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Self-organized criticality in a deterministic mechanical model

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We investigate a mechanical model consisting of a one-dimensional chain of blocks connected by springs. Each block is in contact with a lower rough surface, and the chain is pulled at one end with a small constant velocity. This system was introduced by Burridge and Knopoff [Bull. Seismol. Soc. Am. **57**, 341 (1967)] to model the dynamics of earthquakes. In a wide range of the parameter space, we observe the existence of self-organized criticality, that is, robust power-law distributions limited only by the size of the chain. The model is completely deterministic.

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I. INTRODUCTION

Very little is understood about the dynamics of nonequilibrium systems with many degrees of freedom. Many interesting phenomena have been reported with respect to such large systems. In particular, increasing attention has been dedicated to the phenomenon of self-organized criticality (SOC) since it was originally described [1]. The models which display SOC represent spatially extended dynamical with driving and dissipation mechanisms. After some initial transient period, these systems evolve naturally to a statistically stationary state with no length or time scales other than those deduced from the size of the system and that of the elementary cell.

Many simulations on cellular automata models have been performed [1–5] in the investigation of SOC. However, few studies have been done on SOC in systems which have inertial elements. Carlson, Langer and coworkers [6,7] have investigated a mechanical model proposed by Burridge and Knopoff [8] to model the dynamics of earthquakes. They showed that, up to a given correlation length, the model presents a power-law distribution of event sizes according to the Gutenberg-Richter law [9] found in real earthquakes. There are big events in the mechanical model which obey a separate distribution and dissipate most of the eleastic energy.

The system we study here was also proposed by Burridge and Knopoff [8] to model earthquakes. It consists of a chain of blocks connected by linear springs and pulled at one end with a constant small velocity. The blocks are situated on a rough surface. The only study we are aware of on this model is the original work by Burridge and Knopoff [8], which was an experimental investigation of the distribution of event sizes in a chain with a small number of blocks (eight blocks). They found a scaling relation of the Gutenberg-Richter type for the potential energy released in the events. We call this model the "train model," since it has some similarity with a train, where the driving mechanism is applied only at one end of the chain. We do this to distinguish it from the other earthquake model also introduced by Burridge-Knopoff and studied recently by Carlson *et al.* [6,7] in which each block is connected to the driving element.

Here we show that the train model presents sequences of events whose moment distribution obeys a power-law distribution (Gutenberg-Richter law). Also, inspired by two other scaling relations found in real earthquakes, we study the distribution of events involving a given number of blocks (fault length or area) and the relationship between the moment and the number of blocks that take part in the event. We also found power-law relations for these quantities. These two other power laws were also proposed in [10] for friction models in general, consisting of two systems driven against each other with a constant velocity.

We find results consistent with the theory of self-organized criticality [1]. That is, we observe robust power-law distributions, whose extension is limited only by the size of the system. In this way we conclude that inertia does not necessarily destroy SOC (if this were true SOC would be only an academic theory). There is a recent study on cellular automata systems with inertial effects to model sandpiles [3]. It was found that the inertia destroys SOC in the models they considered. This is not the case in the train model. The system studied in [6,7] also has inertia. However, we found [11] that the self-organized critical state occurs only in a very small region of the parameter space in the other Burridge-Knopoff model.

The paper is organized as follows. In Sec. II we give a description of the train model. In Sec. III we present our numerical results on the scaling relations and Sec. IV we dedicated to discussions.

II. MODEL DESCRIPTION

The model we study is shown schematically in Fig. 1. It consists of a chain of N blocks of mass m coupled to each other by harmonic springs of strength k. The blocks are in contact with a rough surface. Between the blocks and the surface there is a velocity-dependent frictional force given by some function F of the blocks velocity. One end of the chain is pulled with a constant velocity v.

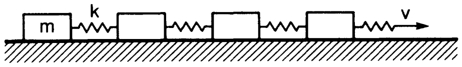

FIG. 1. Mechanical model consisting of a chain of blocks on a frictional surface. The blocks are connected by coil springs and the chain is pulled at one end with a small constant velocity.

The equation of motion for the moving blocks is given by

$$m\ddot{X}_j = k(X_{j+1} - 2X_j + X_{j-1}) - F(\dot{X}_j), \quad \dot{X}_j \neq 0 \tag{2.1}$$

with $j = 1, \ldots, N$, $X_{N+1} = 0$, and $X_0 = vt$. X_j denotes the displacement of the jth block measured from its equilibrium position. We assume a friction force of the type [6]

$$F(\dot{X}) = F_0 \Phi(\dot{X}/v_c) , \tag{2.2}$$

where Φ is a monotonously decreasing function of $\dot{X}$, $\Phi(0) = 1$, and v_c is some velocity that characterizes the dependence on F with $\dot{X}$.

With the introduction of the variables

$$\tau = \omega_p t, \quad \omega_p^2 = k/m, \quad U_j = kX_j/F_0 , \tag{2.3}$$

Eq. (2.1) can be written in the following form:

$$\ddot{U}_j = U_{j+1} - 2U_j + U_{j-1} - \Phi(\dot{U}_j/\nu_c) \tag{2.4}$$

with $U_{N+1} = 0$, $U_0 = \nu\tau$, $\nu = v/V_0$, $\nu_c = v_c/V_0 = (2\alpha)^{-1}$, and $V_0 = F_0/\sqrt{km}$. Dots now denote differentiation with respect to τ. In a system of a single block the quantity F_0/ω_p is the maximum displacement of the pulling spring before the block starts to move; in the absence of dynamical friction, $2\pi/\omega_p$ and V_0 are, respectively, the period of oscillation of the block and the maximum velocity it attains. In the new rescaled system, $V_0 = 1$ is, therefore, the reference velocity.

After rescaling the parameters and variables we see that the model is completely described by two dimensionless parameters ν and α. In our calculations we use a friction force given by

$$\Phi(\dot{x}) = \begin{cases} \dfrac{1}{1+2\alpha\dot{x}} & \text{if } \dot{x} > 0 ; \\ (-\infty, 1] & \text{if } \dot{x} = 0 , \end{cases} \tag{2.5}$$

which is shown schematically in Fig. 2. That is, the static friction can take any value from $-\infty$ to 1. In this way backward motions are not allowed. Larger values of α means less friction and consequently larger events. Note that the only nonlinear element in the system is the velocity-weakening friction force.

In this model a complex behavior is naturally obtained without any kind of embedded randomness, not even in the initial conditions. In other words, this model is *completely* deterministic. The dynamics of the system is as follows: suppose that at the initial instant all the blocks are at rest. As the time evolves, the first spring is stretched by the driving mechanism until the force applied to the first mass exceeds the static frictional force,

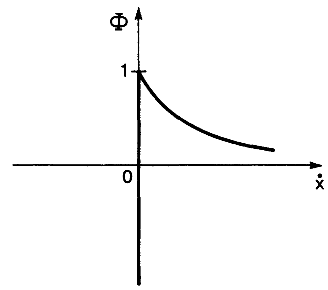

FIG. 2. Schematic representation of the friction force.

at which time the first mass moves. It slips a certain distance and stops. This reduces the extension in the first spring but at the same time stretches the second spring. The one-block events continue to occur until the spring force on the second block exceeds the static frictional force. Then an event involving two blocks is observed and the spring that connects the second to the third block is stretched. Thus, events involving three, four, and more blocks appear during the time evolution. Finally, we see a larger event involving all the blocks of the chain, which rebuilds the system. A new sequence of events starts. In general conditions, the sequence of events are not periodic, since the friction force amplifies instabilities [6]. Note that in this model an event that involves the ith element of the chain necessarily involves all the blocks with $j < i$.

In Fig. 3(a) we show a typical time evolution for the train model where the initial conditions are the blocks stuck and in their equilibrium positions. The blocks that are moving are represented by black dots. A blow up of Fig. 3(a) is shown in Fig. 3(b). The parameters used are $N = 20$, $\nu = 0.1$, and $\alpha = 0.6$. Events of several sizes are observed. Initially, the events involve few blocks, then larger events appear, which are preceded and followed by smaller events.

Some analytical studies can be performed for the center-of-mass motion of the blocks analogous to the studies done in [6]. Suppose that n blocks ($n < N$) are in a spatially homogeneous configuration and assume that this group of blocks slips as a whole. Denoting the coordinate of the center of mass as

$$W_n = \frac{1}{n} \sum_{j=1,n} U_j \tag{2.6}$$

and using Eq. (2.4), we find the equation of motion for W_n, which is given by

$$\ddot{W}_n = -\Omega_n^2 W_n + 1 - \phi(\dot{W}_n/\nu_c) + \nu\tau , \tag{2.7}$$

where $\Omega_n^2 = 2/n$. The solution of (2.7) in a linear approximation with $\phi(\dot{W}_n/\nu_c) \approx 1 - 2\alpha\dot{W}_n$ is

$$W_n(\tau) = \frac{\nu \exp(\alpha\tau)}{2\Omega_n^4 i\Gamma_n}[\Omega_-^2 \exp(i\Gamma_n\tau) - \Omega_+^2 \exp(-i\Gamma_n\tau)] + \frac{2\alpha\nu}{\Omega_n^4} + \frac{\nu\tau}{\Omega_n^2} , \tag{2.8}$$

where $\Omega_\pm = \alpha \pm i\Gamma_n$, $\Gamma_n = (\Omega_n^2 - \alpha^2)^{1/2}$. In the solution

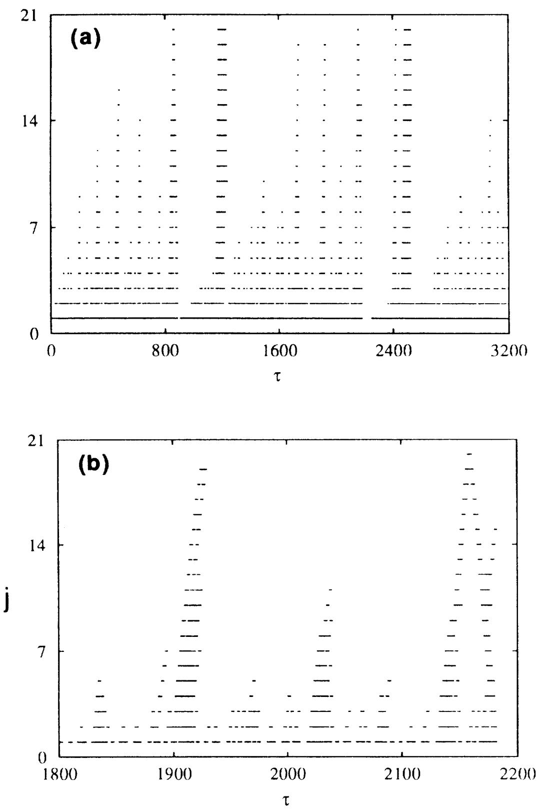

FIG. 3. Temporal evolution of the events in a chain of $N=20$ blocks, $\alpha=0.6$ and $\nu=0.1$. The ordinate refers to the block index j and the abscissa to the rescaled time τ. A black dot means $\dot{U}_j > 0$.

(2.8) we considered the initial conditions $W(\tau=0)=\dot{W}(\tau=0)=0$. For $\alpha \approx 0$, we have the approximate solution

$$W_n(\tau) \approx \frac{\nu}{\Omega_n^2}\left[\tau - \frac{\sin(\Omega_n \tau)}{\Omega_n}\right] . \tag{2.9}$$

The time duration of the events and the displacement of the center of mass are, respectively,

$$\delta\tau = 2\pi/\Omega_n \tag{2.10}$$

and

$$\delta W_n = 2\pi\nu/\Omega_n^3 . \tag{2.11}$$

From Eqs. (2.9)–(2.11) we see that the maximum velocity attained by the blocks is greater than the pulling velocity and that their average velocity is $\delta W_n/\delta\tau = n\nu/2$, which is larger than the pulling velocity for $n>2$. The linear approximation underestimates the motion of the blocks; for nonvanishing α the numerical calculations give larger values for the displacement of the blocks, as well as for the maximum velocity they attain.

III. SCALING RELATIONS

Our numerical simulations show the existence of events of several sizes in the train model which obey scaling relations. If α is small, we may see events which have a maximum velocity smaller than the characteristic velocity. These events have negligible nonlinear effects and the displacement of the blocks is proportional to the pulling speed, as predicted in the linear approximation [Eq. (2.11)]. We also observe events of intermediate sizes where the blocks attain velocities of the order or larger than the characteristic velocity. Finally, we see the largest events, that involve all the blocks of the chain, and where the blocks attain very large velocities. For the two last types of events, the pulling velocity has practically no influence on the scaling relations we find, since we are considering a region of the parameter space where $\nu \ll \nu_c$. We choose a typical example of our simulations to show in Fig. 4(a), the maximum velocity $\dot{U}_{\max}$ attained by the blocks versus the number of blocks n moved in the event. In this example we have $\nu=0.1$, $\alpha=0.6$, and $N=100$. We do not see a one-one relationship between $\dot{U}_{\max}$ and n (except for $n=1$) which reflects the fact that several solutions for $\dot{U}_{\max}$ exist depending on the initial positions of the blocks at the beginning of the slipping event. In the example we see events with $\dot{U}_{\max} < \nu_c$, $\dot{U}_{\max} \geq \nu_c$, and $\dot{U}_{\max} \gg 1$ (when all the blocks are displaced). Figure 4(a) indicates the existence of a power-law relation for $\dot{U}_{\max}$ versus n (which we do not investigate here in detail).

A particularly important quantity in earthquake models is the moment associated with an event. The moment M is a measure of the size of the event, and it is defined as

$$M = \sum_j \delta U_j , \tag{3.1}$$

where the sum is over the blocks displaced during the event. We investigate the moment M as a function of n. Again, as we show in Fig. 4(b), we find no one-one relationship between these two quantities (except for $n=1$), as in the case of $\dot{U}_{\max}$.

From Fig. 4(b) it is clear that a power-law scaling seems to exist between M and n. In fact, this becomes even more evident if we calculate the average moment $\overline{M}$ of the events involving n blocks. We find that the relation

$$\overline{M}(n) = M_0 n^{\lambda} \tag{3.2}$$

holds for a wide range of event sizes. In Fig. 4(c) we display the results for two typical cases with $\alpha=0.6$ and $N=100$ with $\nu=0.1$ (solid line) and $\nu=0.01$ (dashed line). For this example, $\lambda \approx 2.0$. We see that the sizes of the smallest events are affected by the pulling velocity and therefore they have separate scalings for the two different pulling speeds. The events with $\dot{U}_{\max} \gtrsim \nu_c$ (in this example this corresponds to $n \gtrsim 10$) have negligible influence from ν and for the two different pulling speeds they show the same scaling. So there is a crossover from the region where the event sizes scale with ν to a region where they are independent of the pulling velocity. The events with $n=N$ attain very large velocities ($\dot{U}_{\max} \gg 1$) and do not fall in the power law governed by Eq. (3.2).

Now we study the statistical distributions associated with the slipping events of the train model. Generally, we start the system with all the blocks at their equilibrium position and at rest. Before we start to compile statistics we let the system evolve until the first event involving all the blocks occurs. That is, we neglect a transient period to allow the system to reach a statistical stationary state. We compute the distributions for about 100 loading cycles (that is, 100 events involving all the blocks).

First, we calculate the frequency of events $\rho(M > M')$ that have moment greater than M'. For a wide range of parameters, we find a power-law distribution of the Gutenberg-Richter type,

$$\rho(M > M') = AM'^{-B} . \tag{3.3}$$

Figure 5(a) shows the results for $\rho(M > M')$ for the two typical examples specified above. For $\nu = 0.1$ (solid line), the simulation involves 20 000 events and for $\nu = 0.01$ (dashed line), it involves 150 000 events. This corresponds, respectively, to 120 and 100 loading cycles. For both cases there are clear power-law distributions for several decades with $B \approx 0.60$. As ν decreases, the smallest events deviate from the scaling obeyed by the events of intermediate sizes, for the reasons discussed in the previous paragraphs. Surprisingly, we see that the events that involve all the blocks seem to have the same B exponent as the events of intermediate size (with $n < N$).

This model obeys the sum rule [10]

$$\frac{1}{N}\int_{\Delta}^{\Theta} M\rho(M)dM = \nu , \tag{3.4}$$

where Δ and Θ are the smallest and largest moments of the system, respectively. The sum rule is a conservation law which says that the blocks move with an average ve-

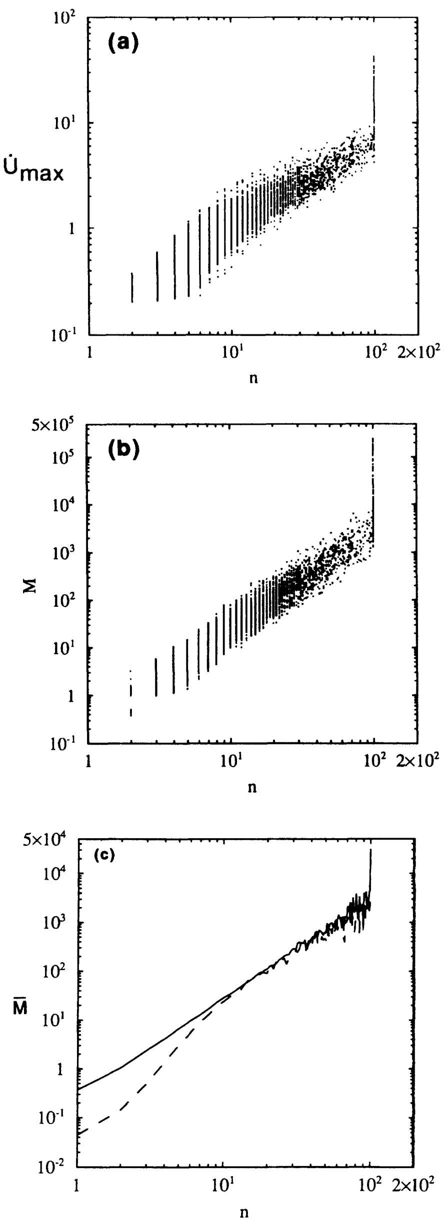

FIG. 4. Power-law scalings associated with (a) maximum velocity attained by the blocks (b) moment, and (c) average moment as a function of the number of blocks displaced in the event. The simulations are for a chain of $N = 100$ blocks, $\alpha = 0.6$ and $\nu = 0.1$. In (c) the dashed line corresponds to $\nu = 0.01$. $\dot{U}_{max}$ and M for $n = 1$ are not shown. The respective values are $\dot{U}_{max} = 0.2040$ and $M = 0.3680$.

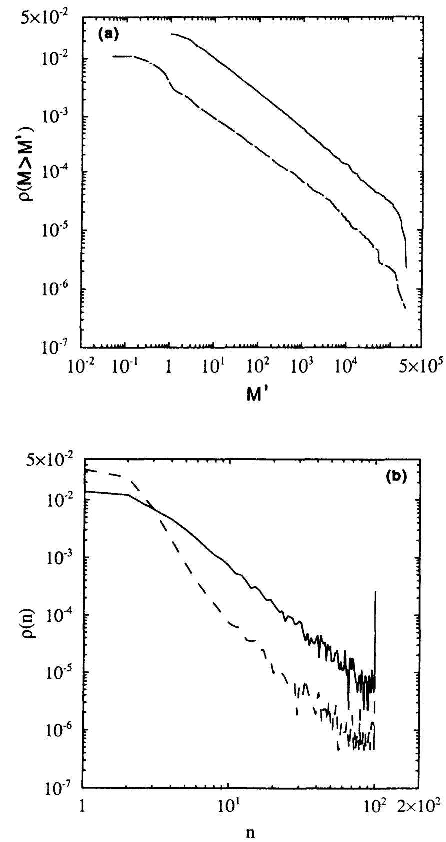

FIG. 5. Power-law distributions associated with (a) frequency of events $\rho(M > M')$ with moment greater than M' and (b) frequency of events $\rho(n)$ that involve n blocks. The simulations are for a chain of $N = 100$ blocks, $\alpha = 0.6$ and $\nu = 0.1$ (solid) and $\nu = 0.01$ (dashed). The events involving only one and two blocks are not included in the statistics shown in (a).

locity equal to the pulling speed. In [10] we have shown that if ρ is given by Eq. (3.3) then, in order to have convergence on the left-hand side of (3.4) in the limit $\Delta \to 0$, one must have $B \leq 1$. This result should be valid for real earthquakes, as indeed is the case, and for friction models in general where there are two surfaces sliding against each other. Thus, our results are in agreement with the theory developed in [10].

We also investigate the frequency of the events involving n blocks and observe a region with a power-law relation given by

$$\rho(n) = Wn^{-\omega} \tag{3.5}$$

for the events with $n < N$, as Fig. 5(b) shows. In this example, $\omega \approx 2.2$. Again, the statistics of the smallest events are affected by v. Now the events with $n = N$ obey a separate distribution.

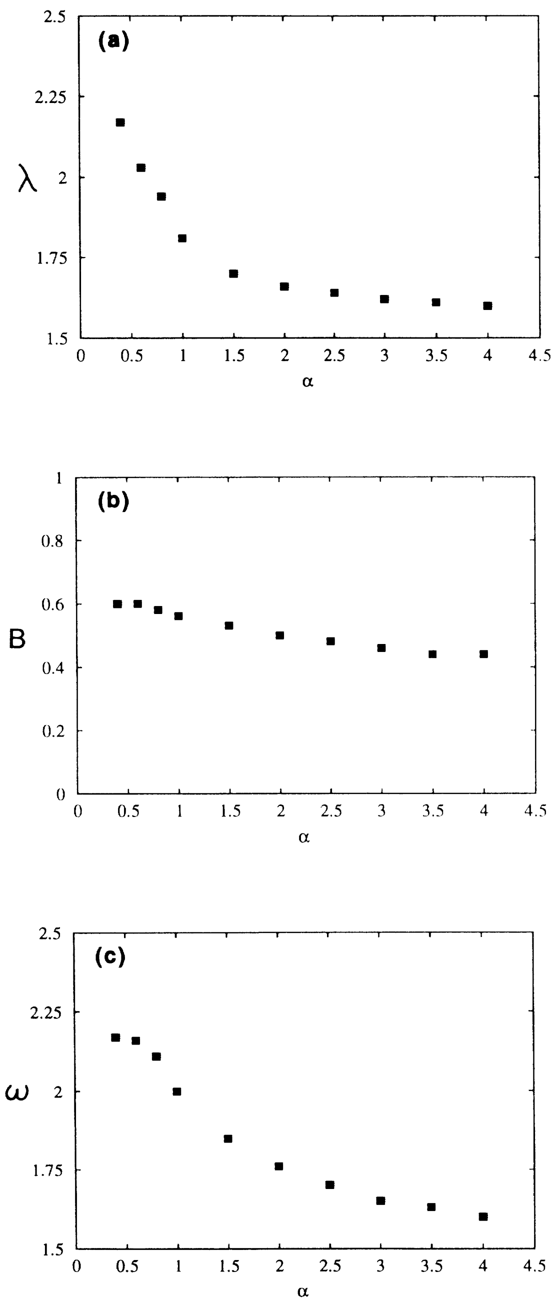

FIG. 6. Dependence on α of the scaling exponents for the events of intermediate sizes for (a) λ, (b) B, and (c) ω in a chain a of $N = 100$ blocks and $v = 0.1$.

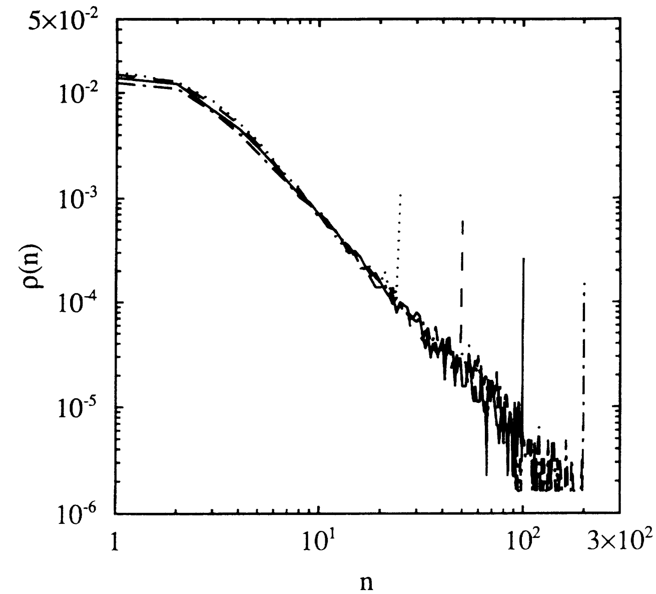

FIG. 7. $\rho(n)$ for $N = 25$ (dotted), $N = 50$ (dashed), $N = 100$ (solid), and $N = 200$ (dotted-dashed) with $\alpha = 0.6$ and $v = 0.1$.

We have studied the train model in a wide variety of parameter values with system sizes up to $N = 200$ to investigate the dependence of the scaling exponents λ, B, and ω with v, α, and N (here we do not concentrate much attention on the coefficients of the power laws). As we have shown, v affects only the scaling of the microscopic events. The scaling exponents λ, B, and ω as a function of α for the events of intermediate sizes are shown in Figs. 6(a), 6(b), and 6(c), respectively, for a chain of $N = 100$ blocks and pulling velocity $v = 0.1$. The estimated errors for the scaling exponents are approximately $\pm 5\%$ of the corresponding ordinate values. The three exponents are smoothly decreasing functions of α. As α becomes larger all the exponents seem to reach a plateau.

We have seen no variation of the scaling exponents λ, B, and ω with the size of the chain, if the chain is not too small. We also have found that the coefficients A, M_0, and W do not depend on N. As an example, Fig. 7 shows $\rho(n)$ for four different system sizes ($N = 25, 50, 100, 200$) with $\alpha = 0.6$ and $v = 0.1$.

We have also investigated the train model for smaller values of α. We have found that in the region $\alpha \lesssim 0.4$ there is a transition to a different regime, where the power laws are of limited extent. The behavior of the model in this parameter region is currently under investigation.

IV. DISCUSSIONS

Perhaps the train model is too simple to be called an "earthquake model." However, we would like to present some comparisons between the results found in real earthquakes and in the train model. The three power laws we have studied in the previous sections are also observed in real faults. The distribution governed by Eq. (3.3) (the Gutenberg-Richter law) has been studied extensively in nature. The latest studies show that $B \approx \frac{2}{3}$ for earthquakes with magnitude smaller than 7.5 [12] (the magnitude of an event is proportional to the logarithm of the moment). For larger magnitudes, a different exponent seems to exist [12]. There are fewer studies for the two other scaling laws in real earthquakes [Eqs. (3.2) and (3.5)]. In this case, n would represent the size (area) of the fault. Kanamori and Anderson [13] found that for large earthquakes in several regions of the Earth, $\lambda \approx \frac{3}{2}$. In a recent study, Scholz and Cowie [14] reported values for ω. They found for two region in Japan $\omega \approx 2.1$.

We note several differences between the train model and the Burridge-Knopoff model studied by Carlson and coworkers [6,7]. In first place, they observe a complex dynamics only if there is randomness in the initial positions of the blocks. This is not necessary in the system studied here. There is a correlation length in the model they investigated which limits the extent of the power laws. Only with a fine tuning of the parameters is it possible to obtain power-law distributions with large correlation lengths [11]. The events that belong to the scaling region are very numerous; however, they contribute less than 1% for the forward motion of the fault and more than 99% of them have maximum velocity smaller than the pulling velocity [11]. In the train model the power laws are limited only by the size of the chain. The events involving all the blocks, which do not always belong to the scaling regime, have a contribution for the sum rule which is less strong. For a chain of 100 blocks and $\nu = 0.1$, the motion due to the events with $n < N$ for $\alpha = 0.6$ is about 16%, whereas for $\alpha = 4$ it decreases to approximately 10%.

The other two power laws we found in the train model, and which are also found in real earthquakes, are not observed in the model investigated in [6,7] if the parameters are not fine tuned.

One may ask which system, the train model or the model studied by Carlson *et al.*, is more realistic to describe the dynamics of earthquakes? Do they model different types of faults? Our response is that we do not know. Our point of view is that both systems are extremely simple and several important features of the seismic dynamics are neglected, as, for example, the complicated geometry and interactions of faults of any real region. We believe that these systems should be taken only as toy models in the investigation of the stick slip processes of real earthquakes.

In summary, we have shown that the deterministic mechanical model introduced by Burridge and Knopoff, where the driving mechanism is only at one point of the chain, shows robust power-law distributions according to the self-organized criticality theory. The only correlation length in the model is the size of the chain, if the dissipation is not too large ($\alpha \gtrsim 0.4$).

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