Quantum-field model of the injected atomic beam in the micromaser

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A general theory of the micromaser is described. The model is based on treating the input atomic beam as a two-component quantum field so that the two-level atoms in the beam are "quanta" of this field. This approach makes it possible to formulate a general quantum Langevin description of the dynamics with the input atomic field as a source of quantum noise. The passage to a master-equation description is then effected by use of an adjoint-operator method, and by introducing a general class of statistical states for the atomic beam known as generalized shot noise. The result is a (non-Markovian) master equation for the field inside the cavity which is valid for a broad range of statistical properties of the input atomic beam. The approximate steady-state solutions to this master equation for the photon statistics of the cavity field for sub-Poissonian (antibunched) atomic beams found by other researchers are regained. The theory is then extended to treat super-Poissonian (bunched) atomic beams. An exact result is found in the limit of a strongly bunched beam in which the cavity-field state is shown to converge to a mixture of a thermal-field state and the state produced by a random beam of twice the intensity. A physical explanation of this result is also presented.

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I. INTRODUCTION

Recent experimental developments in the field of cavity quantum electrodynamics have made it possible to investigate the quantum-mechanical dynamics of a single atom, or perhaps only a few atoms, interacting with the (quantized) electromagnetic field. One such experimental setup in which the interaction of small numbers of atoms with a quantum-mechanical electromagnetic field is directly subject to study is the so-called micromaser. The experimental realization [1–3] of the micromaser essentially consists of a beam of Rydberg atoms prepared in states with very high principal quantum number passing through a superconducting microwave cavity of extremely high Q. The atomic flux is sufficiently low that at most one atom is present in the cavity at any time. The theory of such a device, developed by Filipowicz, Javanainen, and Meystre [4], predicted the existence of novel features such as sub-Poissonian photon statistics for the cavity field and the presence of abrupt transitions in the field state [5], which are averaged out in usual masers and lasers because of the large and fluctuating numbers of atoms interacting with the field.

Although macroscopic averaging effects are no longer present, the low-flux atomic beam is still nevertheless a source of noise in that the number of atoms entering the cavity in a given time interval will undergo random fluctuations. In particular, under typical experimental conditions, the number of atoms entering the cavity during a given time interval obeys, to a very good approximation, a Poisson distribution. Although it was this case with which Filipowicz, Javanainen, and Meystre were principally concerned, they also pointed out that if the atomic-beam noise is suppressed, the sub-Poissonian nature of the cavity field is further enhanced. The limiting case is of course that of a "quiet" or noise-free beam in which the atoms are equally spaced—so-called regular pumping. The fact that regular pumping of lasers can lead to the generation of sub-Poissonian light had earlier been shown by Golubev and Sokolov [6] and more recently by Haake, Tan, and Walls [7]; Kennedy and Walls [8]; and Marte and Zoller [9] and demonstrated experimentally for a regularly pumped semiconductor laser by Yamamoto and co-workers [10]. Even earlier, theoretical and experimental studies had shown that pump fluctuations in dye-laser systems [11–18] resulted in the light generated being more noisy than that expected on the basis of standard laser theory.

In the specific case of the micromaser, a number of theoretical investigations have been undertaken by Bergou *et al.* [19] and Benkert *et al.* [20] which have also looked more closely at the role of atomic-beam statistics. While the principal aim of these investigations was, once again, to determine the effects of atomic-beam-noise suppression on the statistical character of the cavity field, dealing with this specific problem initially requires looking at a far more general one: how to take account, in the theory, of the random nature of the incident beam. This is, in its own right, an interesting problem in the theory of open systems, and a number of different methods have been proposed to attack it. Apart from direct computer simulation of the cavity pumping [21,22], these methods have taken the form of a generalized master-equation treatment for the cavity-field-density operator [19], or a Langevin equation description [20]. As well as employing different mathematical language, these two approaches treat the problem of modeling the statistical properties of the atomic beam in different ways. The overall equivalence of the two resultant formalisms remains to be established. Nevertheless, each of these investigations concluded that for the micromaser there is further narrowing of the photon distribu-

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tion in the limit of regular pumping.

In this paper another very general method of formulating the micromaser problem is presented. The central feature of the method is the treatment of the atomic beam as a quantum field so that the statistics of the atomic beam are embodied in the quantum state of this field. The field then acts as a quantum-noise source in a manner completely analogous to the more familiar radiation field counterparts that arise in the usual Langevin equation methods of quantum optics. A modified version of Gardiner's adjoint operator [23] is then employed to derive a generalized master equation for the cavity-field-density operator that is valid for a very broad class of statistical states of the atomic beam ranging from regular to Poissonian to super-Poissonian. Under certain conditions, the results of Ref. [19] are regained. The theory is then extended to the case of atomic beams with super-Poissonian statistics for which it turns out to be possible to obtain exact results.

The rest of the paper is organized as follows. In order to introduce the main ideas underlying the method to be described here and to show how it relates to the work of others, it is necessary to give a general theoretical overview of the micromaser. This is done in Sec. II. Section III is specifically concerned with introducing the quantum-field model for the atomic beam. In particular, the space and time evolution of this field is determined for a cavity of finite width leading to equations describing the propagation of the atomic fields through the microwave photon "medium." The adjoint operator for the micromaser is introduced in Sec. IV. The "point-cavity limit" is defined in Sec. V and the generalized master equation derived for the cavity field for a class of atomic-beam states known as generalized shot noise. It is here that contact is made with the results of others. The general method is then used in Sec. VI to determine the steady-state properties of the cavity field for a super-Poissonian atomic beam. Some new results, including an exact result in the limit of a strongly bunched atomic beam, are obtained here. In particular, as expected, the photon distribution is found to be super-Poissonian, but this is found not to be due to a simple broadening of the photon distribution peaks: The thermal cavity field arising from the coupling of the cavity to a thermal reservoir is shown to play an important role in producing this result. A summary and discussion is presented in Sec. VII, and a number of mathematical results are derived in the Appendixes.

II. GENERALIZED MODELS OF THE MICROMASER

In the standard model of a micromaser, a beam of two-level atoms, assumed to be all prepared in their excited states, is incident on a high-Q single-mode cavity. Each atom crosses the cavity in a time τ_c during which time the atom-field system evolves according to the standard Jaynes-Cummings model. Thus, if immediately prior to the arrival of the ith atom, the field in the cavity is in a state described by the density operator $\rho(t_i)$, then as this atom leaves the cavity, the field is in the state

$$\rho(t_i+\tau_c)=[1+\hat{F}]\rho(t_i) , \tag{2.1}$$

where the superoperator $1+\hat{F}$ describes the Jaynes-Cummings evolution (see Sec. IV) and is identical to the operator $\hat{M}$ in Ref. [19].

The time of flight τ_c is assumed to be sufficiently small that cavity damping can be ignored as each atom crosses the cavity. However, between successive atoms, cavity damping is taken into account, this being done in the usual way by assuming the cavity field is coupled to a thermal bath of simple harmonic oscillators. Thus, immediately prior to the arrival of the $(i+1)$th atom in the cavity, the field-density operator is

$$\rho(t_{i+1})=\exp[-\hat{L}(t_{i+1}-t_i-\tau_c)][1+\hat{F}]\rho(t_i) , \tag{2.2}$$

where $\hat{L}$ is the cavity-field damping superoperator. At this point an approximation is made [4], specifically that

$$t_{i+1}-t_i-\tau_c \approx t_{i+1}-t_i ,$$

so that we write Eq. (2.2) as

$$\rho(t_{i+1})=\exp[-\hat{L}(t_{i+1}-t_i)][1+\hat{F}]\rho(t_i) . \tag{2.3}$$

This approximation is equivalent to treating the cavity field as a point harmonic oscillator (see Sec. V) that is kicked at the time t_i in such a manner that the change in the field-density operator effected by each kick is, for infinitesimal ϵ,

$$\rho(t_i+\epsilon)=[1+\hat{F}]\rho(t_i-\epsilon) , \tag{2.4}$$

with the oscillator decaying between kicks. Repeated application of the above operations then gives the state of the cavity field resulting from a sequence of kicks and decays.

The arrival times t_i are, in general, random. Two ways of taking this into account have been employed, each motivated by one or the other of two essentially complementary ways in which the statistical character of the beam manifests itself. In the work of Ref. [19], the focus is on the fact that in the time interval $(0,t)$, the total number of atoms that will pass through the cavity is a randomly fluctuating quantity. The state of the cavity field at time t will then depend on the number of atoms that entered the cavity during the time interval $(0,t)$. The density operator $\rho(t)$ for the cavity field is then obtained by taking an average of the field state over the probability distribution for the number of atoms. Cavity damping is initially ignored, and a continuous-time limit approximation is made, appropriate for a high-flux beam [see Sec. V]. The loss contribution due to cavity damping is then added in, thereby yielding a master equation for $\rho(t)$ given by

$$\dot{\rho}=-\hat{L}\rho+\frac{R}{p}\ln[1+p\hat{F}]\rho , \tag{2.5}$$

where R is the atomic flux intensity (atom/s) and p is the probability of an atom in the beam entering the beam in an excited state, and appears in the formalism through the way in which the statistics of the atomic beam are modeled. For $p=0$, corresponding to the case of a random (Poissonian) atomic beam, the master equation takes

the form, familiar from Scully-Lamb theory [24],

$$\dot{\rho}=-\hat{L}\rho+R\hat{F}\rho \ . \tag{2.6}$$

At the other extreme, that of regular arrivals where $p=1$, the equation for ρ becomes

$$\dot{\rho}=-\hat{L}\rho+R\ \ln[1+\hat{F}]\rho \ . \tag{2.7}$$

This work has shown that (nonclassical) sub-Poissonian fields are produced if noise is suppressed in the atomic pump beam, the extreme case of this, of course, being that of regular arrival times. An equation similar to Eq. (2.7) has also been obtained in the case of regular pumping of a laser [6,7].

The alternative approach [20] is to treat the arrival times t_i as random variables and to take the average over the joint probability distribution for the t_i's. The simplest such averaging procedure arises if the fluctuations can be characterized by a waiting-time distribution, which is the conditional probability density $f(\tau)$ that given that an atom has entered the cavity at a time $t=0$, the next one will arrive at time $t=\tau$. It is this second approach that is eventually adopted here although initially the statistical properties of the beam are left largely unspecified. Furthermore, a master-equation-type approach is followed in contrast to Ref. [20] where Langevin methods are used.

Essentially, the method developed here entails modeling the atomic beam as a stream of very sharp wave packets, all moving with the same mean velocity. The initial positions of these atomic wave packets, as specified by the choice of initial state of the beam, typically a mixed state as these initial positions will be random in general, then fix the time interval between successive atoms entering the cavity. For such a multiparticle system, the use of second-quantized fields is a natural step to take and one which also allows the problem to be cast in the quantum-optical language of quantum-noise theory in which the atomic field acts as a Langevin noise term. This introduction of quantum degrees of freedom to describe the atomic-beam pump parallels a similar approach used by Marte and Zoller [9] in which a quantum description is used for the radiation field pumping of a laser.

The theory is also formulated in such a way that the point-cavity approximation, leading to Eq. (2.4), is not made initially. The distribution of the atoms in the input atomic beam across the width of the cavity, i.e., the possibility of there being more than one atom in the cavity at a time, is included under an adiabatic approximation. This more general problem will be considered elsewhere, only the point-cavity limit is considered here.

III. QUANTUM-FIELD MODEL OF THE MICROMASER

A. Quantum-field model for the atomic beam

An important feature of the work to be described below is that the atomic beam is modeled, using second-quantized formalism, as a one-dimensional space- and time-dependent quantum field. This is done for three reasons. First, such an approach presents a direct way of introducing the possible statistics of the input beam through the choice of the initial state of the beam. Second, the spatial dependency of the beam as it propagates across the cavity can be treated in a very compact manner. Finally, the field picture also leads to a useful formalism for calculating the properties of the atomic beam as it exits from the cavity. It must be borne in mind that the use of a quantum-field description is not intended to imply any treatment of a fundamental nature of the properties of the beam. The field formalism here is used more in the way of a convenient calculational device. Such wave-mechanical phenomena as wave-packet spreading are not included in the theory: In fact, it is removed from the problem, in the present approximation. Indeed, in the final analysis these fields take on more the character of space- and time-dependent classical processes. The question of the true quantum properties of the beam as implied by the role of the quantum statistics of the identical particles making up the atoms is a difficult one and is not addressed here.

With these provisos in mind, we can define the quantum fields appearing in the problem as follows. The atoms are taken to be two-state systems, with the lower state $|1\rangle$, energy $\hbar\omega_1$ and upper state $|2\rangle$, energy $\hbar\omega_2$. Two one-dimensional fields are introduced:

$$\Psi_s(x)=\int_{-\infty}^{+\infty}dk\ \psi_k(x)b_{ks}, \quad s=1,2 \tag{3.1}$$

where

$$\psi_k(x)=(2\pi)^{-1/2}e^{ikx} \tag{3.2}$$

is the free-particle wave function for a particle of momentum $\hbar k$ and energy $\hbar\omega_k$ with

$$\hbar\omega_k=\frac{\hbar^2k^2}{2m} \ . \tag{3.3}$$

The quantum nature of the fields $\Psi_s(x)$ enters through the annihilation operators b_{ks}, whose effect is to annihilate an atom in state $|s\rangle$ with center-of-mass momentum $\hbar k$. These operators are assumed to satisfy the boson-commutation rules

$$[b_{k's'},b_{ks}^{\dagger}]=\delta_{ss'}\delta(k-k') \ . \tag{3.4}$$

With this choice, it is possible to model atomic-beam states that range from bunched through to regular (antibunched) beams.

The Hamiltonian for the atomic beam, including the center-of-mass motion, is then

$$H_{\rm beam}=\sum_s\int_{-\infty}^{+\infty}dk\ \hbar(\omega_k+\omega_s)b_{ks}^{\dagger}b_{ks} \ . \tag{3.5}$$

The atomic beam will be assumed to be prepared in many-atom states consisting of well-defined wave packets, positioned along the x axis, and all with the same mean center-of-mass velocity. Attention will be confined to those states in which all the atoms in the beam are excited. The details of how these states are constructed are found in Sec. V and in Appendix C.

B. Total system Hamiltonian

The atomic beam is assumed to interact with the cavity field over a region $0<x<L$. The interaction Hamiltonian is therefore taken to be of the form

$$V=\hbar K\int_0^L dx\,[\Psi_1^\dagger(x)\Psi_2(x)a^\dagger+\Psi_2^\dagger(x)\Psi_1(x)a]\ , \quad (3.6)$$

where K is an atom-field coupling constant and $a,a^\dagger$ are the annihilation and creation operators for the cavity-field mode. The cavity-field Hamiltonian is

$$H_{\text{field}}=\hbar\omega_0 a^\dagger a\ , \quad (3.7)$$

where the detuning from resonance is given by

$$\Delta=\omega_0-\omega_{21}\ . \quad (3.8)$$

Including the coupling of the cavity field to a thermal bath of simple harmonic oscillators then gives the final form for the total Hamiltonian

$$H=H_{\text{field}}+H_{\text{beam}}+V+\int dq\,\hbar q c_q^\dagger c_q +\sqrt{\gamma_c/2\pi}\int dq[c_q^\dagger a+c_q a^\dagger]\ , \quad (3.9)$$

where c_q and $c_q^\dagger$ are the boson annihilation and creation operators for the bath oscillators. Following Gardiner and Collett [25], the q values are assumed to range over $+\infty<q<+\infty$ so that the Markov approximation can be applied directly and shift terms are automatically excluded.

C. Intracavity evolution of atomic-beam fields

The Heisenberg equations of motion for the atomic-beam operator b_{ks} is given, after integrating over time, by

$$b_{k1}(t)=\exp[-i(\omega_k+\omega_1)t]b_{k1}(0) -iK\int_0^t d\tau\exp[-i(\omega_k+\omega_1)(t-\tau)] \times\int_0^L dx\ \psi_k^*(x)\Psi_2(x,\tau)a^\dagger(\tau)\ , \quad (3.10)$$

with a similar equation for $b_{k2}(t)$. From the defining expression Eq. (3.1), we find for the Heisenberg operator $\Psi_1(x,t)$:

$$\Psi_1(x,t)=\Psi_1^{(0)}(x,t) -iK\int_0^t d\tau\int_0^L dx' a^\dagger(t-\tau)\exp[-i\omega_1\tau] \times\Psi_2(x',t-\tau)G(x-x',\tau)\ , \quad (3.11)$$

where

$$G(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\,\exp[i(kx-\omega_k t)] \quad (3.12)$$

is the free-particle Green's function and

$$\Psi_1^{(0)}(x,t)=\int_{-\infty}^{+\infty}dk\ \psi_k(x)b_{k1}(0)\exp[-i(\omega_k+\omega_1)t] \quad (3.13)$$

is the free-field operator for the atomic beam for atoms in state $|1\rangle$. A similar result follows for $\Psi_2(x,t)$.

It is at this point that an important approximation is made. It is assumed that the atomic beam is prepared in a state in which the center-of-mass momentum of the atom is sharply peaked around a mean value $\hbar k_0$, corresponding to a mean velocity v_0. The mass of the atoms is assumed to be sufficiently large that any change in their momentum as they cross the cavity can be ignored and that any wave-packet spreading can also be ignored (i.e., the center-of-mass motion is treated classically). In view of the assumed sharpness of the distribution of k values around k_0, this approximation can be incorporated into the analysis by linearizing ω_k in the Green's function around k_0. The result obtained is

$$G(x,t)=\exp[\tfrac{1}{2}ik_0x]\delta(x-v_0t)\ . \quad (3.14)$$

Under this approximation, the integral in Eq. (3.10) can now be performed. In terms of the operators

$$\Phi_n(x,t)=\Psi_n(x,t)\exp[i(\omega_n t-\tfrac{1}{2}k_0x)]\ ,\quad n=1,2$$
$$A(t)=a(t)\exp[i\omega_{21}t]\ , \quad (3.15)$$

the result Eq. (3.11) becomes for $0<x<L$ and for $t>L/v_0$:

$$\Phi_1(x,t)=\Phi_1^{(0)}(x,t) -iK\int_{t-x/v_0}^t d\tau\ A^\dagger(\tau)\Phi_2(x+v_0(\tau-t),\tau)\ , \quad (3.16a)$$

$$\Phi_2(x,t)=\Phi_2^{(0)}(x,t) -iK\int_{t-x/v_0}^t d\tau\ A(\tau)\Phi_1(x+v_0(\tau-t),\tau)\ . \quad (3.16b)$$

Effectively, the atoms (i.e., the "quanta" of the atomic fields) under this approximation all move with the same unchanging velocity v_0 and all in the same direction. However, there still remains the freedom to define wave packets (which do not spread) with a limited extension in space.

Equations (3.16a) and (3.16b) can be transformed into wave equations for Φ_1 and Φ_2 to give, for $0<x<L$,

$$\frac{\partial\Phi_1}{\partial t}+v_0\frac{\partial\Phi_1}{\partial x}=-iKA^\dagger(t)\Phi_2\ , \quad (3.17a)$$

$$\frac{\partial\Phi_2}{\partial t}+v_0\frac{\partial\Phi_2}{\partial x}=-iKA(t)\Phi_1\ , \quad (3.17b)$$

so that in effect the fields Φ_1 and Φ_2 are propagating through a medium of photons, in contrast to the more usual situation in which a photon field propagates through an atomic medium.

The free fields $\Phi_1^{(0)}(x,t)$ and $\Phi_2^{(0)}(x,t)$ will both satisfy homogeneous equations also of the form Eq. (3.17) so that we can immediately write

$$\Phi_n^{(0)}(x,t)=\Phi_n^{(0)}(x-v_0t,0)\ ,\quad n=1,2\ . \quad (3.18)$$

The program now is to solve Eq. (3.17) for Φ_1 and Φ_2 in-

side the cavity. This is made difficult by the presence of the cavity-field source terms on the right-hand sides of Eqs. (3.17a) and (3.17b) which are coupled to the atomic fields. However, a solution can be obtained under the approximation that the cavity field is slowly varying over a time interval less than the time of flight (τ_c) of the atoms across the cavity. This approximation is expected to hold true provided the mean number of atoms in the cavity at any one time is small compared to the mean number of photons and provided the time of flight is much less than the damping time of the cavity, i.e., $\tau_c \ll \gamma_c^{-1}$. Details of this calculation are to be found in Appendix A. The following explicit expressions (in normal order) for the atomic fields $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are obtained:

$$\Psi_1(x,t)=C(N(t),x)\Psi_1^{(0)}(x,t) - iS(N(t),x)a^\dagger(t)\Psi_2^{(0)}(x,t) , \tag{3.19}$$

$$\Psi_2(x,t)=-ia(t)S(N(t),x)\Psi_1^{(0)}(x,t) + C(N(t)+1,x)\Psi_2^{(0)}(x,t) , \tag{3.20}$$

where exact resonance ($\Delta=0$) has been assumed and where

$$N(t)=a^\dagger(t)a(t) \tag{3.21}$$

is the photon-number operator in the Heisenberg picture and

$$C(N,x)=\cos[KN^{1/2}x/v_0] , \tag{3.22}$$

$$S(N,x)=N^{-1/2}\sin[KN^{1/2}x/v_0] . \tag{3.23}$$

These solutions exhibit Rabi oscillations of the atomic fields through the presence of the operator functions $S(N,x)$ and $C(N,x)$, though as a function of position rather than time. The fact that simple Rabi oscillations are still present when the possibility exists of there being more than one atom simultaneously in the cavity suggests that the approximations leading to Eqs. (3.19) and (3.20) amount to assuming that the atoms each couple independently to the ambient cavity field, i.e., there is no atom-atom communication via exchange of photons with this field. This is certainly a reasonable approximation if the number of photons in the cavity is, on average, much larger than the number of atoms. If this condition were not met, the possibility exists of significant changes in the field state, leading to the more complex oscillatory behavior exhibited by the multiatom Tavis-Cummings model [26]. It is also worthwhile to note from Eqs. (3.19) and (3.20) that

$$\langle \Psi_1^\dagger(x,t)\Psi_1(x,t)+\Psi_2^\dagger(x,t)\Psi_2(x,t)\rangle = \langle \Psi_1^{(0)\dagger}(x,t)\Psi_1^{(0)}(x,t)+\Psi_2^{(0)\dagger}(x,t)\Psi_2^{(0)}(x,t)\rangle , \tag{3.24}$$

i.e., the flux of atoms is conserved.

IV. ADJOINT EQUATION FOR CAVITY FIELD

The atomic-beam free-field operators $\Psi_n^{(0)}(x,t)$ play the role here of noise sources, due to the incident atomic beam, which affect the quantum state of the cavity field. This can be best seen if we write down the Heisenberg equations of motion for an arbitrary cavity field operator:

$$\dot{Y}=\frac{i}{\hbar}[H_{\text{field}},Y] + iK\int_0^L dx[\Psi_2^\dagger(x,t)[a,Y]\Psi_1(x,t) + \Psi_1^\dagger(x,t)[a^\dagger,Y]\Psi_2(x,t)] + i\sqrt{\gamma_c/2\pi}\int dq[c_q^\dagger a + c_q a^\dagger, Y] . \tag{4.1}$$

Upon substituting for Ψ_1 and Ψ_2 from Eqs. (3.19) and (3.20), we obtain a Langevin-type equation for Y with terms containing the $\Psi_n^{(0)}$'s acting as noise sources. This result is quite general in that by appropriate choice of input beam states, $\rho_B(0)$ atomic beams with different possible statistical properties can be represented, and Eq. (4.1) averaged over this state. A typical choice of state would be one in which all the incident atoms are excited so that

$$\rho_B(0)=|(\text{vac})_1\rangle\langle(\text{vac})_1|\rho_2(0) , \tag{4.2}$$

i.e., the upper atomic-state component of the beam field is in the state $\rho_2(0)$, and the lower atomic-state component of the beam field is in the state $|(\text{vac})_1\rangle\langle(\text{vac})_1|$, the vacuum state. As a consequence of the normal ordering of the atomic-field operators, all the $\Psi_1^{(0)}(x,t)$ operators will act to the right [and $\Psi_1^{(0)\dagger}(x,t)$ operators to the left] on the vacuum state $|(\text{vac})_1\rangle$. Thus, the contributions due to the terms involving $\Psi_1^{(0)}(x,t)$ and $\Psi_1^{(0)\dagger}(x,t)$ will all vanish. More general states, including those in which the atoms carry initial atomic coherence, would lead to contributions due to all the noise terms in Eq. (4.1).

The Heisenberg equation, Eq. (4.1), is not the most convenient to work with. It would be preferable to have a master equation for the density operator ρ of the cavity field, but a more readily obtained equation is that for a quantity related to ρ, the so-called adjoint operator μ introduced by Gardiner [23]. This operator is defined here such that if the total system of cavity field, reservoir, and atomic beam is initially in the product state $\rho(0)\rho_R(0)\rho_B(0)$, then for an arbitrary cavity-field operator X,

$$\text{Tr}_{CR}[X(t)\rho(0)\rho_R(0)\rho_B(0)]=\text{Tr}_C[X(0)\mu_S(t)]\rho_B(0) , \tag{4.3}$$

where the trace on the left-hand side is over states of the cavity field and the reservoir. The subscript S indicates a Schrödinger-picture operator. This definition differs from Gardiner's in that on the left-hand side the trace is taken over the cavity-field states and an additional trace is also taken over the reservoir states. On the right-hand side the trace is only over cavity-field states. Thus, the Schrödinger-picture operator $\mu_S(t)$ acts only on the Hilbert space of the cavity field and atomic beam. This operator is related to the cavity-field density operator (in the Schrödinger picture) by

$$\text{Tr}_B[\mu_S(t)\rho_B(0)]=\rho_S(t) . \tag{4.4}$$

From the defining condition Eq. (4.3), we also find for an

arbitrary cavity-field operator X that

$$\mathrm{Tr}_{CR}[\Psi_n^{(0)\dagger}(x,t)X(t)\Psi_m^{(0)}(x,t)\rho(0)\rho_R(0)\rho_B(0)] = \Psi_n^{(0)\dagger}(x,t)\mathrm{Tr}_C[X(0)\mu_S(t)]\Psi_m^{(0)}(x,t)\rho_B(0) \quad (4.5)$$

and

$$\mathrm{Tr}_{CR}[\dot{X}(t)\rho(0)\rho_R(0)\rho_B(0)] = \mathrm{Tr}_C[X(0)\dot{\mu}(t)]\rho_B(0) \ . \quad (4.6)$$

An equation for $\mu_S(t)$ can be derived from Eq. (4.1). As no trace over the beam states is taken, this equation retains explicit dependence on the atomic-beam noise operators $\Psi_n^{(0)}(x,t)$. The equation for μ follows from Eq. (4.1) by multiplying by $\rho(0)\rho_R(0)\rho_B(0)$, taking a partial trace over cavity-field and reservoir states, substituting for the $\Psi_n(x,t)$ from Eqs. (3.19) and (3.20), using cyclic permutation under the trace to bring the various terms to the form Eq. (4.3), and using Eqs. (4.5) and (4.6). A further step required in the evaluation of the reservoir interaction term is the elimination under a Markov approximation of the reservoir operators arising in the third term in Eq. (4.1). The state of the reservoir, $\rho_R(0)$ is, as usual, assumed to be in thermal equilibrium, resulting in a nonzero (for finite temperature of the reservoir) cavity-field mean thermal photon number n_b. As this kind of calculation is not considered in Ref. [23], the details are given here in Appendix B. Finally, for later purposes, it is more convenient to work in the interaction picture defined by

$$\mu(t) = U(t)\mu_S(t)U^\dagger(t) \ , \qquad \rho(t) = U(t)\rho_S(t)U^\dagger(t) \ , \quad (4.7)$$

where

$$U(t) = \exp(-iH_{\text{field}}t/\hbar) \ . \quad (4.8)$$

The net result of the above operations is an equation of the form

$$\mathrm{Tr}_C[Y(0)\dot{\mu}(t)]\rho_B(0) = \mathrm{Tr}_C[Y(0)\cdots\mu(t)\cdots]\rho_B(0) \ , \quad (4.9)$$

where the ellipses indicate terms involving cavity-field operators and free-field operators for the atomic beam. Since the operator Y is arbitrary, an equation for μ can be extracted:

$$\begin{aligned}\dot{\mu} = &-\tfrac{1}{2}\gamma_c n_b(\mu aa^\dagger + aa^\dagger\mu - 2a^\dagger\mu a) - \tfrac{1}{2}\gamma_c(n_b+1)(\mu a^\dagger a + a^\dagger a\mu - 2a\mu a^\dagger) \\ &+ K\int_0^L dx\{-\Psi_1^{(0)\dagger}(x,t)[C(N,x)\mu S(N,x)N - C(N+1,x)a\mu a^\dagger S(N+1,x)]\Psi_1^{(0)}(x,t) \\ &+ ie^{i\omega_0 t}\Psi_1^{(0)\dagger}(x,t)[S(N,x)a^\dagger\mu S(N,x)N - (N+1)S(N+1,x)\mu a^\dagger S(N+1,x)]\Psi_2^{(0)}(x,t) \\ &+ ie^{-i\omega_0 t}\Psi_2^{(0)\dagger}(x,t)[C(N,x)\mu aC(N,x) - C(N+1,x)a\mu C(N+1,x)]\Psi_1^{(0)}(x,t) \\ &+ \Psi_2^{(0)\dagger}(x,t)[S(N,x)a^\dagger\mu aC(N,x) - (N+1)S(N+1,x)\mu C(N+1,x)]\Psi_2^{(0)}(x,t) + \text{H.c.}\} \ . \end{aligned} \quad (4.10)$$

This equation for μ is the general equation to be solved in order to determine the cavity-field density operator $\rho(t)$ using Eq. (4.4). The quantity μ is obviously a functional of the noise operator terms $\Psi_n^{(0)}(x,t)$ associated with the input atomic beam.

V. MASTER EQUATION FOR CAVITY FIELD

A. Point-cavity limit

Equation (4.10) is made difficult to deal with by the presence of the spatial integral which gives rise to a memory effect, though by further exploiting the coarse-grain averaging used in deriving Eqs. (3.19) and (3.20), it is possible to make some progress in dealing with this general result. However, here we will be confining attention to the limiting case of a "point" cavity in which the limits $L \to 0$ and $K \to \infty$ are taken in such a way that $KL = \xi$, a constant. Taking this limit amounts to a formal way of guaranteeing the "one atom in the cavity at a time" condition without losing the Jaynes-Cummings dynamics associated with the interaction of the atoms with the cavity field. We then find that Eq. (4.10) can be written

$$\dot{\mu} = -\hat{L}\mu + \sum_{n,m=1,2}\hat{J}_{nm}(0,t)\hat{F}_{nm}\mu \ , \quad (5.1)$$

where the cavity-damping superoperator $\hat{L}$ is given by

$$\begin{aligned}\hat{L}\mu = &\tfrac{1}{2}\gamma_c n_b(\mu aa^\dagger + aa^\dagger\mu - 2a^\dagger\mu a) \\ &+ \tfrac{1}{2}\gamma_c(n_b+1)(\mu a^\dagger a + a^\dagger a\mu - 2a\mu a^\dagger)\end{aligned} \quad (5.2)$$

and the superoperators representing the pumping effect of the atomic beam are given by

$$\begin{aligned}(1+\hat{F}_{11})\mu = &\cos[N^{1/2}\xi/v_0]\mu\cos[N^{1/2}\xi/v_0] \\ &+ \frac{\sin[(N+1)^{1/2}\xi/v_0]}{(N+1)^{1/2}}a\mu a^\dagger \\ &\times\frac{\sin[(N+1)^{1/2}\xi/v_0]}{(N+1)^{1/2}} \ , \end{aligned} \quad (5.3)$$

$$\hat{F}_{12}\mu = i\cos[(N+1)^{1/2}\xi/v_0]\mu a^\dagger \frac{\sin[(N+1)^{1/2}\xi/v_0]}{(N+1)^{1/2}} - i\frac{\sin[N^{1/2}\xi/v_0]}{N^{1/2}} a^\dagger \mu \cos[N^{1/2}\xi/v_0] , \quad (5.4)$$

$$(1+\hat{F}_{22})\mu = \cos[(N+1)^{1/2}\xi/v_0]\mu\cos[(N+1)^{1/2}\xi/v_0] + \frac{\sin[N^{1/2}\xi/v_0]}{N^{1/2}} a^\dagger \mu a \frac{\sin[N^{1/2}\xi/v_0]}{N^{1/2}} \quad (5.5)$$

[$\hat{F}_{21}\mu$ is the Hermitean conjugate of Eq. (5.4)], and

$$\hat{J}_{nm}(x,t)\mu = v_0 \Psi_n^{(0)\dagger}(x,t)\mu\Psi_m^{(0)}(x,t) . \quad (5.6)$$

Equation (5.1) describes a damped oscillator, kicked instantaneously and randomly by a beam of atoms whose internal states and arrival statistics are embodied in the states of the fields $\Psi_1^{(0)}$ and $\Psi_2^{(0)}$. In particular, if the incident beam consists solely of excited atoms, then, as a consequence of each kick, the state of the cavity field evolves according to the prescription in Eq. (5.5), which is the usual expression for the change in the field due to a single initially excited atom interacting with the field over a time interval $\tau = \xi/Kv_0$ [4]. We can note here that in order to arrive at Eq. (5.1), the above limiting procedure is necessary, i.e., the cavity cannot be treated initially as a point oscillator in the Hamiltonian as incorrect treatment of the δ-function potential that is thereby implied can lead to inconsistent results [27].

In this form, it is possible, in principle, to treat atomic beams of arbitrary statistical properties, including those in which individual atoms are in coherent superposition states [28], in which case the cross terms, Eq. (5.4), will contribute. Another interesting possibility is that in which there is spatial coherence along the beam. However, in the following we will be dealing with the usual micromaser case in which the atoms are all initially in their excited state. Under these circumstances, only Eq. (5.5) will contribute so that we can write the adjoint equation in the simpler form

$$\dot{\mu} = -\hat{L}\mu + \hat{F}\hat{J}(t)\mu , \quad (5.7)$$

where

$$\hat{F} \equiv \hat{F}_{22} \quad (5.8)$$

and

$$\hat{J}(t) \equiv \hat{J}_{22}(0,t) . \quad (5.9)$$

For a particular class of input beam states, of sufficient generality that bunched, random, and antibunched atomic beams can be modeled, it is possible to obtain from Eq. (5.7) the master equation for the cavity-field density operator ρ. This class of beam states is now described.

B. Generalized shot noise

The statistical properties of the atomic beam could be defined through the choice of state, or through the appropriate choice of the correlation functions of the field operators [9]. It is the former approach that is followed here. In Appendix C, quantum states are constructed based on the picture of the atomic beam as a stream of excited atoms, represented by well-localized wave packets initially positioned at the points $x_1, x_2, x_3, \ldots$, i.e., δ-function wave packets which, under the approximations discussed in Sec. III, do not spread as they propagate. The properties of the incident atomic beam are then described by a hierarchy of intensity correlation functions defined in terms of the intensity operator

$$J(x,t) = v_0 \Psi_2^{(0)\dagger}(x,t)\Psi_2^{(0)}(x,t) , \quad (5.10)$$

which is distinguished from the associated superoperator defined in Eq. (5.6) by the absence of a caret.

A general correlation function is then

$$\langle :J(x_1,t_1)J(x_2,t_2)\cdots J(x_n,t_n): \rangle = \mathrm{Tr}_B[\hat{J}(x_1,t_1)\hat{J}(x_2,t_2)\cdots\hat{J}(x_n,t_n)\rho_B(0)] , \quad (5.11)$$

where as usual the : : indicates normal ordering. Further, since the free field $\Phi_n^{(0)}(x,t)$, related to $\Psi_n^{(0)}(x,t)$ by Eq. (4.6), is the homogeneous solution of the wave Eq. (4.8), we can conclude that

$$J(x,t) = J(0,t-x/v_0) \equiv J(t-x/v_0) . \quad (5.12)$$

The mean intensity of the beam is shown in Appendix C to be

$$\langle J(t) \rangle = v_0 \sum_n \delta(x_n + v_0 t) , \quad (5.13)$$

while the nth-order intensity correlation function is

$$\langle :J(t_1)J(t_2)\cdots J(t_n): \rangle = v_0^n \sum_{l_1,l_2,\ldots,l_n}{}' \delta(x_{l_1}+v_0t_1)\delta(x_{l_2}+v_0t_2) \times \cdots \delta(x_{l_n}+v_0t_n) , \quad (5.14)$$

where the prime on the summation sign serves to indicate that equal summation indices are excluded from the sum.

The positions of the atoms in the beam, or, equivalently, the time of arrival of the atoms in the cavity, are now assumed to be classical random variables. Their statistical properties will be specified by the conditions that define a renewal process (see, e.g., Bhat [29]). Thus, we let $f_1(t_1)dt_1$ be the probability that the first atom arrives in the cavity in the time interval (t_1, t_1+dt_1), $t_1 > 0$, and then introduce a further probability distribution, the waiting-time distribution $f(t)$ defined such that the probability of an atom arriving in the cavity in the time interval $(t, t+dt)$ after the preceding atom reached the cavity is given by $f(t)dt$. The beam therefore is described as a classical stochastic process consisting of a random sequence of δ-function pulses whose statistical properties are specified by these probability distributions. Such a process belongs to a general class termed generalized shot noise by Rice [30]. This model for the beam is sufficiently general that atomic beams with statistical properties ranging from antibunched (regular) arrivals to random and bunched arrival times can be treated. Furthermore, the intensity correlation functions possess a useful factor-

ization property which makes it possible to obtain a formal solution to Eq. (5.7) in the steady-state limit.

The beam will have stationary statistics if we choose [29]

$$f_1(t)=T^{-1}\left[1-\int_0^t f(\tau)d\tau\right], \tag{5.15}$$

where

$$T=\int_0^\infty \tau f(\tau)d\tau \tag{5.16}$$

is the mean time between atoms. In the stationary case we can readily show that

$$\langle J(t)\rangle = R = T^{-1}, \tag{5.17}$$

as it should be.

From the general formulation by Rice, it follows that the two-time intensity correlation function is given, for a stationary renewal process, by (see Appendix D)

$$\langle :J(t_2)J(t_1):\rangle = E\left[\sum_{\substack{l,m\\ l\neq m}} \delta(t_2+x_l/\nu_0)\delta(t_1+x_m/\nu_0)\right]$$
$$= g(t_2-t_1)R^2, \quad t_2\geq t_1 \tag{5.18}$$

where $E[\ ]$ denotes a (classical) average over the renewal process and where the intensity correlation function (or renewal density [29]) $g(t)$ satisfies the integral equation

$$g(t)=Tf(t)+\int_0^t d\tau f(\tau)g(t-\tau). \tag{5.19}$$

The higher-order correlation functions are found to possess the convenient factorization property

$$\langle :J(t_n)J(t_{n-1})\cdots J(t_1):\rangle, \quad t_n\geq t_{n-1}\geq\cdots\geq t_1$$
$$= g(t_n-t_{n-1})g(t_{n-1}-t_{n-2})\cdots g(t_2-t_1)R^n$$
$$= Rg(t_n-t_{n-1})\langle :J(t_{n-1})J(t_{n-2})\cdots J(t_1):\rangle . \tag{5.20}$$

Thus, given the waiting-time distribution, we can obtain from Eq. (5.19) the intensity correlation function and from Eq. (5.20) any higher-order intensity correlation functions. Equations (5.19) and (5.20) can also be derived by performing explicitly the averages over the sums of δ functions.

Of particular interest is the case in which

$$f(t)=R^{-1}\exp(-Rt), \tag{5.21}$$

which leads to

$$g(t)=1. \tag{5.22}$$

In that case, the trace over the beam state in Eq. (5.7) for μ can be readily taken, the result being just Eq. (2.6). The other interesting example is that of a regular beam for which

$$f(t)=\delta(t-T). \tag{5.23}$$

This yields for the intensity correlation function:

$$g(t)=\sum_{n=1}^\infty \delta(Rt-n), \tag{5.24}$$

a result somewhat different from the same quantity given in Ref. [20], Eq. (A28), for $p=1$. The case of a regular beam is considered further below.

C. General form of master equation

Integrating the equation for μ over the range $(0,t)$ yields

$$\mu(t)=\exp(-\hat{L}t)\mu(0)$$
$$+\int_0^t d\tau\exp(-\hat{L}\tau)\hat{J}(t-\tau)\hat{F}\mu(t-\tau). \tag{5.25}$$

At $t=0$, the cavity is taken to be in the state $\rho(0)$ and, hence,

$$\mu(0)=\rho(0)\otimes 1_B, \tag{5.26}$$

where 1_B is the unit operator in the Hilbert space of the beam states. Now define the operator $\sigma(t)$, acting in the Hilbert space of the cavity field, by

$$R\sigma(t)=\mathrm{Tr}_B[\hat{J}(t)\mu(t)\rho_B(0)]. \tag{5.27}$$

This operator has unit trace

$$\mathrm{Tr}_C[\sigma(t)]=1, \tag{5.28}$$

this following from Eq. (5.17).

From Eq. (5.25), and on using Eq. (5.27), $\sigma(t)$ is seen to satisfy the equation

$$R\sigma(t)=R\exp(-\hat{L}t)\rho(0)$$
$$+\int_0^t d\tau\exp(-\hat{L}\tau)$$
$$\times\mathrm{Tr}_B[\hat{J}(t)\hat{J}(t-\tau)\hat{F}\mu(t-\tau)\rho_B(0)]. \tag{5.29}$$

As the operator $\mu(t-\tau)$ appearing under the integral sign in Eq. (5.29) will depend on $\hat{J}$ evaluated at times earlier than $t-\tau$ (and hence also earlier than t), the factorization property of the intensity correlation functions, Eq. (5.20), can be used to write Eq. (5.29) as

$$\sigma(t)=\exp(-\hat{L}t)\rho(0)+R\int_0^t d\tau\exp(-\hat{L}\tau)g(\tau)\hat{F}\sigma(t-\tau). \tag{5.30}$$

The Laplace transform of Eq. (5.30) is then

$$\tilde{\sigma}(s)=(s+\hat{L})^{-1}\rho(0)+R\tilde{g}(s+\hat{L})\hat{F}\tilde{\sigma}(s), \tag{5.31}$$

where $\tilde{g}(s)$ is the Laplace transform of $g(t)$. Further, from Eq. (5.25) and on using Eqs. (4.4) and (5.27), we find, for the Laplace transform of $\rho(t)$,

$$\tilde{\rho}(s)=(s+\hat{L})^{-1}\rho(0)+R(s+\hat{L})^{-1}\hat{F}\tilde{\sigma}(s). \tag{5.32}$$

Eliminating $\tilde{\sigma}(s)$ from these equations yields

$$[1-R\hat{F}\tilde{g}(s+\hat{L})][(s+\hat{L})\tilde{\rho}(s)-\rho(0)]$$
$$=R\hat{F}(s+\hat{L})^{-1}\rho(0). \tag{5.33}$$

This result can be cast in a number of different forms. For instance, after some rearranging of terms and upon inverting the Laplace transform, the density operator $\rho(t)$

can be shown to satisfy the following integro-differential equation:

$$\dot{\rho}(t)=-\hat{L}\rho(t)+\int_0^t d\tau\,\hat{K}(t-\tau)\rho(\tau)\ . \tag{5.34}$$

The Laplace transform of the memory kernel is given by

$$\hat{K}(s)=[1-R\hat{F}\tilde{G}(s+\hat{L})]^{-1}R\hat{F}\ , \tag{5.35}$$

in which there appears the Laplace transform of the function $G(t)$ defined by

$$G(t)=g(t)-1=\frac{\langle :J(t)J(0):\rangle-\langle :J(0)^2:\rangle}{\langle :J(0)^2:\rangle}\ , \tag{5.36}$$

where Eqs. (5.17) and (5.18) have been used. This is just the normalized intensity correlation function for the atomic beam.

Equation (5.34) immediately shows the general non-Markovian character of the evolution of the cavity field. The exception is when the input atomic beam has Poisson statistics. In that case $g(t)=1$ so that $\hat{K}=R\hat{F}$ and Eq. (5.34) reduces to the expected result given by Eq. (2.6).

Alternatively, Eq. (5.33) can be formally solved for $\tilde{\rho}(s)$. In doing so, it is convenient to make use of an expression for the Laplace transform of $g(t)$ which follows directly from the integral equation Eq. (5.19) relating $g(t)$ to the waiting-time distribution $f(t)$. The required expression is

$$\tilde{g}(s)=\frac{T\tilde{f}(s)}{1-\tilde{f}(s)}\ , \tag{5.37}$$

which when substituted into Eq. (5.33) leads to the result

$$\begin{aligned}\tilde{\rho}(s)=&T\tilde{f}_1(s+\hat{L})(1+\hat{F})\\&\times[1-\tilde{f}(s+\hat{L})(1+\hat{F})]^{-1}\tilde{f}_1(s+\hat{L})\rho(0)\\&+(s+\hat{L})^{-1}[1-\tilde{f}_1(s+\hat{L})]\rho(0)\ ,\end{aligned} \tag{5.38}$$

where

$$\tilde{f}_1(s)=Rs^{-1}(1-\tilde{f}(s)) \tag{5.39}$$

is the Laplace transform of $f_1(t)$, the probability of arrival of the first atom in the cavity, given by Eq. (5.15) for stationary statistics.

A formal expression for $\rho(t)$ is then obtained by inverting the Laplace transform in Eq. (5.38). If the inverse operator appearing in the first term on the right-hand side of Eq. (5.38) is expanded as a power series, then $\rho(t)$ can be expressed as an infinite sum of convolution integrals:

$$\begin{aligned}\rho(t)=&T\sum_{n=0}^{\infty}\exp(-\hat{L}t)f_1(t)(1+\hat{F})\\&\times[\circ\exp(-\hat{L}t)f(t)(1+\hat{F})]^n\\&\circ\exp(-\hat{L}t)f_1(t)\rho(0)\\&+\exp(-\hat{L}t)\int_t^{\infty}d\tau\,f_1(\tau)\rho(0)\ ,\end{aligned} \tag{5.40}$$

where "$\circ$" indicates a convolution is to be taken, i.e.,

$$x(t)\circ y(t)=\int_0^t d\tau\,x(\tau)y(t-\tau)\ .$$

This result contains alternating factors of $(1+\hat{F})$, representing the change in the cavity field due to the passage of a single atom, and $\exp(-\hat{L}t)$, representing the free decay of the cavity field between each atom, these factors weighted by the probability $f(t)$ of the arrival of successive atoms. While the appearance of these alternating factors is perhaps to be expected, it is nevertheless satisfying that it is an automatic consequence of the formalism, i.e., it is not introduced by hand.

D. Master equation for a regular atomic beam

This structure becomes most clear, and assumes a particularly simple form, in another special case of interest, specifically that of a regular, antibunched atomic beam. For such a beam,

$$f(t)=\delta(t-T)\ , \tag{5.41}$$

so that

$$\tilde{f}(s)=\exp(-sT)\ . \tag{5.42}$$

Taking the inverse Laplace transform of Eq. (5.38) is then straightforward. It is convenient to write

$$t=nT+\epsilon T\ , \tag{5.43}$$

where n is an integer ≥ 0 and $0\leq\epsilon<1$. A lengthy calculation then yields

$$\begin{aligned}\rho(nT+\epsilon T)=&R\int_{\epsilon T}^{T}d\tau\exp\{-\hat{L}[(\epsilon+1)T-\tau]\}[(1+\hat{F})\exp(-\hat{L}T)]^{n-1}(1+\hat{F})\exp(-\hat{L}\tau)\rho(0)\\&+R\int_0^{\epsilon T}d\tau\exp[-\hat{L}(\epsilon T-\tau)][(1+\hat{F})\exp(-\hat{L}T)]^n(1+\hat{F})\exp(-\hat{L}\tau)\rho(0)\ .\end{aligned} \tag{5.44}$$

The evolution of ρ induced by kicks due to the arrival of an atom in the cavity [represented by $(1+\hat{F})$] separated by free decay of the cavity field during the time interval T between the kicks [represented by the $\exp(-\hat{L}T)$ factors] is clearly present in this result. The integrals represent an average over the random arrival of the first atom, evenly distributed over the interval $[0,T]$.

For $\epsilon=0$, we find that

$$\begin{aligned}\rho(nT)=&R\int_0^T d\tau\exp[-\hat{L}(T-\tau)][(1+\hat{F})\exp(-\hat{L}T)]^{n-1}\\&\times(1+\hat{F})\exp(-\hat{L}\tau)\rho(0)\ .\end{aligned} \tag{5.45}$$

If we ignore that fact that $\hat{L}$ and $\hat{F}$ do not commute, this result becomes

$$\rho(nT)=[(1+\hat{F})\exp(-\hat{L}T)]^n\rho(0) . \tag{5.46}$$

Following Ref. [19], we can introduce a continuous-time variable $t=nT$, a valid approximation if the decay time is not too long. This gives

$$\rho(t)=[(1+\hat{F})\exp(-\hat{L}T)]^{t/T}\rho(0) . \tag{5.47}$$

On differentiating this with respect to t and once again exploiting the assumption that $\hat{L}$ and $\hat{F}$ commute, we obtain the master equation derived in Ref. [19] under the same assumptions and approximations, viz., Eq. (2.7). It is worthwhile noting that as a consequence of the "smoothing" effect of the continuous-time approximation, there results a Markovian master equation for ρ in contrast to the general non-Markovian result of Eq. (5.34).

E. Steady-state limit

The final result of this section is the steady-state limit for $\rho(t)$. This is obtained by calculating the limit

$$\rho(\infty)=\lim_{s\to 0} s\tilde{\rho}(s) \tag{5.48}$$

in Eq. (5.32) and a similar limit for $\sigma(t)$ in Eq. (5.31). Equation (5.32) is straightforward, yielding

$$\hat{L}T\rho=\hat{F}\sigma , \tag{5.49}$$

where, from now on, ρ and σ are understood to mean $\rho(\infty)$ and $\sigma(\infty)$, respectively. In Eq. (5.31) it turns out to be better to introduce the Laplace transform of the waiting-time distribution $f(t)$ by use of Eq. (5.37). It then follows that

$$\sigma-\tilde{f}(\hat{L})(1+\hat{F})\sigma=\lim_{s\to 0} s(s+\hat{L})^{-1}[1-\tilde{f}(s+\hat{L})]\rho(0) . \tag{5.50}$$

The right-hand side can be shown to vanish. This follows from the fact that for any arbitrary density operator η,

$$\exp(-\hat{L}t)\eta=\rho_{\rm eq}+\text{exponentially decaying terms} , \tag{5.51}$$

which is equivalent in the limit of $t\to\infty$ to

$$\lim_{s\to 0} s(s+\hat{L})^{-1}\eta=\rho_{\rm eq} . \tag{5.52}$$

This result can be used to show that both of the terms on the right-hand side of Eq. (5.50) equal $\rho_{\rm eq}$ and hence cancel. Overall, then, we have the result

$$\sigma=\tilde{f}(\hat{L})(1+\hat{F})\sigma . \tag{5.53}$$

The quantity

$$\tilde{f}(\hat{L})=\int_0^\infty \exp(-\hat{L}\tau)f(\tau)d\tau \tag{5.54}$$

is obviously the cavity-damping operator averaged over the waiting-time distribution $f(t)$. Equation (5.54) shows that σ is a steady-state operator invariant under the combination of a kick $(1+\hat{F})$ followed by averaged damping $\tilde{f}(\hat{L})$. The appearance of the averaged damping in this steady-state result corresponds closely to expectations based on the analysis presented in Ref. [4]. However, here an additional step (5.49) relating σ to ρ is required. This extra step can be shown to be related to averaging over the time of arrival of the first atom.

A more useful result is obtained by eliminating σ from Eqs. (5.49) and (5.53) and reintroducing the Laplace transform of the quantity $G(t)$ defined in Eq. (5.36). We find that ρ then satisfies

$$\hat{L}T\rho=\hat{F}\rho+\hat{F}\hat{L}\tilde{G}(\hat{L})\rho . \tag{5.55}$$

The statistical properties of the beam enter this equation through the second term on the right-hand side which depends on the intensity correlation function of the incident atomic beam [Eq. (5.36)]. This term vanishes for an atomic beam with Poissonian statistics and hence represents the effects due to deviation of the beam away from being purely Poissonian. In general, this term cannot be dealt with exactly, but a useful series expansion can be made of $\hat{L}\tilde{G}(\hat{L})$ in powers of $\hat{L}$ which leads to an approximate treatment in the limit of a sufficiently intense atomic beam. This expansion takes the form

$$\hat{L}\tilde{G}(\hat{L})=1+C_1(\hat{L}T)+C_2(\hat{L}T)^2+\cdots , \tag{5.56}$$

where

$$C_n=\frac{(-1)^{n-1}}{T^n(n-1)!}\int_0^\infty t^{n-1}G(t)dt . \tag{5.57}$$

These coefficients can be expressed in terms of the moments of $f(t)$ defined by

$$\langle t^n\rangle=\int_0^\infty dt\, t^n f(t) . \tag{5.58}$$

In particular, we have

$$\begin{aligned} C_1&=R\int_0^\infty G(t)dt=\frac{\langle t^2\rangle}{2T^2}-1 , \\ C_2&=R^2\int_0^\infty tG(t)dt=\frac{\langle t^2\rangle^2}{4T^4}-\frac{\langle t^3\rangle}{6T^3} , \end{aligned} \tag{5.59}$$

where Eq. (5.19) has been used to express $G(t)$ in terms of $f(t)$. Thus, Eq. (5.55) becomes

$$\hat{L}T\rho=(1-C_1\hat{F})^{-1}\hat{F}(1+C_2(\hat{L}T)^2+\cdots)\rho . \tag{5.60}$$

As shown in Refs. [19] and [31], the actions of the superoperators $\hat{F}$ and $\hat{L}T$ each give rise to contributions of order $1/N_{\rm ex}$, where, in the notation of Ref. [4],

$$N_{\rm ex}=1/(\gamma_c T) \tag{5.61}$$

is the number of atoms passing through the cavity during the decay time of the cavity. Thus, to first order we get

$$\hat{L}T\rho=\hat{F}\rho+C_1\hat{F}^2\rho , \tag{5.62}$$

a result expected to be valid for $N_{\rm ex}\gg 1$. This last equation is the steady-state limit of a similar result found in Ref. [9] with the parameter C_1 identical to their quantity $Q_P(\infty)$. It is this latter quantity that provides a measure

of the degree of bunching of the atomic beam. For a fully antibunched beam, $\langle t^2\rangle = T^2$ so that $C_1 = -\frac{1}{2}$, for a random (Poissonian) beam $C_1 = C_2 = \ldots = 0$, and for a bunched beam we have $C_1 > 0$. These examples are consistent with the fact that $2C_1$ can be shown to be the variance in the number of atoms entering the cavity over a very long time interval, normalized relative to the mean number of atoms that entered the cavity during this time interval. In other words, $2C_1$ can be interpreted as the Q parameter characterizing the departure of the atomic beam from being purely Poissonian. Furthermore, it will be seen later that C_1 can also be identified with $-\frac{1}{2}p$, where p is the parameter introduced in Ref. [19] to characterize the statistics of the atomic beam.

To this order, the noncommutation of $\hat{F}$ and $\hat{L}$ plays no role. At the next highest order, we obtain

$$\hat{L}T\rho = \hat{F}\rho + C_1\hat{F}^2\rho + C_1^2\hat{F}^3\rho + C_2\hat{F}(\hat{L}T)\hat{F}\rho \ , \tag{5.63}$$

in which the last term will depend on the ordering of $\hat{L}$ and $\hat{F}$ and on the higher-order coefficient C_2. Thus, the neglect of operator ordering in deriving Eq. (2.7) will be a satisfactory approximation in the limit of $N_{\rm ex} \gg 1$, a point discussed in more detail in Ref. [21]. By writing out Eq. (5.55) in the number-state basis, we obtain, in general, an infinite-term recurrence relation for the diagonal elements ρ_{nn}. However, the approximate result Eq. (5.62) results in a more readily solved recurrence relation:

$$\begin{aligned} n(n_b+1)\rho_{nn} = {} & C_1 N_{\rm ex}\beta_n\beta_{n-1}\rho_{n-2,n-2} \\ & + [nn_b + N_{\rm ex}\beta_n(1-C_1\beta_n)]\rho_{n-1,n-1} \ , \end{aligned} \tag{5.64}$$

where, in the notation of [4],

$$\beta_n = \sin^2[n^{1/2}\xi/\nu_0] \ , \tag{5.65}$$

with ξ/ν_0 being identified with $\frac{1}{2}\kappa t_{\rm int}$. The result Eq. (5.64) is the same recurrence relation obtained in Ref. [19] with C_1 identified as $-\frac{1}{2}p$. It has been the basis of a number of studies of the steady-state photon distribution in the case of a Poissonian and a sub-Poissonian atomic beam for which the coefficient C_1 is small and negative. The limitations of the above approximate results [21] and the effects of higher-order corrections [32] have also been studied. On the other hand, for a super-Poissonian (bunched) atomic beam for which C_1 is positive, and possibly large for a strongly bunched atomic beam, the above approximate result is less satisfactory as the coefficient $(1-C_1\beta_n)$ can become negative if $C_1 > 1$. In that case, Eq. (5.64) predicts negative values for ρ_{nn}. In Sec. VI it is shown how the general formalism developed above can be applied in the case of a super-Poissonian beam in a way that avoids the above approximations. In fact, in the limit of a strongly bunched atomic beam, an exact limiting form for ρ can be found that predicts some interesting properties of the photon distribution which are quite distinct from those found in the antibunched or random beam cases.

VI. STEADY-STATE PHOTON DISTRIBUTION FOR A SUPER-POISSONIAN ATOMIC BEAM

A bunched atomic beam can be characterized by the intensity correlation function

$$g(t) = e^{-\Gamma t} + 1 \ , \tag{6.1}$$

which is analogous to the corresponding correlation function for a thermal radiation field. The bunched character of the beam as a function of the product ΓT is illustrated by the simulations presented in Fig. 1. The intensity of the beam is such that $N_{\rm ex} = 50$. The pattern of intervals between atoms is depicted over a time interval of $10^3 T$ corresponding to 20 cavity lifetimes (γ_c^{-1}), except in Fig. 1(d), which has been rescaled and is over a time interval of $5\times 10^4 T$ or 10^3 cavity lifetimes. To obtain these simulations, the waiting-time distribution $f(t)$ is derived from Eq. (6.1) by use of the relation Eq. (5.19). The cumulative distribution $F(t)$ defined by

$$\begin{aligned} F(t) &= \int_0^t f(t')dt' \\ &= 1 - A_+ e^{-(\alpha-\beta)t} - A_- e^{-(\alpha+\beta)t} \ , \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} A_\pm &= \frac{1}{2}\left[\frac{a-b/2}{([a-b/2]^2+ab)^{1/2}} \pm 1\right] \ , \\ \alpha \pm \beta &= \gamma_c[a \pm ([a-b/2]^2+ab)^{1/2} + b/2] \ , \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} a &= \tfrac{1}{2}\Gamma/\gamma_c = \tfrac{1}{2}N_{\rm ex}/C_1 \ , \\ b &= 2N_{\rm ex} \ , \end{aligned} \tag{6.4}$$

can then be employed to simulate the random spacing of the atoms in the beam. The procedure [33] is to generate repeatedly a random number ν from a uniform distribution over the interval (0,1) and then to invert $F(t)$ to give

FIG. 1. Simulations of the arrival times of atoms in an atomic beam with super-Poissonian statistics. Lines representing individual atoms in (a) become merged together in (b)–(d). $N_{\rm ex} = 50$ in all cases. The simulation is over a time interval of 20 cavity lifetimes (approximately 1000 atoms) except in (d) where the interval is 1000 cavity lifetimes. (a) $\Gamma T = 0.04$; (b) $\Gamma T = 0.02$; (c) $\Gamma T = 0.004$; (d) $\Gamma T = 0.000\,04$.

the random time intervals

$$t=F^{-1}(v) \tag{6.5}$$

between successive atoms.

For large ΓT, the limiting form for $F(t)$ is just that which would be found for a random beam with intensity of N_{ex} atoms per cavity lifetime γ_c^{-1} [see Fig. 1(a)]. For decreasing ΓT, the tendency for the beam to become more bunched becomes apparent. In this case, for small ΓT, $F(t)$ can be written

$$F(t)=1-e^{-2t/T-\frac{1}{4}\Gamma T e^{-\Gamma t/2}}, \tag{6.6}$$

which, apart from the third term, is what would be found for a random atomic beam of intensity $2N_{ex}$. However, the correction term dominates $F(t)$ for $t \gg T$ and its effect is to enhance the probability for there to be a substantial gap between two successive atoms. This effect is seen, in particular, in Fig. 1(d), which has been rescaled by a factor of 50. Here there are long gaps of up to $1500T$ (in the example given), i.e., 30 cavity lifetimes between successive "bunches," which have a similar duration. Within each bunch it is found numerically that the atoms are arriving with an average intensity of $2N_{ex}$ per cavity lifetime, consistent with expectations based on Eq. (6.6). The picture that then emerges in the limit of extreme bunching is that of a beam consisting of long intervals in which atoms arrive randomly at twice the average rate separated by, on average, equal intervals in which no atoms arrive. This kind of structure has a clear effect on the state of the field inside the cavity, as discussed below.

Other choices of $g(t)$ are possible, e.g., $g(t)=\exp(-\Gamma t^2)+1$, but the principal advantage of the above choice is that it yields a simple form for $\hat{G}(\hat{L})$, viz.,

$$\hat{G}(\hat{L})=(\hat{L}+\Gamma)^{-1}. \tag{6.7}$$

Substituting Eq. (6.7) into Eq. (5.52) then yields the operator equation

$$\hat{L}T\rho=\hat{F}[1+\hat{L}(\hat{L}+\Gamma)^{-1}]\rho. \tag{6.8}$$

The inverse appearing on the left-hand side is most readily dealt with by rewriting Eq. (6.8) as a pair of coupled equations in ρ and σ:

$$\begin{aligned}&\hat{L}T\rho=\hat{F}\sigma,\\ &(\hat{L}+\Gamma)\sigma=(2\hat{L}+\Gamma)\rho.\end{aligned} \tag{6.9}$$

The fact that these equations can be written in this fashion leads to a four-term recurrence relation for ρ_{nn} rather than an infinite-term relation that would follow directly from Eq. (5.55). It is actually more convenient computationally to work with the recurrence relation for a related quantity z_n defined by

$$\rho_{nn}=\left[\frac{n_b}{1+n_b}\right]^n \prod_{k=1}^{n}\left[1+\frac{2N_{ex}\beta_k}{kn_b}\right] z_n, \tag{6.10}$$

where n_b is the mean number of thermal photons in the cavity-field reservoir. The recurrence relation is of the form

$$A_n z_{n-1}+B_n z_n+C_n z_{n+1}+D_n z_{n+2}=0, \tag{6.11}$$

where the coefficients are given by

$$\begin{aligned}A_n&=4N_{ex}^2\beta_{n+1}\beta_{n+2}n_b n^2(n_b+1),\\ B_n&=8aN_{ex}^3\beta_n\beta_{n+1}\beta_{n+2}-A_n-C_n-D_n,\\ C_n&=4N_{ex}^2\beta_n\beta_{n+2}[n_b(n+1)+n(n_b+1)+2a]\\ &\quad\times[(n+1)n_b+2N_{ex}\beta_{n+1}]-D_n,\\ D_n&=-4N_{ex}^2\beta_n\beta_{n+1}[(n+2)n_b+2N_{ex}\beta_{n+2}]\\ &\quad\times[(n+1)n_b+2N_{ex}\beta_{n+1}].\end{aligned} \tag{6.12}$$

This recurrence relation can be solved numerically by use of matrix continued fraction methods [34]. The behavior of the mean photon number and standard deviation of the cavity field as a function of the pumping parameter θ [4] defined by

$$\theta=\sqrt{N_{ex}}\xi/\nu_0 \tag{6.13}$$

yields a result, at least for the mean photon number, that is generally similar to that obtained for antibunched or random atomic beams. This is illustrated in Fig. 2 where

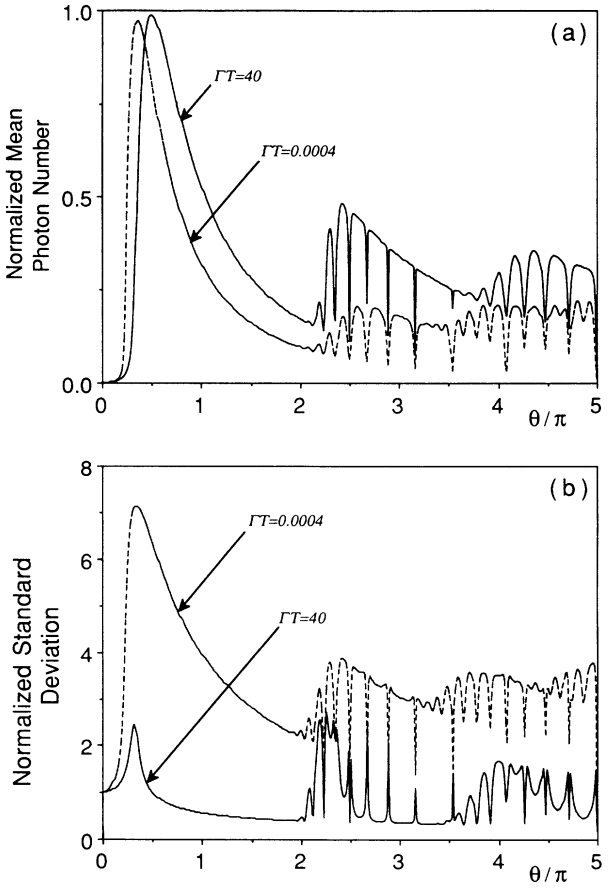

FIG. 2. (a) Normalized mean photon number as a function of the pump parameter θ for $\Gamma T=40$ (i.e., Poissonian beam) and $\Gamma T=0.0004$ (strongly bunched beam); (b) normalized standard deviation for the same parameter values.

the normalized mean photon number $\langle n \rangle / N_{ex}$ and the normalized standard deviation

$$\sigma = \frac{(\langle n^2 \rangle - \langle n \rangle^2)^{1/2}}{\langle n \rangle^{1/2}} \tag{6.14}$$

are plotted as functions of θ with $N_{ex}=50$ and $n_b=0$. The two cases of an extreme bunched beam with $\Gamma T = 4 \times 10^{-4}$ and, for comparison, a random beam with $\Gamma T = 40$ are plotted together. In Fig. 2(a) the two curves for the mean photon number show that the threshold is

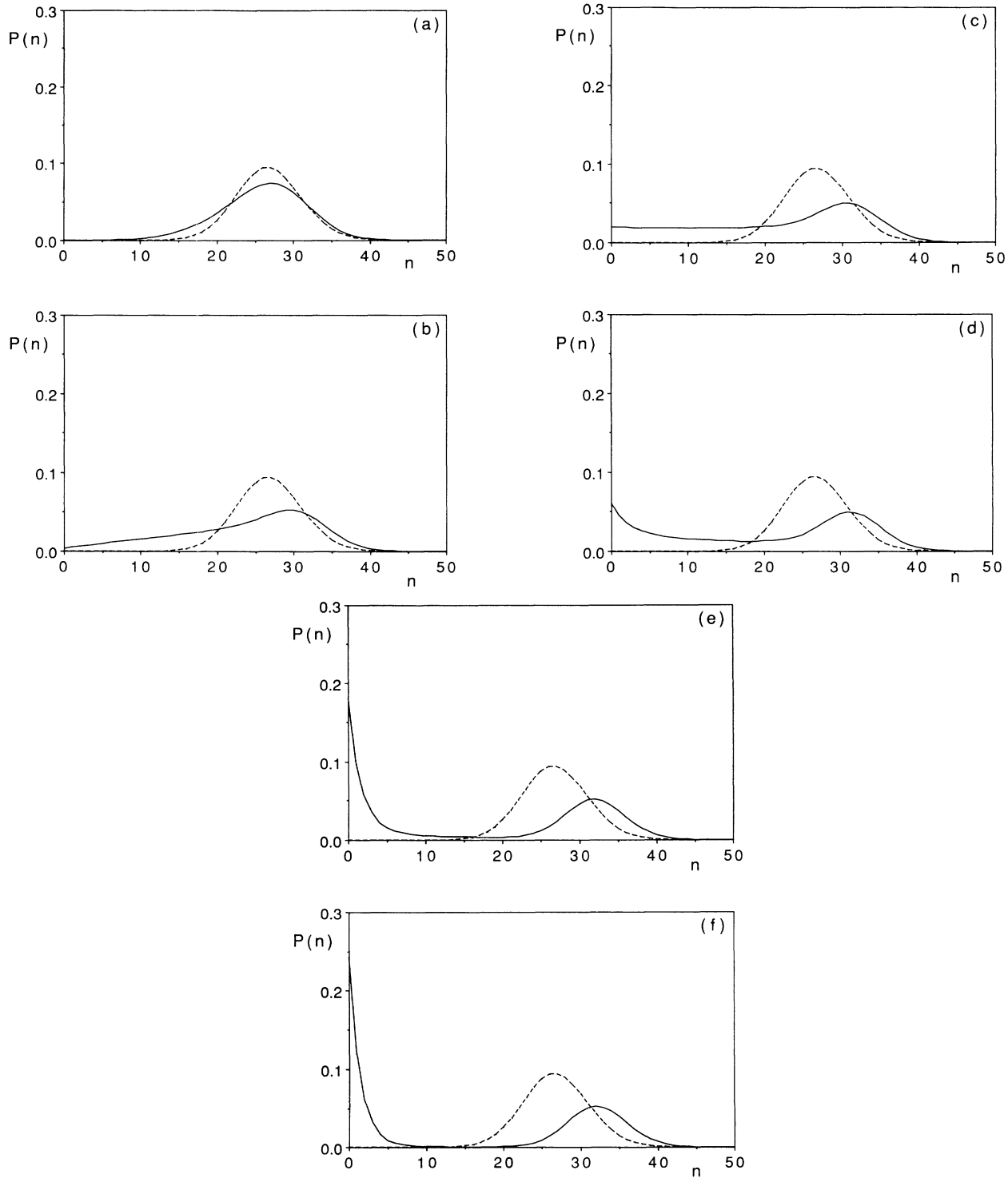

FIG. 3. Photon distribution for pump parameter $\theta = 1.5\pi$, thermal photon number $n_b = 1.0$, and $N_{ex} = 50$ as a function of bunching parameter: (a) $\Gamma T = 0.04$, (b) 0.008, (c) 0.004, (d) 0.002, (e) 0.0004, and (f) 0.000 04. The dotted curve is the photon distribution for a Poissonian atomic beam with the same values of θ, n_b, and N_{ex}. The steady increase of the thermal contribution is seen, becoming clearly established in the extreme bunched limit $\Gamma T = 0.000\,04$.

displaced to a lower θ value with, overall, the average number of photons in the cavity about half the corresponding result for a random atomic beam for θ values greater than 2π. There is also a substantial effect on the standard deviation, Fig. 2(b), which is seen to be always greater than unity for a strongly bunched beam, i.e., the cavity field has super-Poissonian statistics. While this property of the cavity field is generally expected, it does not have its origins in a simple broadening of the peaks of the photon distribution. The distribution assumes a much more unusual form in which the thermal field associated with the cavity reservoir is found to play a crucial role in giving rise to a super-Poissonian distribution.

In Figs. 3 and 4 the photon distributions are plotted for

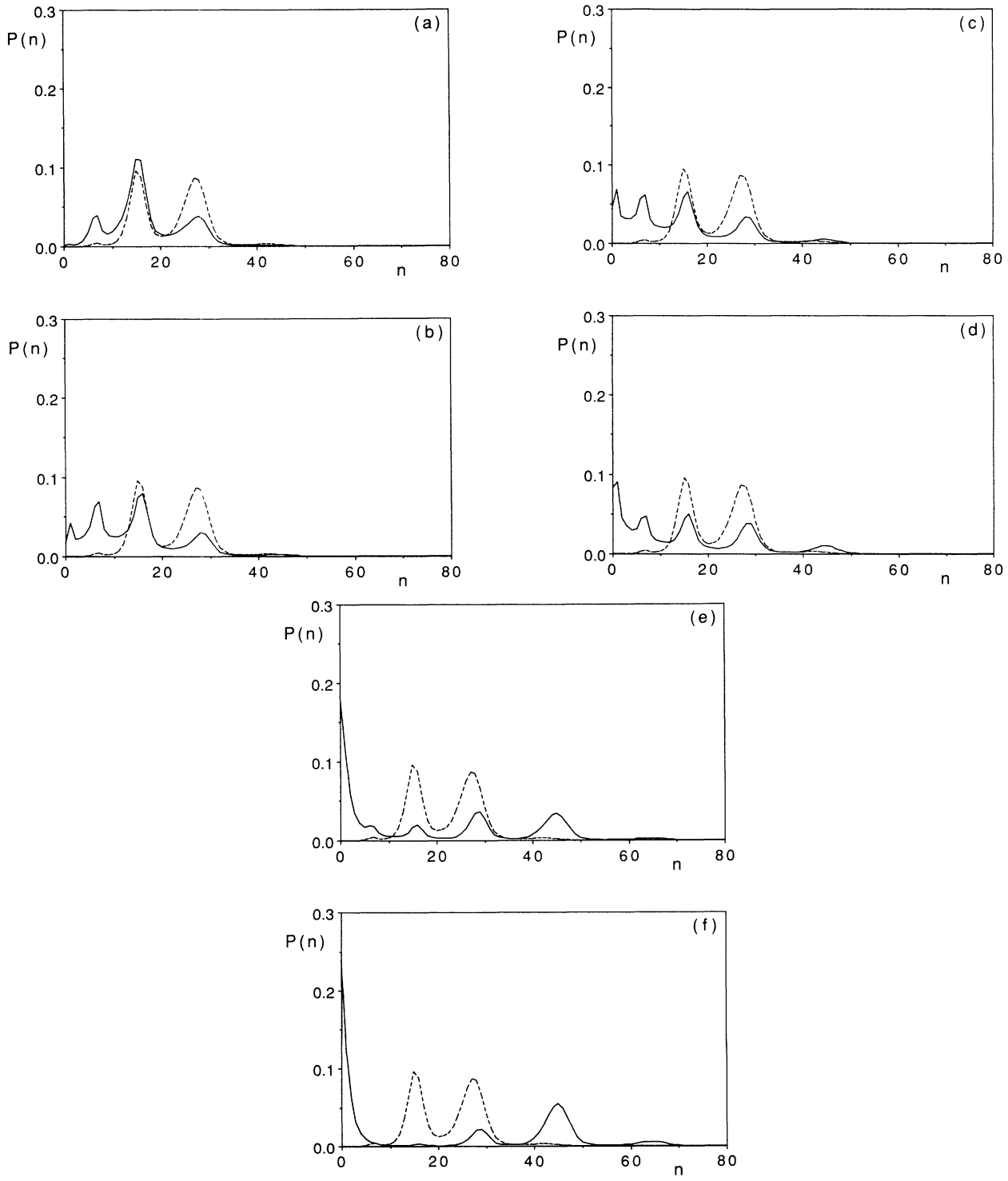

FIG. 4. As for Fig. 3, except $\theta=5\pi$.

various values of ΓT for $\theta=1.5\pi$ and 5π, respectively, the latter to illustrate the effects when the photon distribution is multipeaked. What can be noted from these figures is that for small values of ΓT, there clearly develops a thermal contribution to the photon distribution, while the remaining peak(s) can be shown numerically to be identical to the photon distribution (scaled down by a factor of 2) that would be produced by an atomic beam of twice the current intensity. The exact nature of these contributions readily follows from Eq. (6.8). Writing Eq. (6.8) in the form

$$\hat{L}T\rho=\hat{F}[2-\Gamma(\hat{L}+\Gamma)^{-1}]\rho \tag{6.15}$$

and making use of the result, Eq. (5.52),

$$\lim_{\Gamma\to 0}\Gamma(\hat{L}+\Gamma)^{-1}\rho=\rho_{\rm eq}\ , \tag{6.16}$$

we obtain

$$\hat{L}T\rho=\hat{F}(2\rho-\rho_{\rm eq})\ . \tag{6.17}$$

The solution to this is

$$\rho=\tfrac{1}{2}(\rho_{\rm eq}+\rho^{(2)})\ , \tag{6.18}$$

where

$$\hat{L}T\rho^{(2)}=2\hat{F}\rho^{(2)}\ , \tag{6.19}$$

i.e., $\rho^{(2)}$ is the density operator appropriate for an atomic beam of *twice* the intensity of the atomic beam used here. This result can be readily confirmed numerically, yielding results which are indistinguishable from those plotted in Figs. 3 and 4 for small ΓT.

Physically, this result can be understood from the statistical properties of a strongly bunched atomic beam, as indicated by the simulations presented in Fig. 1 and discussed above. During the periods of bunching, which last many cavity decay times, the atoms are arriving at twice the overall average rate for the beam, thus giving rise to the $\rho^{(2)}$ contribution to ρ. These periods are separated by intervals that are also much longer than a decay time of the cavity during which no atoms pass through the cavity. The cavity field therefore decays to its equilibrium thermal state. The result is an equal weight mixture of the $\rho_{\rm eq}$ and $\rho^{(2)}$ states for the field.

VII. SUMMARY AND DISCUSSION

The work presented here represents a method by which the micromaser problem can be treated in which the atomic beam is treated as a quantum field. The central result is Eq. (4.10) which applies for a very general class of statistics for the incident atomic beam, specified by the quantum state of this beam. By choice of a particular class, generalized shot noise, most of the important kinds of statistical properties for the incident beam can be modeled, in which case the results of others [19,20] are shown to follow for a beam with Poissonian or sub-Poissonian statistics. The method is also applicable to super-Poissonian atomic beams, for which the cavity-field photon state is found to assume a mixture of a thermal state due to the cavity reservoir, and a state produced by a Poissonian atomic beam of effective intensity double that of the beam itself.

Other important results that follow from the analysis include the propagation equations for the atomic fields as they pass through a medium of photons [Eqs. (3.17a) and (3.17b)]. Only the point-cavity limit has been considered here; the effect of a finite region over which the atoms and field can interact is yet to be investigated. Mathematically, the method used here represents the generalization of the adjoint-operator method to a system in contact with essentially two reservoirs, the thermal reservoir and the reservoir provided by the atomic beam [Sec. IV], and its application to a problem in which noise sources arise through quantum fields other than the usual radiation field. A characterization of the beam in terms of its correlation properties through its modeling as generalized shot noise enabled the formulation of a general master equation for the cavity-field density operator, for which a formal steady-state solution could be found [Sec. V].

Further work needs to be done in analyzing the properties of the cavity field as the photon-number distribution only gives one picture of the field statistics. Of particular interest are the temporal properties of the field—in particular, the intensity fluctuations in the case of a super-Poissonian atomic beam. The marked fluctuations in the atomic beam in this case would lead to substantial fluctuations in the intensity of the micromaser field. Given that noisy pumping has been shown to lead to an enhancement of the noise fields generated by dye lasers [11–18], it would therefore be of some value to evaluate the intensity correlation of the micromaser field and compare its properties to those found in the case of dye lasers.

However, in a micromaser, the properties of the field inside the cavity are not determined directly, but rather by measuring the properties of the atomic beam as it emerges from the cavity [1–3]. Thus further work will be directed towards calculating the properties of the atomic beam as it emerges from the cavity. By use of the quantum-field formalism developed here, it has already been shown that this beam is spatially coherent and carries information on the spectrum of the cavity field [35]. This may be important in the understanding of the properties of the coupled micromaser problem [36]. Finally, the Langevin methods implicit here can be exploited to show the correspondence between the Langevin results of Ref. [20] and master-equation methods [37].

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APPENDIX A

The coupled equations for the atomic-beam fields are, for $0<x<L$,

$$\frac{\partial\Phi_1}{\partial t}+v_0\frac{\partial\Phi_1}{\partial x}=-iKA^\dagger(t)\Phi_2\ ,$$
$$\frac{\partial\Phi_2}{\partial t}+v_0\frac{\partial\Phi_2}{\partial x}=-iKA(t)\Phi_1\ , \tag{A1}$$

which, on making the substitution

$$\Phi_n=(2\pi)^{-1}\int_{-\infty}^{+\infty}dk\,\exp[ik(x-v_0t)]f_n(k,t)\ , \tag{A2}$$

reduce to

$$\frac{\partial f_1}{\partial t}=-iKA^\dagger(t)f_2\ ,$$
$$\frac{\partial f_2}{\partial t}=-iKA(t)f_1\ . \tag{A3}$$

Consider now the evolution of f_1 and f_2 over a time interval (s,t) such that $t-s\le\tau_c$, which is sufficiently short that $A(t)$ evolves freely:

$$A(t)=A(s)\exp[-i\Delta(t-s)]\ . \tag{A4}$$

Then from Eq. (A3) we can separate out the equations for f_1 and f_2:

$$\frac{\partial^2 f_1}{\partial t^2}-i\Delta\frac{\partial f_1}{\partial t}+K^2A^\dagger(s)A(s)f_1=0\ ,$$
$$\frac{\partial^2 f_2}{\partial t^2}+i\Delta\frac{\partial f_2}{\partial t}+K^2A(s)A^\dagger(s)f_2=0\ . \tag{A5}$$

These equations can be solved for $f_1(k,t)$ and $f_2(k,t)$ in terms of $f_1(k,s)$ and $f_2(k,s)$ and the results substituted into Eq. (A2) to give

$$\Phi_1(x,t)=\{\lambda_2\exp[\lambda_1(t-s)]-\lambda_1\exp[\lambda_2(t-s)]\}(\lambda_2-\lambda_1)^{-1}\Phi_1(x-v_0(t-s),s)$$
$$+iK\{\exp[\lambda_1(t-s)]-\exp[\lambda_2(t-s)]\}(\lambda_2-\lambda_1)^{-1}A^\dagger(s)\Phi_2(x-v_0(t-s),s) \tag{A6}$$

and

$$\Phi_2(x,t)=\{\mu_2\exp[\mu_1(t-s)]-\mu_1\exp[\mu_2(t-s)]\}(\mu_2-\mu_1)^{-1}\Phi_2(x-v_0(t-s),s)$$
$$+iK\{\exp[\mu_1(t-s)]-\exp[\mu_2(t-s)]\}(\mu_2-\mu_1)^{-1}A(s)\Phi_1(x-v_0(t-s),s) \tag{A7}$$

where, with $n=1,2$,

$$\lambda_n=\tfrac{1}{2}i\Delta-\tfrac{1}{2}i(-1)^n[\Delta^2+4K^2N(s)]^{1/2}\ ,$$
$$\mu_n=-\tfrac{1}{2}i\Delta-\tfrac{1}{2}i(-1)^n\{\Delta^2+4K^2[N(s)+1]\}^{1/2}\ , \tag{A8}$$

and

$$N(s)=A^\dagger(s)A(s)\ . \tag{A9}$$

These results are now to be substituted into the integral expressions for Φ_1 and Φ_2 given by Eqs. (3.16a) and (3.16b). Thus, for example, in the equation for Φ_2,

$$\Phi_2(x,t)=\Phi_2^{(0)}(x,t)-iK\int_{t-x/v_0}^{t}d\tau\,A(\tau)\Phi_1(x+v_0(\tau-t),\tau)\ , \tag{A10}$$

the range of integration satisfies the condition for the evolution of $A(\tau)$ to be approximated by its freely evolving form,

$$A(\tau)=A(t)\exp[-i\Delta(\tau-t)]\ , \tag{A11}$$

and for Eq. (A6) to be used to write $\Phi_1(x+v_0(\tau-t),\tau)$ in terms of operators at time t. The integral can be readily performed to eventually yield

$$\Phi_2^{(0)}(x,t)=iKA(t)(\lambda_2-\lambda_1)^{-1}[\exp(\lambda_2x/v_0)-\exp(\lambda_1x/v_0)]\Phi_1(x,t)$$
$$-K^2A(t)(\lambda_2-\lambda_1)^{-1}[\lambda_1\exp(\lambda_2x/v_0)-\lambda_2\exp(\lambda_1x/v_0)](\lambda_1\lambda_2)^{-1}A^\dagger(t)\Phi_2(x,t)\ . \tag{A12}$$

A similar procedure can be followed for Φ_1, leading to

$$\Phi_1^{(0)}(x,t)=iKA^\dagger(t)(\mu_2-\mu_1)^{-1}[\exp(\mu_2x/v_0)-\exp(\mu_1x/v_0)]\Phi_2(x,t)$$
$$-K^2A^\dagger(t)(\mu_2-\mu_1)^{-1}[\mu_1\exp(\mu_2x/v_0)-\mu_2\exp(\mu_1x/v_0)](\mu_1\mu_2)^{-1}A(t)\Phi_2(x,t)\ . \tag{A13}$$

In both Eqs. (A12) and (A13), the quantities λ_n and μ_n are to be evaluated at the time t. We now specialize to on-resonance, $\Delta=0$, and solve Eqs. (A12) and (A13) for Φ_1 and Φ_2, the result being Eqs. (3.19) and (3.20).

APPENDIX B

The third term in Eq. (4.1) leads to the following expression after multiplying by $\rho(0)\rho_R(0)\rho_B(0)$ and taking the partial trace over cavity-field and reservoir states:

$$i\sqrt{\gamma_c/2\pi}\int dq\ \mathrm{Tr}_{CR}\{[c_q^\dagger(t)a(t)+c_q(t)a^\dagger(t),Y(t)]\rho(0)\rho_R(0)\rho_B(0)\}\ . \tag{B1}$$

Evaluation of this quantity amounts to evaluating expressions of the general form

$$T_n = i\sqrt{\gamma_c/2\pi}\int dq\ \mathrm{Tr}_{CR}[S(t)c_q(t)\rho(0)\rho_R(0)\rho_B(0)] \times \exp(-n\beta\hbar\omega_q)\ , \tag{B2}$$

where S is an arbitrary cavity-field operator, $\beta=1/kT$, and, in Eq. (B1), we have $n=0$. The reservoir operator $c_q(t)$ can be eliminated in favor of cavity operators by, first of all, integrating the Heisenberg equation of motion for $c_q(t)$, which yields

$$c_q(t)=c_q(0)\exp(-i\omega_q t) - i\sqrt{\gamma_c/2\pi}\int_0^t d\tau \exp[-i\omega_q(t-\tau)]a(\tau)\ . \tag{B3}$$

Substituting this into Eq. (B2) yields

$$T_n = i\sqrt{\gamma_c/2\pi}\int dq\ \mathrm{Tr}_{CR}[S(t)c_q(0)\rho(0)\rho_R(0)\rho_B(0)] \times \exp(-i\omega_q t)\exp(-n\beta\hbar\omega_q) + \tfrac{1}{2}\gamma_c \mathrm{Tr}_{CR}[S(t)a(t)\rho(0)\rho_R(0)\rho_B(0)]\exp(-n\beta\hbar\omega_0)\ , \tag{B4}$$

where the Markov approximation has been used to give the second term. Furthermore, since

$$c_q(0)\rho(0)\rho_R(0)\rho_B(0) = c_q(0)\frac{\exp(-\beta H_R)}{\mathrm{Tr}[\exp(-\beta H_R)]}\rho(0)\rho_B(0) = \rho(0)\rho_R(0)\rho_B(0)c_q(0)\exp(-\beta\hbar\omega_q)\ , \tag{B5}$$

then by cyclic permutation of $c_q(0)$ (an operator that acts only on the Hilbert space of the reservoir states), under the partial trace over reservoir states we get

$$\mathrm{Tr}_{CR}[S(t)c_q(0)\rho(0)\rho_R(0)\rho_B(0)] = \mathrm{Tr}_{CR}[c_q(0)S(t)\rho(0)\rho_R(0)\rho_B(0)]\exp(-\beta\hbar\omega_q)\ .$$

Substituting for $c_q(0)$ from Eq. (B3) and inserting the result into Eq. (B4) then gives

$$T_n = T_{n+1} - \tfrac{1}{2}\gamma_c \mathrm{Tr}_{CR}[a(t)S(t)\rho(0)\rho_R(0)\rho_B(0)] \times \exp[-(n+1)\beta\hbar\omega_0] + \tfrac{1}{2}\gamma_c \mathrm{Tr}_{CR}[S(t)a(t)\rho(0)\rho_R(0)\rho_B(0)] \times \exp(-n\beta\hbar\omega_0)\ , \tag{B6}$$

where the Markov approximation has been used once again to yield the second term. We are after the term T_0 which can be obtained from Eq. (B6) by iteration. This yields two infinite geometric series which can be summed to give

$$T_0 = \lim_{n\to\infty} T_n - \tfrac{1}{2}\gamma_c n_b \mathrm{Tr}_{CR}[a(t)S(t)\rho(0)\rho_R(0)\rho_B(0)] + \tfrac{1}{2}\gamma_c(n_b+1)\mathrm{Tr}_{CR}[S(t)a(t)\rho(0)\rho_R(0)\rho_B(0)]\ , \tag{B7}$$

where

$$n_b = [\exp(\beta\hbar\omega_0)-1]^{-1} \tag{B8}$$

is the mean number of thermal photons in the cavity.

The limit in Eq. (B7) will vanish, leaving an expression for T_0 in which traces appear over cavity-field operators only. The defining condition for the adjoint operator $\mu_S(t)$ [Eq. (4.3)] then yields

$$T_0 = -\tfrac{1}{2}\gamma_c n_b \mathrm{Tr}_C[a(0)S(0)\mu_S(t)]\rho_B(0) + \tfrac{1}{2}\gamma_c(n_b+1)\mathrm{Tr}_C[S(0)a(0)\mu_S(t)]\rho_B(0)\ . \tag{B9}$$

The general result can then be applied to the various terms that appear in Eq. (B1), leading finally to the usual cavity-damping terms to be found in Eq. (4.10).

APPENDIX C

States of the beam containing excited atoms localized in space can be generated through the action of the operator

$$f^\dagger_{\eta 2}(x) = (2\eta)^{-1/2}\int_{-\eta}^{+\eta} dk\ \exp(-ikx)b^\dagger_{k2} \tag{C1}$$

on the vacuum state $|(\mathrm{vac})_2\rangle$ for atoms in their excited state $|2\rangle$. The limit $\eta\to+\infty$ is understood to be taken at the end of any calculation. The single atom state is then

$$|\psi_\eta\rangle = (2\eta)^{-1/2}\int_{-\eta}^{+\eta} dk\ \exp(-ikx)b^\dagger_{k2}|(\mathrm{vac})_2\rangle\ , \tag{C2}$$

and the probability density for finding the atom at x' is

$$\langle\psi_\eta(x)|\Psi_2^{(0)\dagger}(x')\Psi_2^{(0)}(x')|\psi_\eta(x)\rangle = \frac{\sin^2[\eta(x-x')]}{\pi\eta(x-x')}\ , \tag{C3}$$

which in the limit of $\eta\to+\infty$ becomes $\delta(x-x')$, i.e., this is the probability distribution for an atom positioned with certainty at $x=x'$. Many particle states are then given by

$$|\psi_\eta(x_1,x_2,\ldots)\rangle = \prod_j f^\dagger_{\eta 2}(x_j)|(\mathrm{vac})_2\rangle\ . \tag{C4}$$

The operator $f^\dagger_{\eta 2}(x)$ satisfies the commutation rules

$$[f^\dagger_{\eta 2}(x), f_{\eta 2}(x')] = \frac{\sin[\eta(x-x')]}{\eta(x-x')}\ , \qquad [f^\dagger_{\eta 2}(x), \Psi_2^{(0)}(x')] = \frac{\sin[\eta(x-x')]}{(\pi\eta)^{1/2}[\eta(x-x')]}\ . \tag{C5}$$

From the first of these it can be shown that the states defined in Eq. (C4), in the limit of $\eta\to+\infty$, are normalized to unity, and orthogonal if the states differ through different positions of the atoms.

From Eq. (5.10), the mean beam intensity is given, for the state given in Eq. (C4), by

$$\langle J(x,0)\rangle = v_0\langle \psi_\eta(x_1,x_2,\ldots)|\Psi_2^{(0)\dagger}(x')\Psi_2^{(0)}(x)|\psi_\eta(x_1,x_2,\ldots)\rangle$$

$$= \sum_{m,n}\left\langle (\text{vac})_2 \left| \prod_{l\,(\neq m)} f_{\eta 2}(x_l) \prod_{j\,(\neq n)} f_{\eta 2}^\dagger(x_j) \right| (\text{vac})_2 \right\rangle (\pi\eta)^{-1} \frac{\sin[\eta(x-x_m)]}{(x-x_m)} \frac{\sin[\eta(x-x_n)]}{(x-x_n)} \quad \text{(C6)}$$

If $m\neq n$, the inner product vanishes and the sum reduces to

$$\sum_m \frac{\sin^2[\eta(x-x_m)]}{\pi\eta(x-x_m)^2} ,$$

so that in the limit of $\eta\to+\infty$, we find

$$\langle J(x,0)\rangle = \sum_m \delta(x-x_m) . \quad \text{(C7)}$$

Intensity correlations follow from

$$\langle :J(x,0)J(x',0):\rangle = v_0^2\langle \Psi_\eta(x_1,x_2,\ldots)|\Psi_2^{(0)\dagger}(x)\Psi_2^{(0)\dagger}(x')\Psi_2^{(0)}(x')\Psi_2^{(0)}(x)|\psi_\eta(x_1,x_2,\ldots)\rangle$$

$$= \sum_{\substack{l,m\\ l\neq m}} \sum_{\substack{r,s\\ r\neq s}} \left\langle (\text{vac})_2 \left| \prod_{p\neq r,s} f_{\eta 2}(x_p) \prod_{j\neq l,m} f_{\eta 2}^\dagger(x_j) \right| (\text{vac})_2 \right\rangle$$

$$\times (\pi\eta)^{-2} \frac{\sin[\eta(x-x_r)]}{(x-x_r)} \frac{\sin[\eta(x-x_s)]}{(x-x_s)} \frac{\sin[\eta(x-x_l)]}{(x-x_l)} \frac{\sin[\eta(x-x_m)]}{(x-x_m)} . \quad \text{(C8)}$$

The inner product vanishes unless $r=l$, $s=m$ or $r=m$, $s=l$. The second possibility always gives zero as there is only one atom present at any point in the beam and hence Eq. (C8) becomes

$$\sum_{\substack{l,m\\ l\neq m}} (\pi\eta)^{-2} \frac{\sin^2[\eta(x-x_l)]}{\pi\eta(x-x_l)^2} \frac{\sin^2[\eta(x-x_m)]}{\pi\eta(x-x_m)^2} ,$$

so that in the limit of $\eta\to+\infty$ we find

$$\langle :J(x,0)J(x',0):\rangle = \sum_{\substack{l,m\\ l\neq m}} \delta(x-x_l)\delta(x'-x_m) . \quad \text{(C9)}$$

A proof based on induction leads to the result Eq. (5.14).

APPENDIX D

Rice [30] considers a class of stochastic process, referred to as "generalized shot noise," that is formed by the superposition of a sequence of randomly spaced pulses of identical shape. The shot-noise process is then given by

$$X(t) = \sum_i S(t-t_i)$$

$$= \int S(t-u)\,dN(u) , \quad \text{(D1)}$$

where $N(u)$ is a stochastic process such that

$$\frac{dN(u)}{du} = \sum_i \delta(t-t_i) \quad \text{(D2)}$$

and where the t_i are random variables (the realization of the stationary stochastic point process N) and $S(t)$ is the "shape" of an individual pulse.

In the particular case in which the statistics of the t_i's are described by a renewal process [29] characterized by a waiting-time distribution $f(t)$, Rice describes the following properties possessed by N:

$$E[dN(s)dN(t)] = R^2 g(t-s)\,ds\,dt , \quad s<t \quad \text{(D3)}$$

$$E[dN(s)dN(t)dN(u)] = R^3 g(t-s)g(u-t)\,ds\,dt\,du , \quad s<t<u \quad \text{(D4)}$$

where $E[\ ]$ represents an average over the renewal process, $g(t)$ is the renewal density defined in Eq. (5.18), and R is the mean rate of the renewal process. From Eq. (D2) these results are just the averages of generalized shot noise for which each pulse is a δ function, i.e., $S(t)=\delta(t)$, so that we can write, for example, from Eq. (D3),

$$E[X(s)X(t)] = R^2 g(t-s) , \quad s<t . \quad \text{(D5)}$$

The strict inequality on the time arguments is important. If $s=t$, the correlation function on the left-hand side of Eq. (D5) is singular. In fact, including the case $t=s$ by direct calculation of $E[X(t)^2]$ from the infinite series yields the result

$$E[X(s)X(t)] = R^2 g(|t-s|) + R\delta(t-s) . \quad \text{(D6)}$$

Such singular contributions do not arise in the intensity correlation functions required here as they differ from those dealt with by Rice in a small but important way. In the intensity autocorrelation function, Eq. (5.18),

$$E\left[\sum_{\substack{l,m\\ l\neq m}} \delta(t_2-t_l)\delta(t_1-t_m)\right] ,$$

the condition $l\neq m$, whose origin can be traced back to the commutation properties of the quantum-field operators in terms of which the correlation function was originally defined, implies that the singular condition will not

arise. Thus, for the intensity correlation functions required here, the result Eq. [D5] continues to apply even for equal-time arguments. This conclusion can be confirmed from first principles in a long calculation in which the multitime correlation functions are evaluated directly from the infinite series expressions, taking into account the restrictions imposed on the summation indices. In particular, the result Eq. (5.20), a generalization of Eq. (D4) to n time arguments, follows by a proof based on induction.

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