Squeezed states and the Franck-Condon overlap

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A simple and direct approach to studying squeezed states is proposed. It is based on treating the corresponding differential equations and their solutions directly on the Fock-Bargmann space. The method is exemplified by giving quite general solutions to the case of amplitude-squared squeezed states. We show the importance of this approach by deriving the overlap between these solutions and connecting them with the overlap integral associated with the Franck-Condon principle.

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Although the Fock-Bargmann approach was suggested many years ago by Glauber to treat coherent states [1], and by Yuen [2] to treat "two-photon coherent states." The method has not been exploited fully and it is specially suited for the case of amplitude-squared squeezed states recently discussed by Bergou, Hillery, and Yu [3].

In this Brief Report we will show that the Fock-Bargmann approach [4] is the natural space to treat squeezed states, which comes from the squared amplitude of the field, and that it can easily be extended to other more complicated mathematical forms for the corresponding differential equations. In passing we obtain straightforwardly the recent results of Bergou, Hillery, and Yu, and obtain more general solutions than the previous works on squeezed states.

Our starting point is the equation associated with the squared amplitude of the field [3]:

$$\left\{a^2+\frac{1-\lambda}{1+\lambda}a^{\dagger 2}\right\}\psi=\frac{2\beta}{1+\lambda}\psi\ , \tag{1}$$

where λ is a real parameter and β any complex number. Making the change of variable

$$z^2=\left[\frac{1-\lambda}{1+\lambda}\right]^{1/2}a^{\dagger 2}\ , \tag{2}$$

we obtain

$$\left[\frac{d^2}{dz^2}-z^2+K\right]\psi=0\ , \tag{3}$$

where $K=-2\beta/[\lambda^2-1]^{1/2}$. This equation has well-known solutions in terms of the Hermite polynomials if $K=2n+1$, where n is a non-negative integer number. This restriction of the values of K gives

$$\beta_n=-[\lambda^2-1]^{1/2}(n+1/2)\ . \tag{4}$$

Since we have not made any assumptions on the values of the parameter λ, we notice that for fixed values of λ these eigenvalues are in one-to-one correspondence with those of the harmonic oscillator; for $\lambda=0$ we obtain the harmonic-oscillator eigenvalues except for a phase factor of $e^{i\pi/2}$. For $|\lambda|<1$ the eigenvalues are imaginary numbers, while for $|\lambda|>1$ they are real numbers. These cases can be discussed in connection with the solutions given by the power-series method, as has been done by Bergou, Hillery, and Yu [3]. Indeed, this result is expected since in the Fock-Bargmann space, Eq. (3) is almost identical to the real harmonic-oscillator differential equation.

Another interesting result that can be obtained is a general solution for the wave function [5]

$$\psi_n(a^\dagger)=N_n e^{g(\lambda)a^{\dagger 2}}H_n(\gamma(\lambda)a^\dagger)\ , \tag{5}$$

where $g(\lambda)=-i\vartheta^2/2$ and $\gamma(\lambda)=-i\vartheta e^{i\pi/4}$, with $\vartheta=[(1-\lambda)/(1+\lambda)]^{1/4}$, and N_n is a normalization constant. This solution is written in the Fock-Bargmann space; however, it can be translated to the configuration space immediately by using the concept of the standard ket developed by Dirac many years ago [6].

The representation in the configuration space is very appropriate to work with the derivation of matrix elements using the normal ordering properties of functions of the creation and annihilation operators. As an example we can derive the overlap between two squeezed state

$$\langle m\lambda|n\lambda\rangle=N_m^*N_n m!n![1-|\vartheta|^4]^{-1/2}\left[\frac{1+|\vartheta|^4}{1-|\vartheta|^4}\right]^{(m+n)/2}\times\sum_{k=0}^{[m,n]}\left[\frac{4|\vartheta|^2}{1+|\vartheta|^4}\right]^k\frac{H_{m-k}(0)H_{n-k}(0)}{k!(m-k)!(n-k)!}\ , \tag{6}$$

where $[m,n]$ is the smaller of m or n.

This is a quite general formula that is very closely related to the closed formula for the Franck-Condon factors derived by analytic methods by Ansbacher [7] and rederived by algebraic methods by Palma and Morales [8]. Furthermore, our last result reveals the mathematical structure of the conjecture advanced a few years ago by Wheeler about the parallelism between the squeezed states and the Franck-Condon principle [9].

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