Statistical properties of resonances in two-dimensional quantum-mechanical point scattering

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Based on the close relations between statistical properties of quantum dissipative systems and scattering systems, we conjecture that for quantum chaotic scattering the distribution of the resonance poles of the S matrix is generic and follows the predictions of Ginibre's ensemble of random (non-Hermitian) matrices. This will be demonstrated on a simple example of a single particle being scattered by (a fixed number of) point obstacles distributed randomly in two dimensions.

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Surely, irregular quantum-mechanical scattering is one of the most interesting fields of today's mathematical physics; indeed, the question is, as far as quantum chaos is concerned, which quantum-mechanical criteria for chaotic behavior do exist? Many recent investigations showed the essential role of random matrix theory for this task; for example, the S matrix was recently shown to be describable in the case of a classically irregular limit by the Dyson ensemble of random matrices [1]. So it is appropriate to ask if any universal predictions based on random matrix theory can be made for the distribution of the resonance poles in the complex plane, too.

In 1957, M. S. Livšic introduced [2] a dissipative operator (later called "Livšic matrix" [3]) $B(z)$ as some kind of restriction of a given quantum Hamiltonian H: Let P be an (orthogonal) projection onto a finite-dimensional subspace $P(\mathfrak{H})$ of the given Hilbert space $\mathfrak{H}$. Then $B(z)$ is defined via

$$[B(z)-z]^{-1}=P(H-z)^{-1}P$$

and thus becomes an operator on $P(\mathfrak{H})$ or, in other words, a finite-dimensional (complex) matrix. Livšic showed the resonances of H to be well defined as the eigenvalues of $B(z)$, i.e., the solutions of the eigenvalue equation

$$B(z)\Psi=z\Psi,\quad z\in\mathbb{C}^{+},\quad \Psi\in P(\mathfrak{H})\ .$$

This proceeding is very similar to the idea that $B(z)$ can be understood as the generator of a dissipative system on $P(\mathfrak{H})$, whereas the rest of $\mathfrak{H}$ behaves like the "heat bath [4]."

The influence of random matrix theory upon generators of dissipative systems was investigated by Grobe and Haake [5,6]. They analyzed the (complex) eigenvalues of generators and were able to show the universality of linear and cubic "level" repulsion, for classically (predominantly) regular and chaotic motion, respectively; i.e., the spacing S, defined as the Euclidean distance s between an eigenvalue and its next neighbor in the complex plane, rescaled by a local density ρ, thus defined as $S=s\sqrt{\rho}$ and seen as a random variable, can be computed into a next-neighbor distribution $\mathcal{P}(S)$ and into a distribution function

$$\mathcal{I}(S)=\int_0^S \mathcal{P}(\mathcal{S})d\mathcal{S}\ .$$

Now linear or cubic repulsion means that

$$\mathcal{P}(S)\sim\begin{cases}S, & \text{"regular"}\\ S^3, & \text{"chaotic"}\end{cases}\quad\text{as } S\to 0\ .$$

The idea behind this formula is quite clear: Eigenvalues that obey cubic level repulsion strongly repel each other, so that they cannot easily be made to cross when varying some underlying parameter; thus, they may be regarded as correlated.

For some strongly damped dissipative systems, Grobe and Haake even found, in the case of chaotic classical motion, a distribution agreeing with the predictions of Ginibre's ensemble of random matrices, and in the case of regular classical motion a distribution corresponding to a Poisson process in the complex plane. Ginibre defined that ensemble of complex matrices in 1965 by dropping the requirement of Hermiticity, which is the crucial feature of the Dyson ensemble. Therefore, the matrices of Ginibre's ensemble (strictly speaking, $N\times N$ matrices) have complex eigenvalues; moreover, these eigenvalues have a next-neighbor distribution $\mathcal{P}(S)$ that yields cubic level repulsion, while, of course, a Poisson process leads to linear repulsion and to uncorrelated eigenvalues.

To be more precise, generators of dissipative systems obey some antiunitary symmetries, so they do not really fulfill the restriction of being totally free of underlying symmetries, as is essential for Ginibre's ensemble. Therefore, the eigenvalues of such generators always come along in complex-conjugate pairs, so that it was essential for Grobe and Haake to take into account only those eigenvalues that were separated from the real axis by at least some distance $\bar{s}$.

Because of the close correspondence between these eigenvalues and the resonance poles of scattering systems, one might expect a similar statistics for these. In order to investigate this, John *et al.* constructed scattering systems out of eight point obstacles randomly placed in space, and calculated numerically the resonances and their next-neighbor distributions [7]. The crucial advantage of this kind of system is its rather easy mathematical handling combined with its obviously total lack of any underlying symmetries. John *et al.* found a very good congruity between the resulting distribution and the Ginibre distribution (for 2×2 matrices). In the present paper we describe results obtained for scattering systems constructed from point obstacles randomly placed in the two-dimensional plane, regaining the same advantages as before in the case of three-dimensional scattering.

The basic idea of point potentials (placed at the positions y_i, $i\in I$) is to gain a Hamiltonian formally written as

$$H=-\Delta+\sum_{i\in I}\alpha_i\delta(x-y_i)\ .$$

A mathematical proper description of such Hamiltonians can be found in [8]. The scattering process is described there completely analytically; the scattering amplitude results from the formula

$$\mathscr{f}(\omega,\omega')=\text{factor}\times\sum_{i,j\in I}[\Gamma_{i,j}^{-1}\exp(\hat{\imath}k\mathscr{t}_{i,j})]\ ,$$

$$\mathscr{t}_{i,j}\equiv\langle\omega',y_i\rangle-\langle\omega,y_j\rangle\ .$$

Here ω and ω' are the unit vectors in the incoming and outgoing directions, respectively, "factor" depends on the energy $E=k^2$ and on the space dimension $\mathfrak{d}$ ($=2$ here; $=3$ in [7]), but is independent of ω, ω', and the y_i. Γ denotes an $N\times N$ matrix (an $I\times I$ matrix, strictly speaking, since N is the number of point scatterers) with elements

$$\Gamma_{ij}=\begin{cases}\alpha_i+\dfrac{1}{2\pi}\left[\gamma+\ln\left[\dfrac{k}{2\hat{\imath}}\right]\right], & i=j\\[2ex] -\dfrac{\hat{\imath}}{4}H_0^{(1)}(k|y_i-y_j|), & i\neq j\end{cases}$$

for dimension $\mathfrak{d}=2$, where $H_0^{(1)}$ denotes the Hankel function of first kind and order zero, and $\gamma=0.577\,215\,664\,9\ldots$ is Euler's constant. The resonance poles can now be found by solving the equation

$$\mathrm{Det}[\Gamma(k)]=0\ .$$

The resonances were computed numerically from this equation on an MS-DOS personal computer for different point numbers N. For example, for $N=4$, we computed the resonances for 20 scattering systems and found 2500 poles, shown in Fig. 1, from which the importance of rescaling $S=s\sqrt{\rho}$ when calculating the next-neighbor distribution is obvious. The distribution $\mathscr{P}(S)$ was calculated for each scattering system separately and then put together to one distribution function $\mathscr{I}(S)$. The local densi-

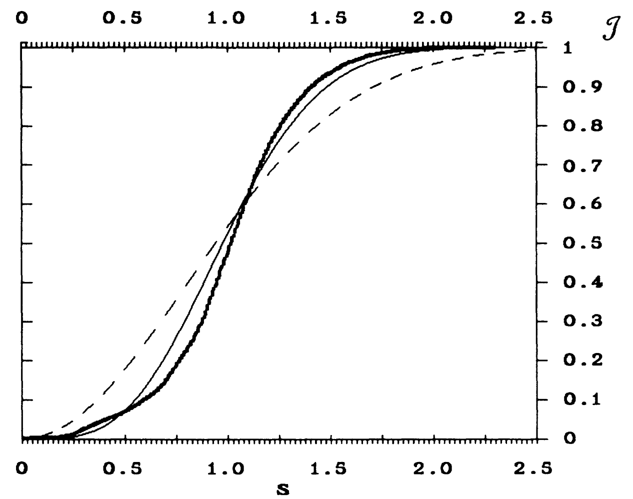

FIG. 2. Next-neighbor distribution for $N=5$. Thick line, calculated resonance-spacing distribution; thin line, prediction of Ginibre's ensemble; dashed line, Poisson process in the plane.

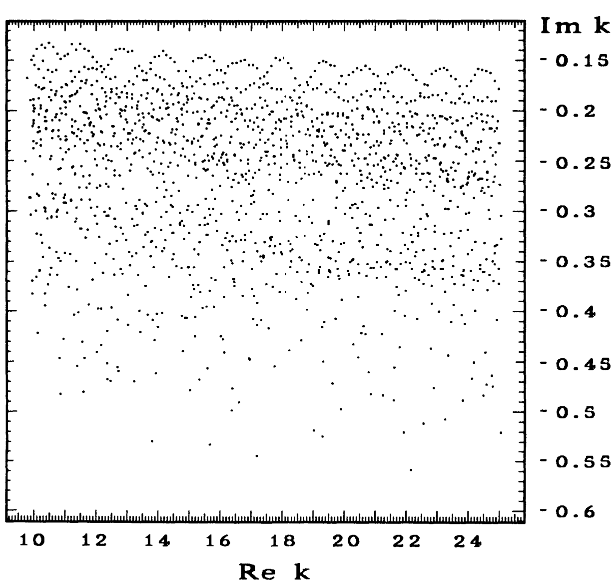

FIG. 1. Positions of the resonance poles of 20 scattering systems, each built of $N=4$ point scatterers.

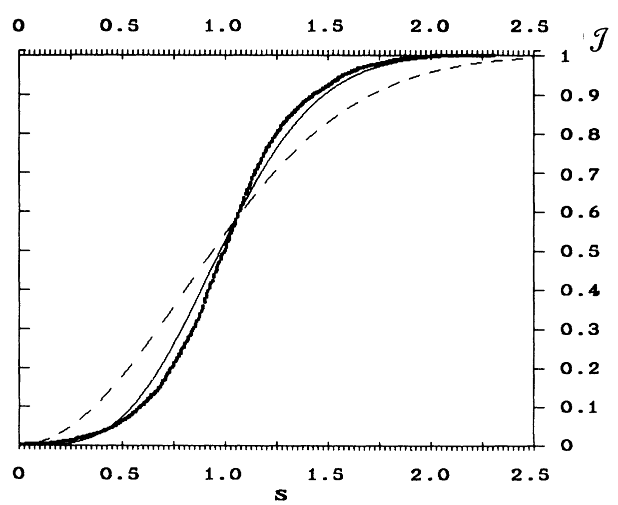

FIG. 3. Next-neighbor distribution for $N=6$. Thick line, calculated resonance-spacing distribution; thin line, prediction of Ginibre's ensemble; dashed line, Poisson process in the plane.

ty ρ needed for rescaling was calculated as a function of the resonance poles, $\rho=\rho(k)$, by choosing a number n of next neighbors; when denoting the n next neighbors of a given $k\equiv k_0(k)$ by $k_1(k),\ldots,k_n(k)$, we define

$$\rho(k)=\frac{n+1}{\sum_{i=0}^{n}\frac{\pi}{4}\{k_1[k_i(k)]-k_i(k)\}^2}\ .$$

(Here we chose $n=5$.)

Afterwards we depicted the resulting distribution function $\mathcal{I}(S)$ together with that which one gets out of the Poisson process

$$\mathcal{P}_P(S)=\tfrac{1}{2}\pi S\exp\left[-\frac{\pi}{4}S^2\right]$$

and out of Ginibre's ensemble of 2×2 matrices,

$$\mathcal{P}_G(S)=2(\tfrac{9}{16}\pi)^2S^3\exp(-\tfrac{9}{16}\pi S^2)\ .$$

One observes a rather good agreement with the prediction of Ginibre's ensemble in Figs. 2 and 3, and especially a clear "level" repulsion for small spacings S, where the agreement becomes better for larger N, i.e., for greater numbers of the scattering points.

In summary, we demonstrated that the described system displays a generic next-neighbor distribution of the resonance poles, which coincides with the predictions of Ginibre's ensemble. Surely, it is still worth analyzing whether cubic pole repulsion can also be found in other scattering systems.

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