ARTICLES

Tracking unstable orbits in an experiment

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We present a continuation method for experimentally tracking unstable periodic orbits by slowly varying an available system parameter in a dynamical system. The method does not depend on an explicit model, but on the signal analysis of a measured time series. Unstable periodic orbits can be tracked through various bifurcations. We apply this to a Duffing-like circuit and compare the results to an approximate model of the circuit.

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INTRODUCTION

When a model exists for a dynamical system, theoretical tools are available which allow the location of steady-state, periodic, and aperiodic phenomena. Furthermore, numerical methods have been developed which allow both stable and unstable phenomena to be located [1] and followed as a function of parameters, creating various theoretical bifurcation diagrams. Here the theorist has access to various continuation or homotopy techniques which enable him to follow the unstable trajectories [2–4].

In a parallel manner, recent progress in the theory of nonlinear dynamical systems has begun to provide the experimentalist with a collection of new tools which have generated new areas of exploration from the measurement of a single time series. For example, if the experiment exhibits deterministic chaos, the relevant dynamics may be constructed from a single time series by using any one of a number of embedding techniques [5–7]. In addition, once the attractor is constructed, unstable periodic orbits contained in the attractor may be located using techniques such as those found in [8].

One new method, in particular, has emerged as a potentially useful experimental tool. This is the Ott-Grebogi-Yorke (OGY) [9] technique for stabilizing the unstable orbits in a chaotic attractor. Using this it is possible to keep the system on an unstable orbit contained in a chaotic attractor by making small changes in some accessible parameter.

We show here that we can combine the OGY technique with the ideas of continuation [10] in parameters to now allow experimentalists to follow unstable orbits over large ranges of parameters and through bifurcations. This provides experimental access to dynamical objects which are usually avoided by the system because of their instability. Hence, experimental mapping of bifurcation diagrams becomes a possibility for unstable as well as stable trajectories even in nonchaotic regimes. We test this approach on a Duffing-like circuit.

An extension of this is that one can now consider the problem where the parameter is changing as a function of time. The location of the fixed point of the orbit, therefore, will also change as a function of time. If the parameter-dynamics time scale is sufficiently slow compared to the state dynamics, then one may implement the technique we present to keep the system stabilized on the unstable orbit.

CONTINUATION METHOD

This method is applicable to flows which are reduced to maps by taking a Poincaré section. For simplicity, let us assume that we want to track a period-1 saddle, i.e., we describe the algorithm for tracking $\xi_F(p)$ as p varies, where $\xi_F(p)=F(\xi_F,p)$ for a given map F, which is not explicitly known. Full details of the algorithm may be found in [10].

The method we are using requires approximate values of the unstable orbit and corresponding eigenvectors and eigenvalues for some initial value of the parameter p. Subsequent values, as we increase (or decrease) p, are determined by a predictor-corrector method. The prediction step can be done in various ways, depending on the amount of computer power in the experiment (see [10]). Here a simple increase in the parameter and taking the corresponding value of the unstable orbit as the predicted value is enough for prediction. The correction step is then able to bring the orbit back close to the correct value.

To control the system, we use OGY's algorithm; i.e., we choose p so that the next iterate will fall on the local stable manifold of the orbit. This amounts to modifying p, by

$$\delta p=C(\xi_n-\xi_F)\cdot f_u \ , \tag{1}$$

where ξ_F is the predicted point, ξ_n is the current iterate, f_u is an eigenvector along the unstable direction, and C is given by

$$C=\frac{\lambda_u}{\lambda_u-1}\,\frac{1}{g\cdot f_u}\ . \tag{2}$$

To determine if ξ_F is close to the true fixed point we determine δp several times, say 100, and then examine the mean of these fluctuations in the parameter p. If the predicted value would be the exact value of the orbit, this mean would be close to zero (to within experimental noise).

The error in the fixed point is given approximately by

$$\frac{\langle \delta p \rangle}{C\cdot f_u}\ . \tag{3}$$

We minimize this error by varying the estimate of the fixed point ξ_F at this new parameter value which minimizes $\langle \delta p \rangle$. The above procedure locates a new point on the curve of fixed points, which is used for further prediction, as the procedure is repeated.

The entire process can be automated giving experimentalists a continuation technique for following unstable orbits in mapping out bifurcation diagrams. Below we show this technique applied to a nonlinear circuit.

EXPERIMENTAL TEST

The Duffing circuit has been described elsewhere [11]. This system may be modeled by the following equations:

$$\frac{dy}{dt}=\alpha[A\cos(\omega t)+C-0.2y-G(x)]\ , \tag{4}$$

$$\frac{dx}{dt}=\alpha y\ , \tag{5}$$

$$G(x)=\begin{cases}0\ \ [\mathrm{abs}(x)<1.2]\\ x-1.2\,\mathrm{sgn}(x)\ \ [1.2\le \mathrm{abs}(x)<2.6]\\ 2x-3.8\,\mathrm{sgn}(x)\ \ [2.6\le \mathrm{abs}(x)],\end{cases} \tag{6}$$

where A is the amplitude of the cosine drive and C is a constant offset that may be added to the drive. The time factor α is $1\times10^4\ \mathrm{s}^{-1}$. The function $G(x)$ is a piecewise linear approximation to x^3 which was used to make the circuit easier to characterize. For this study, the drive frequency ω was 726 Hz, the offset C was 0.3 V while the drive amplitude A was varied.

From this circuit we generated a first return map in the chaotic regime by digitizing the value of the x signal from the circuit as the drive signal crossed zero in the positive direction. We then made a plot of x at the $(n+1)$th zero crossing vs x at the nth zero crossing. We made this map with the drive amplitude $A=7.4$ V. The techniques described in [12] were then used to find the unstable period-1 fixed point from the return map and to determine the parameter control variation necessary to keep the chaotic system on the unstable period-1 orbit by making small changes in the amplitude of the driving signal. In practice, these methods were robust enough that one could actually find periodic orbits by simply scanning through possible fixed-point values and control-parameter-variation factors.

The circuit was kept on the unstable period-1 orbit by using a control circuit very similar to the one described in [13]. The control signal from the control circuit was used to amplitude modulate the output of a sinusoidal function generator. When the control circuit was set to control about the exact fixed point, which was at $x=0.493$ V when $A=7.4$ V, the average control signal was zero. If the circuit was set to control about some value near but not the same as the fixed point, the average of the control signal was not zero.

Once the location of the unstable period-1 fixed point was found for some drive amplitude A, this value was used as a zero-order estimate for the location of the fixed point at some other value of A. Since the actual fixed-point location was not the same as this estimate, the average of the control signal was not zero. The control point for the circuit was then adjusted to find the actual location of the fixed point by zeroing the average control signal. For the drive amplitude $A=7.4$ V, for example, the unstable period-1 fixed point was at $x=0.493$ V. The drive amplitude was then changed to 7.1 V, while the control point was left at 0.493 V. The average of the control signal was then about -50 mV. When the control point was adjusted to 0.523 V, the average control signal was zero, indicating that for $A=7.1$ V, the unstable period-1 fixed point was at $x=0.523$ V.

In this fashion the location of the unstable period-1 fixed point was tracked from the chaotic regime down through the period-doubling cascade to the period-doubling point, where the period-1 orbit was stable. The locations of the unstable period-1 fixed point, the stable period-1 fixed point and the stable period-2 fixed point are plotted as functions of the drive amplitude A in Fig. 1. The black circles show stable fixed points while the open circles show unstable fixed points. The small black

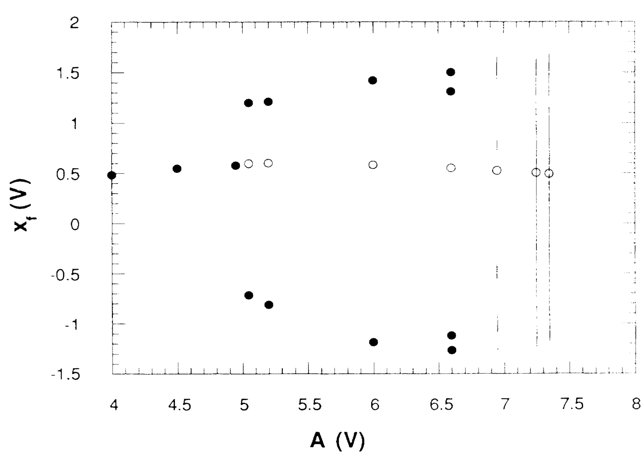

FIG. 1. Bifurcation diagram for a periodically driven Duffing circuit. A is the amplitude of the drive and x_f is the location of a fixed point found by digitizing the value of the signal corresponding to the x variable when the drive signal crossed zero. The dark circles represent stable fixed points while the open circles represent an unstable fixed point tracked by the method described in the text. The broken lines indicate the presence of chaos.

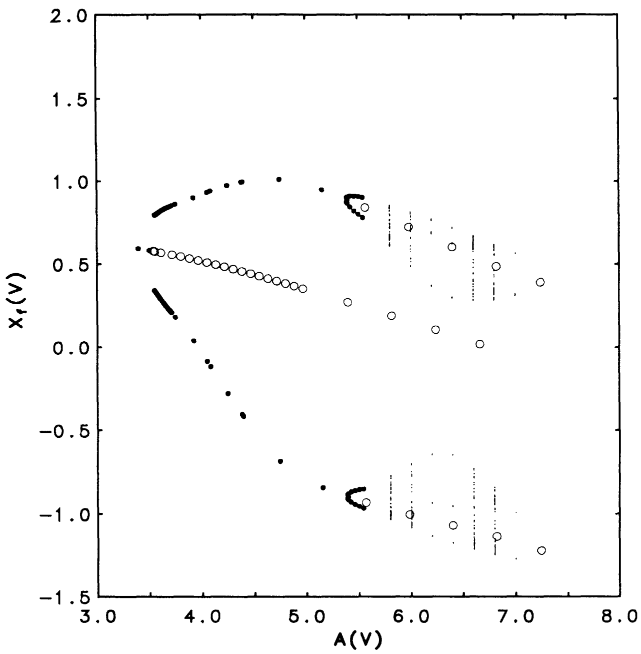

FIG. 2. A bifurcation diagram of x_f as a function of A. Unstable periodic orbits are shown as open circles. All other orbits are either periodic or chaotic attractors. The frequency ω is 0.4575, $\alpha = 1.0$, and $C = 0.3$.

dots at the highest drive voltages correspond to chaos in the uncontrolled Duffing circuit.

A bifurcation diagram similar to the Fig. 1 may be made from Eqs. (4)–(6). The system is periodically driven, and a discrete representation of the dynamics is constructed by sampling the solution (x_f, y_f) every time the drive crosses zero with positive slope. For each drive amplitude A, x_f is plotted in Fig. 2. Unstable period-1 and -2 points are depicted as open circles, while all other points are attractors, both periodic and chaotic. (The unstable branch of period-2 orbits emanating from the period-1 branch is not shown here, since the bifurcation branch is almost vertical, making it very difficult to compute accurately.)

From a comparison of the two bifurcation diagrams, one may see that the techniques described here allow the accurate tracking of unstable periodic orbits in an experiment. This circuit is not an unrealistic physical system that was carefully contrived just to demonstrate this method; rather, the circuit uses many inexpensive (and noisy) operational amplifiers, and the control circuit is simply built with inexpensive and low precision parts.

Finally, the crucial aspect of the correction step is that the error fluctuations Eq. (3) be a monotonic function of the fixed-point estimate ξ_F so that adjustment of ξ_F is guaranteed to minimize $\langle \delta p \rangle$ at the proper ξ_F estimate. Figure 3 shows an experimental plot of the average error as a function of ξ_F at the bifurcation parameter value of $A = 0.588$ showing just that monotonic relation.

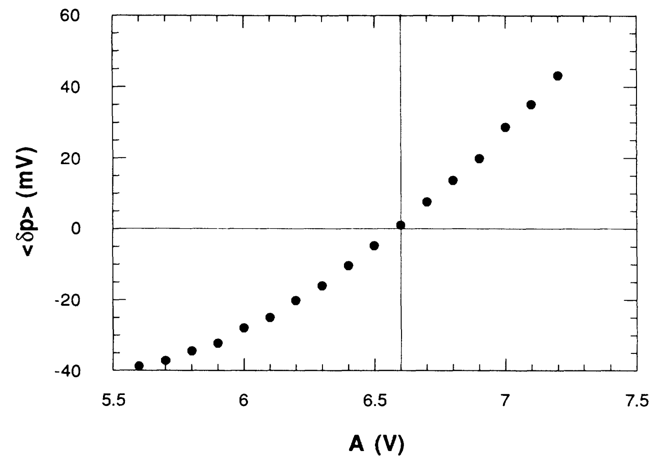

FIG. 3. Mean value of the correction signal $\langle \delta p \rangle$ as the drive amplitude is changed while the control point is fixed at 0.509 V, which is the location of the unstable fixed point when $A = 6.6$ V.

CONCLUSIONS

The important feature of our method consists in the fact that we do not need to know explicitly the equations of the system, which makes it especially useful to experimentalists and applicable to a wide variety of problems. The method also depends only on the application of a small-amplitude method of controlling unstable fixed points. By using other control methods, the tracking technique can be extended to higher-period orbits [13], as well as aperiodic signals [14].

Our technique now gives the experimentalists in dynamical systems an exploratory tool where regions of phase space previously not attainable are now within reach. In a previous study, it was shown how an unstable period-1 orbit kept pseudoperiodically driven period-doubled circuits in phase with each other [11]. This method could be used to locate the unstable period-1 orbit in this experiment, making it easier to study the effects of pseudoperiodic driving. It will be shown elsewhere how this method can be applied to such problems as accurate bifurcation location, branch switching between attractors, and the location of attractors having small basins of attraction.

Note added. After this work was completed, the authors became aware of a paper [15] by Z. Gills *et al.* applying the same technique to a laser experiment.

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