the vortices was at $\mathbf{m}_1=(1,0)$. In our view no particular pattern emerges.

In any finite scheme, the localizability of individual vortices is lost, the vorticity can become redistributed, and it is difficult to model the motion of ideal point vortices in this way, unless possibly N is extremely large. However the evolution for SU(31) shown in Fig. 2 offers no evidence for such a trend.

The results for SU(5) were somewhat atypical and are displayed in Fig. 3. The vortices appear to be rotating around each other. The mode-truncation calculation starting from the same stream function yields results of a generally similar nature. A number of other initial configurations were also propagated with, again, roughly comparable outcomes.

V. STATISTICAL BEHAVIOR

The numerical results so far presented are for relatively short-time evolution. It has been suggested by Kraichnan [43] that two-dimensional turbulence can be statistically modeled on the assumption that the system is ergodic and can, after a sufficiently long time, be described by a microcanonical or even a canonical distribution. In the latter case, two "temperatures" can be introduced as Lagrange multipliers for the conserved quadratic quantities, the energy E, and the enstrophy Ω.

A number of computer experiments have been performed in both the viscid and inviscid cases on the truncated versions of the Euler and Navier-Stokes equations to test this idea. The results are suggestive but not conclusive.

The models discussed in the present work allow a similar numerical analysis. These systems, having more than just the two conserved quantities E and Ω of the truncated theory, might provide a more realistic arena in which to explore the statistical hypothesis.

With this in mind the systems were evolved for long times, at a reduced numerical accuracy for speed purposes. The quantities E and the S_l were found typically to be conserved to one part in 10^5 to 10^6 over the extended time period. Vorticity moments and correlation functions were evaluated by simple time averaging since actu-

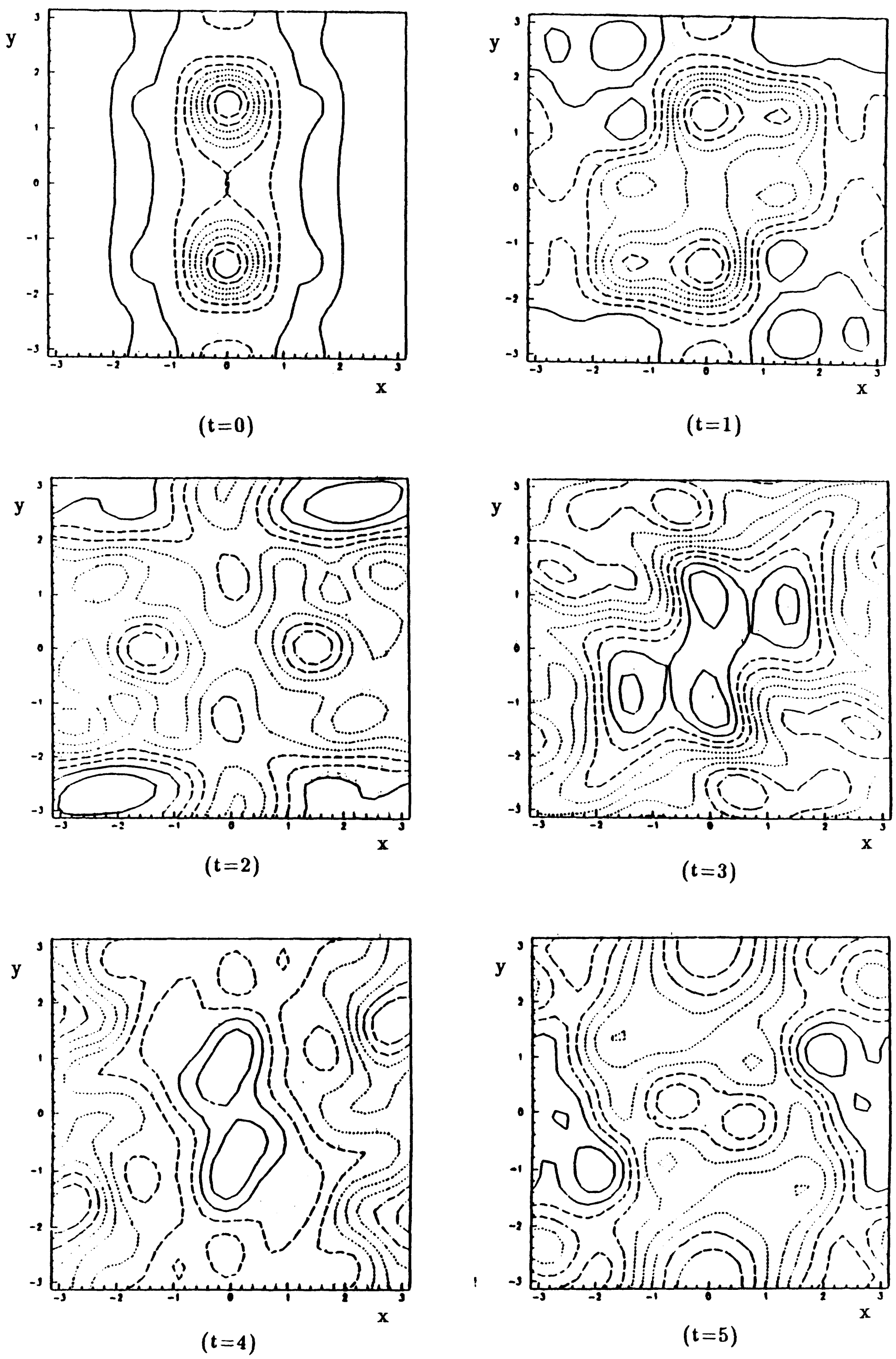

FIG. 1. Double vortex stream function for SU(9) plotted on the torus at evolution times $t = 0$, 1, 2, 3, 4, and 5.

al ensemble averaging was impractical. The results were compared with the canonical distribution values where appropriate. The basic theory can be found in Kells and Orszag [44], for example, and so we need not give many details.

The canonical vorticity distribution is given by

$$P(\zeta)=\frac{1}{Z}\exp\left[-\sum_{\mathbf{n}}(\alpha\Lambda_{\mathbf{n}}^{-1}+\beta)|\zeta^{\mathbf{n}}|^2\right]$$

with the partition function $Z=\int P(\zeta)d\zeta$. The relation with the parameters in Ref. [44] is $\alpha=C^{-1}$ and $\beta=D^{-1}$.

Ensemble averages are

$$\langle F(\zeta)\rangle=\int F(\zeta)P(\zeta)d\zeta \; .$$

In terms of the two temperatures α and β the moments of the vorticity are

$$\langle(\zeta^{\mathbf{n}})^p\rangle=\frac{(2p-1)!!}{2^p}(\alpha\Lambda_{\mathbf{n}}^{-1}+\beta)^{-p} \; . \tag{53}$$

The time-averaged moments are

$$M_p(\mathbf{n},T)=\frac{1}{T}\int_0^T|\zeta^{\mathbf{n}}|^p dt \; ,$$

which, for large T, are to be compared with the ensemble averages (53).

The mode correlation function $C(s)$ is defined by

$$C(\mathbf{n},T;s)=\frac{1}{TM_2(\mathbf{n},T-s)}\int_0^{T-s}\zeta^{\mathbf{n}}(t)\zeta^{\mathbf{n}}(t+s)dt \; .$$

The prefactor is a normalization. If $C(s)$ does not tend to zero with increasing s the system is probably not ergodic and cannot be described by a statistical ensemble.

Three starting distributions were chosen. One was the double vortex discussed in Sec. IV, another was a vortex-antivortex pair, and a third was a more or less random arrangement. In the latter case the coefficients in the truncated model were adjusted to give the same energy and enstrophy as the corresponding SU(N) model so that the calculated canonical temperatures α and β should be the same.

Figures 4 and 5 display some results using the double

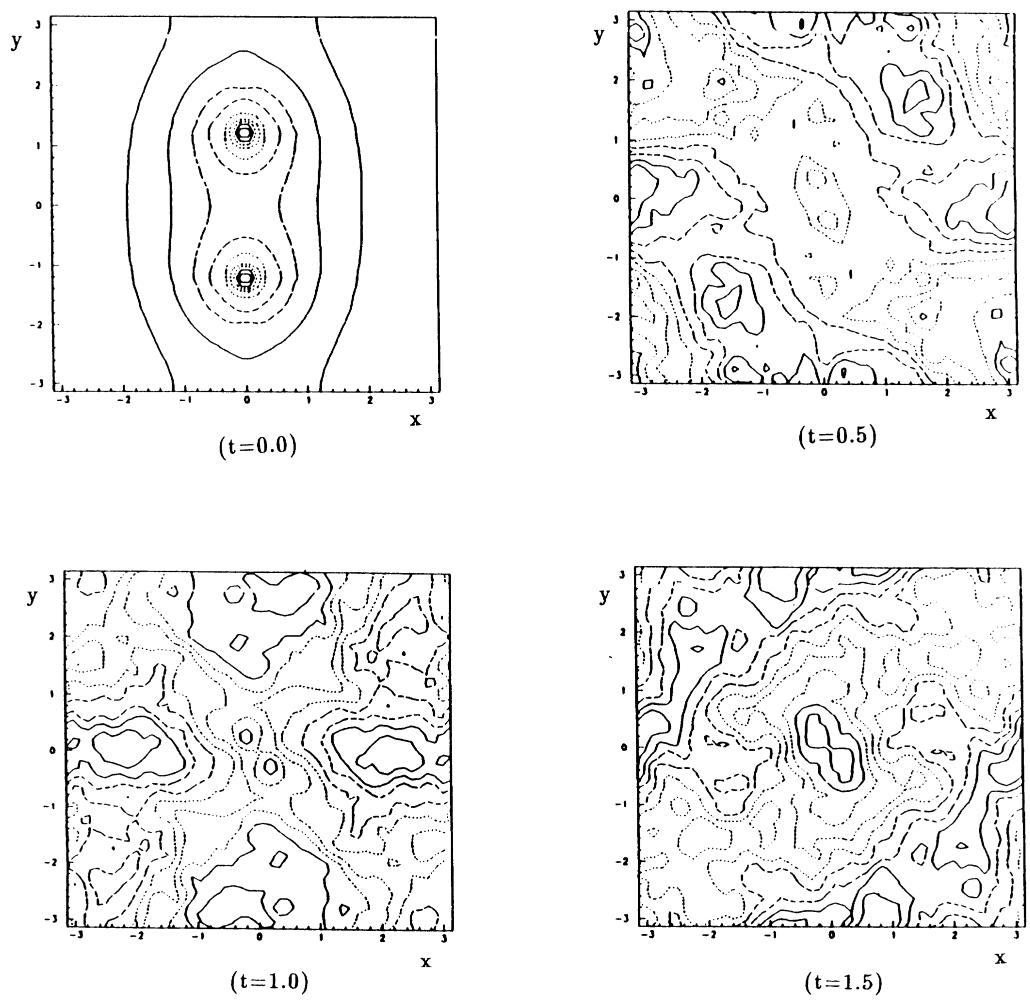

FIG. 2. Double vortex stream function for SU(31) plotted on the torus at evolution times $t=0$, 0.5, 1.0, and 1.5.

vortex starting configuration for SU(9). The figure captions are descriptive. Figure 6 shows the long-time evolution of the corresponding stream function. For the double vortex the values of the two temperatures were calculated to be $\alpha=2.925\,518\,8\times10^{-2}$ and $\beta=0.242\,236\,45$.

The presentation is limited to these data sets for reasons of space and also because one would really like to explore much larger N values. More extensive data and discussion is contained in Ref. [48].

As a measure of the accuracy of the evolution algorithms, we tested the conserved quantities. The following are the values of the SU(9) energy and the first three Casimir invariants S_2, S_3, and S_4, at $t=0$ (the first number in the brackets) and at $t=1000$ (the second number):

$$E(14.754\,915\,2, 14.754\,914\,5)\ ,$$

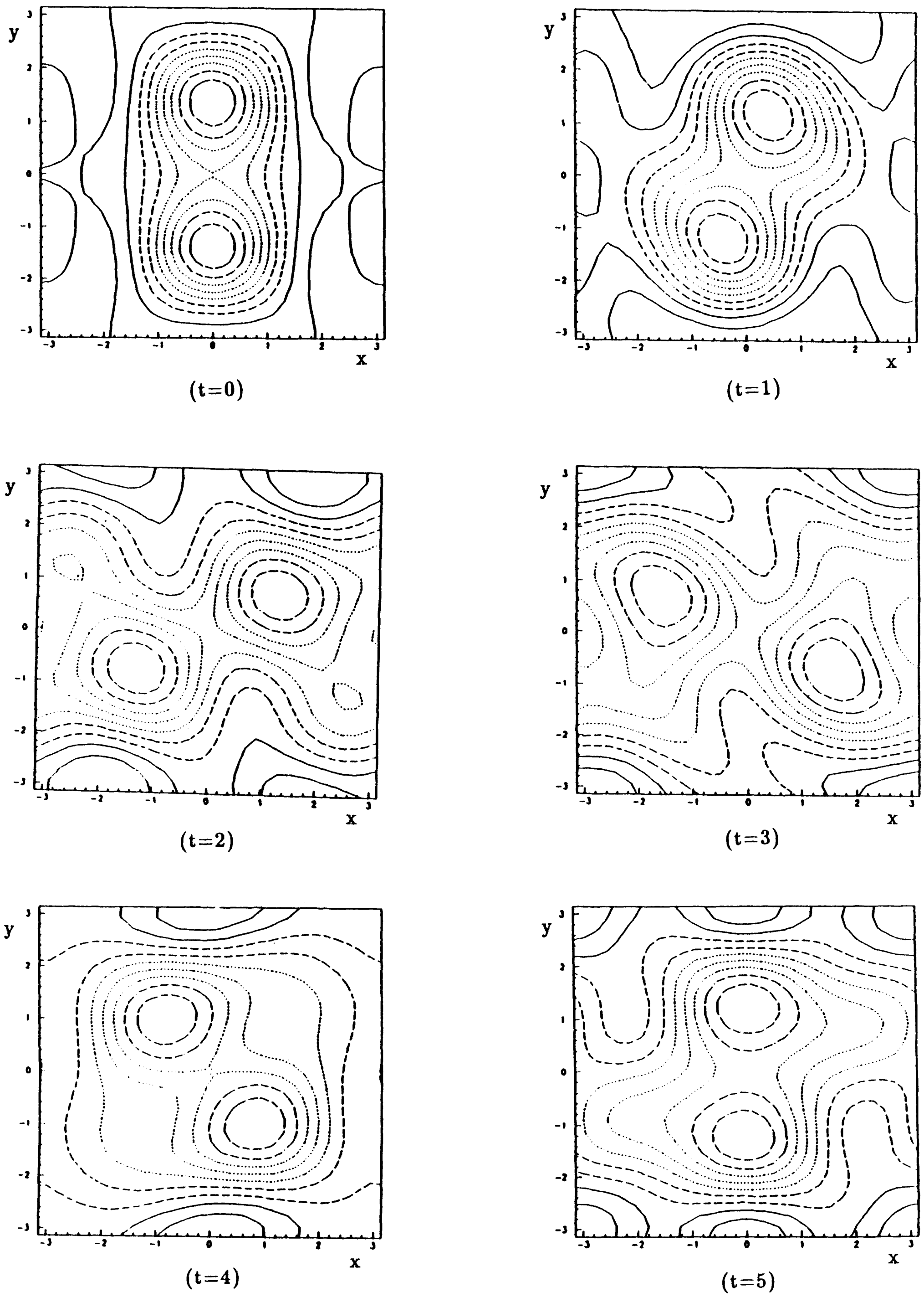

FIG. 3. Double vortex stream function for SU(5) plotted on the torus at evolution times $t=0$, 1, 2, 3, 4, and 5.

$S_2(158.000037, 158.000058)$,

$S_3(-308.0002, -308.0035)$,

$S_4(38022.0096, 38022.0630)$.

The integer parts of the S_l are the exact values. The evaluation of the highest Casimir invariants from the stream function coefficients is very time consuming due to the multiple summations.

The statistical results are inconclusive. We find no indication of nonergodic behavior but the evidence for a statistical description is still only suggestive. The results are perhaps better than one would expect for systems with a small number of modes when treated by a canonical distribution, which, moreover, ignores the other conserved quantities. The microcanonical distribution would be more relevant but the evaluation of the statistical averages is then itself an involved numerical procedure, which we have chosen to avoid.

The truncation results differ in no essential aspect from the corresponding SU(N) cases, except that the vorticity second moment of the highest mode does not relax to the canonical distribution value for large times. There is also nothing in particular that distinguishes the other starting configurations, although our numerical experiments are not yet very extensive in this respect.

The statistical mechanics of systems of this type, with many conserved quantities, remains to be elaborated. Zeitlin [10] makes some relevant remarks on the structure of the phase space.

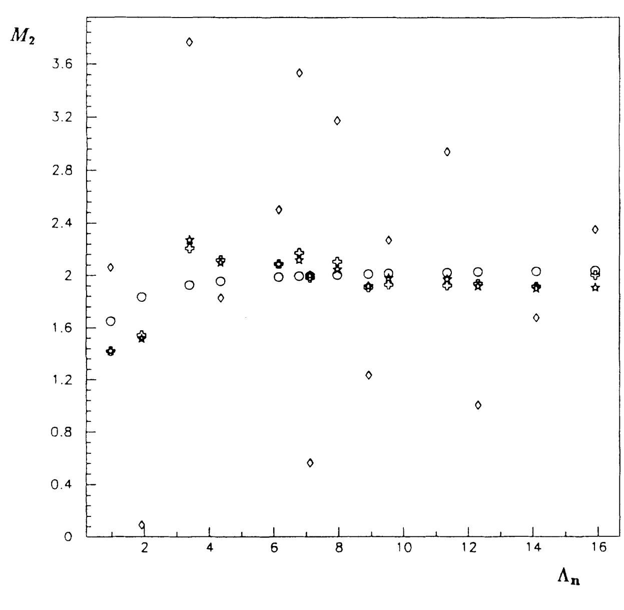

FIG. 4. SU(9) second moment of vorticity $M_2 = \langle \zeta_n^2 \rangle_t$, as a function of energy eigenvalue Λ_n calculated (by time averaging) at several evolution times t. The diamonds correspond to the initial values, the crosses to $t = 750$, and the stars to $t = 1500$. The circles indicate the values calculated on the basis of a two-temperature canonical distribution. The initial configuration was the two vortex one. The relaxation to the canonical values is better than might have been expected in view of the relatively small number of modes.

VI. COMMENTS AND CONCLUSION

In this paper we have been mainly concerned to set up the SU(N) models and to discuss the nature of the infinite N limit, which is not obvious. We have also presented a sample selection of numerical results that should be regarded as exploratory rather than definitive.

Our numerical approach was unsophisticated. For the truncated system (which is sometimes referred to as the

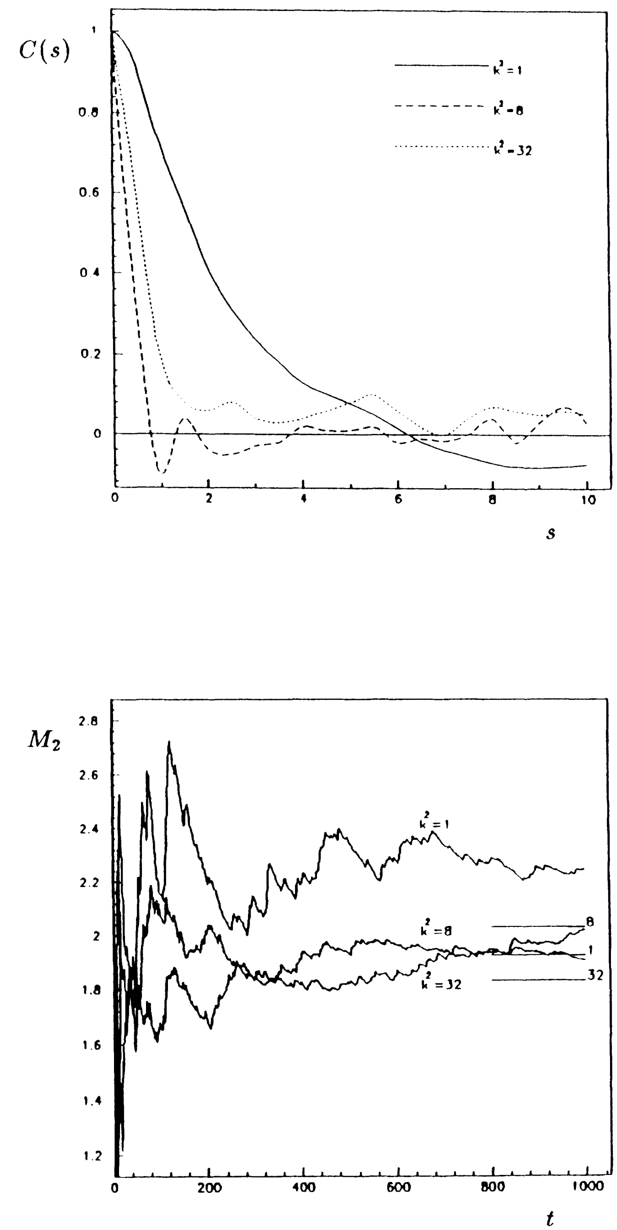

FIG. 5. Second moment of vorticity and correlation function for three modes of the SU(9) model as a function of time ($k^2 = \mathbf{n}^2$). The horizontal levels for M_2 indicate the values expected from a canonical distribution. Note the start of the vertical axis. Although the ordering of the levels is not reproduced, the general agreement is more than acceptable. The behavior of the correlation function is consistent with an ergodic development.

Galerkin approximation) some impressive speed advantages could have been achieved by using pseudospectral [45] or collocation methods combined with fast Fourier transforms. This is, apparently, not possible in the SU(N) models because of the nonlocal coupling terms in the coordinate-space form (25). The whole point of the pseudospectral method is to calculate the mode-coupling terms in the representation in which they are local. For the SU(N) cases we do not seem to have this option. Without some technique corresponding to the pseudospectral one, the SU(N) models could never be viewed as numerical alternatives to the standard truncation or finite element methods.

It might be expected that as N becomes bigger the results for the group model and those for the truncated system should approach one another. There was no evidence for this in the short-time evolutions that we have performed. Also there was no indication that the quantities conserved in the continuum theory were progressively better conserved in the truncated versions as N increased. Perhaps the values of N are still too small or it might be that the $N \to \infty$ limit has not been closely enough considered and that the expectation is unfounded.

More disturbing is the oscillatory behavior of an "entropy" $\sum_n \ln\zeta_n/\Lambda_n$ as a function of time. These evaluations are at a preliminary stage and have not been displayed. They may indicate that the systems are not ergodic or that the number of modes is small.

A corresponding analysis can be performed on the two-sphere. Although the coupling coefficients are more complicated, the eigenvalues are simpler, being the same as in the $N \to \infty$ limit. A discussion will be presented elsewhere.

Calculations have also been done on a triangular lattice corresponding to a regularly slanted torus. For real mode coefficients, the results are relevant for motion on the surface of a regular tetrahedron whereas the square torus discussed in this paper gives a degenerately flat tetrahedron. It would not model the earth too well. This tetrahedron is in fact one of the flat Riemann surfaces

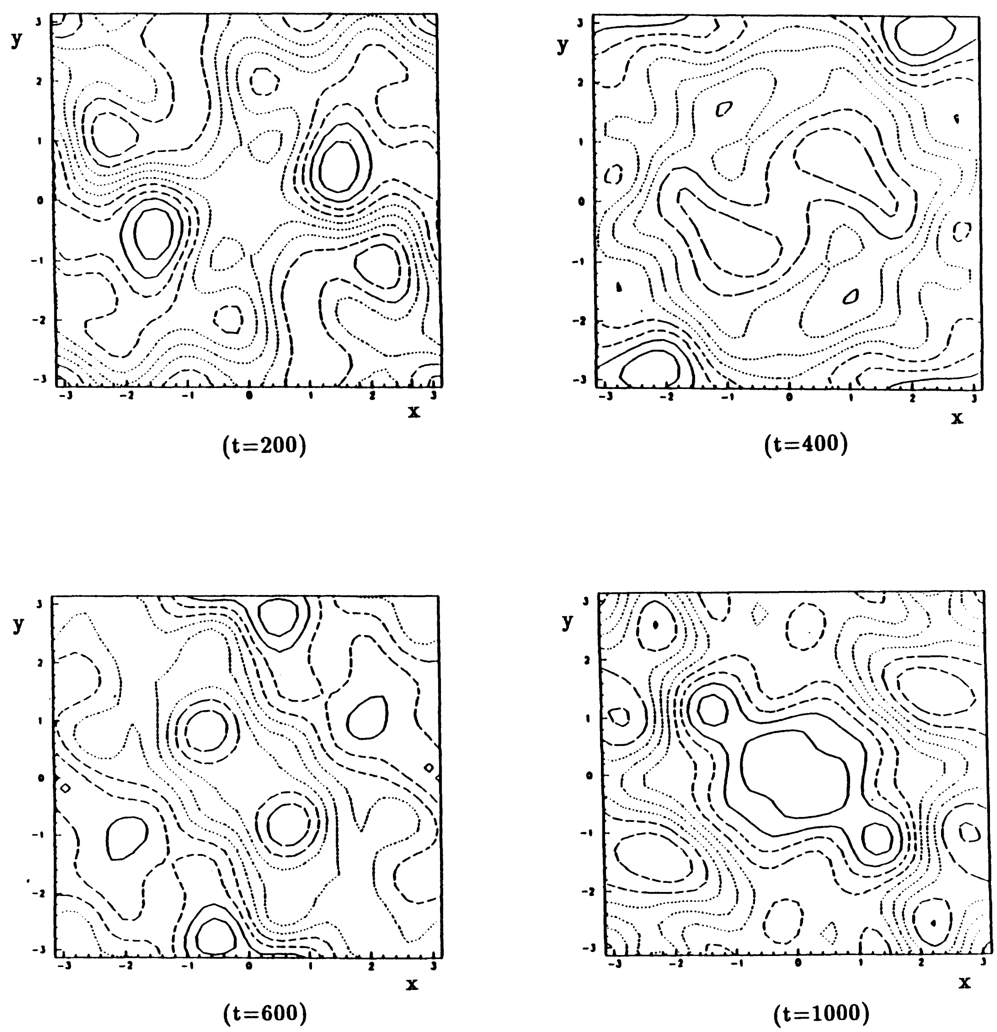

FIG. 6. SU(9) long-time evolution of the stream function starting from the double vortex configuration. The evolution times are $t = 200$, 400, 600, and 1000. See Fig. 1 for the short-time behavior.

discussed elsewhere [37] and it would be possible to extend the present calculation to these.

Whether or not this whole class of models proves to be of use in approximating continuous theories, they at least provide an interesting set of dynamical systems. More realistically, the effect of viscosity could be investigated by analyzing the Navier-Stokes equation.

Another interesting question concerns the Lagrangian stability of the motion, i.e., of the trajectories in the Lie *group.* It is known that the Eulerian (or Lie algebra) motion can be stable, yet the Lagrangian one unstable in the continuum case, being related to the sectional curvature of the group of area-preserving diffeomorphisms. It is of interest to consider the discrete analog of this.

The computations were carried out on a Hewlett-Packard workstation. Transference to a more powerful machine is planned and it is hoped to reach large values of N.

Note added. After this work was completed we were apprised of the paper by Miller, Weichman, and Cross [46], in which this finite class of models is also discussed. No numerical calculations are given and there are a number of formal differences with our setup, particularly the choice of the discrete Laplacian. Also recent work by Rankin [47] contains some information on the SU(∞) limit, mostly in a particle-physics context.

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