Minimum-uncertainty states, a two-photon system, and quantum-group symmetry

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We provide a model for a two-photon system that possesses quantum-group symmetry. The quadrature-phase amplitudes of the model are defined in terms of deformed oscillator operators. We emphasize the investigation of the most general minimum-uncertainty states, i.e., coherent states and squeezed states, for the quadrature-phase amplitudes α_1 and α_2. We show that the squeezed states of the two-photon quantum optics are simultaneous eigenvectors of the operators A_+ and A_- with eigenvalues a_+ and a_-, respectively. For an important subset of the squeezed states, the expectation values and variances of relevant operators are given explicitly.

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I. INTRODUCTION

Squeezing is the noise reduction that can occur in a quantum field theory when the quantum fluctuation in one of the field quadrature phases is reduced below the usual vacuum level. Since the reinvention of squeezed states in 1970 by Stoler [1] for the realm of quantized fields, it was not certain whether they can be realized for the actual electromagnetic field, or if this is just an interesting theoretical idea. But with successful observations of squeezed states of the electromagnetic field at the AT&T Bell Laboratories and other laboratories [2–5], a wide range of applications have been suggested, from gravitational-wave detection to low-noise optical communication. And intensive investigations upon the squeezed light have been made. Recent interesting progress in the topic is the implications of quantum-group symmetry. Quantum group [6] with respect to its Lie counterpart introduces independent parameter q, so it can play an important role in many quantum systems. Chaichian, Ellinas, and Kulish [7] introduced quantum group into the field of quantum optics. They discussed the generalized Jaynes-Cummings model which possesses quantum-group dynamical symmetry. Celeghini, Rasetti, and Vitiello [8] showed that the quantum-group coherent states are related with squeezing for complex q. The modified version of the Biedenharn-Macfarlane states is given by them based on the discussion of the quantum superalgebra $\mathrm{osp}_q(1|2)$. They defined the quantum analog of position (Q_q) and momentum (P_q) operators as

$$P_q \equiv i\left[\frac{m\hbar\omega}{2}\right]^{1/2}(b_q - b_q^\dagger)\ ,\qquad Q_q \equiv \left[\frac{\hbar}{2m\omega}\right]^{1/2}(b_q + b_q^\dagger)\ . \tag{1}$$

Q_q and P_q are Hermitian and have commutation relation

$$[Q_q, P_q] = i\hbar\left[\frac{[N+1]_q[N+1]_{q^*}}{N+1} - \frac{[N]_q[N]_{q^*}}{N+1}\right]\ . \tag{2}$$

With respect to the quantum-group coherent states $\{|a;q\rangle a,q\in C\}$, Celeghini, Rasetti, and Vitiello showed that the squeezing in P_q $[\langle(\Delta P_q)^2\rangle < (\frac{1}{2}|\langle \hbar_q\rangle|)]$ or Q_q $[\langle(\Delta Q_q)^2\rangle < (\frac{1}{2}|\langle \hbar_q\rangle|)]$ appeared.

Because examples of two-photon devices such as four-wave mixers and parametric amplifiers can produce the squeezed mode of the electromagnetic field, a lot of attention has been paid to two-photon quantum optics [9]. The key property of the one-photon device is that its output consists of independent excited modes with time-stationary (TS) noise. The natural quantum-mechanical operator for the mode of one-photon optics is its annihilation operator and natural quantum states for the mode are coherent states. The coherent states are eigenstates of the annihilation operator, thus they have the sharpest complex amplitude permitted by quantum mechanics. The formalism of one-photon optics is founded firmly on the annihilation operator as the fundamental operator and on the coherent states as the fundamental quantum states. The natural quantum-mechanical operators for the modes of the two-photon optics are the quadrature-phase amplitudes $\alpha_1(\varepsilon)$ and $\alpha_2(\varepsilon)$. The natural quantum states are the two-mode squeezed states—the states generated by an ideal two-photon device. So it is important to discuss the generalized squeezed states which are related with quantum-group symmetry in two-photon quantum optics.

In this paper, we investigate the most general minimum-uncertainty states for the two-photon systems which possess quantum-group symmetry. We show that the squeezed states of the two-photon optics are simultaneous eigenvectors of a pair of transformed operators A_+ and A_-. For an important subset of the minimal-uncertainty states, the expectation values and variances of relevant operators are given explicitly. This paper is organized as follows. In Sec. II we review some concepts of quantum group. In Sec. III the two-photon system which possesses quantum-group symmetry is demonstrated. Section IV is devoted to the investigations of the minimum-uncertainty states of two-photon system. We summarize in Sec. V the results of this paper.

II. QUANTUM-GROUP SYMMETRY

Recently more and more physicists and mathematicians are paying attention to the study of quantum group which was introduced by Drinfel'd and Jimbo [6]. Conventionally, quantum group is the Hopf algebra which is neither commutative nor cocommutative. Given an associative algebra A with unity, we say that A is a Hopf algebra if we can define three operations Δ, S, and ϵ on A; Δ: $A \to A\otimes A$ is the comultiplication; S: $A \to A$ is the antipodal map; and ϵ: $A \to C$ is the counit. C is the field over which A is an algebra. The operators Δ and ϵ are algebra homomorphisms $\Delta(ab)=\Delta(a)\Delta(b)$, $\epsilon(ab)=\epsilon(a)\epsilon(b)$, whereas S is an antihomomorphism, $S(ab)=S(b)S(a)$. These three operations must satisfy the following axioms:

$$\begin{aligned} &a,b\in A, \quad (\mathrm{id}\otimes\Delta)\Delta(a)=(\Delta\otimes\mathrm{id})\Delta(a) , \\ &m(\mathrm{id}\otimes S)\Delta(a)=m(S\otimes\mathrm{id})\Delta(a)=\epsilon(a)1 , \\ &(\epsilon\otimes\mathrm{id})\Delta(a)=(\mathrm{id}\otimes\epsilon)\Delta(a)=a , \end{aligned} \tag{3}$$

where m is the multiplication in the algebra; m: $A\otimes A\to A$, $m(a\otimes b)=a\cdot b$. If σ: $A\otimes A\to A\otimes A$ is the permutation map $\sigma(a\otimes b)=b\otimes a$, it is easy to check that $\Delta'=\sigma\circ\Delta$ is another comultiplication in A with antipode $S'=S^{-1}$. A Hopf algebra is a quantum group if the comultiplication Δ and Δ' are related by conjugation

$$\sigma\cdot\Delta(a)=R\Delta(a)R^{-1}, \quad R\in A\otimes A , \tag{4}$$

and the following conditions are satisfied:

$$\begin{aligned} &(\mathrm{id}\otimes\Delta)(R)=R_{13}R_{12} , \\ &(\Delta\otimes\mathrm{id})(R)=R_{13}R_{23} , \\ &(S\otimes\mathrm{id})(R)=R^{-1} . \end{aligned} \tag{5}$$

The motivation for these axioms came originally from the theory of integrable models.

In general, there are two kinds of quantum groups [10]: one is $U_{q,\hbar\to 0}(\alpha)$ realized in classical systems, the another one is $U_{q,\hbar}(\alpha)$ realized in quantum systems. After canonical quantizations, the classical systems become quantum ones, and then $U_{q,\hbar\to 0}(\alpha)$ becomes $U_{q,\hbar}(\alpha)$. So quantum groups with respect to their Lie counterparts introduce independent parameters q.

The simplest example of the quantum group is the quantum version of Weyl-Heisenberg algebra $H_q(4)$,

$$\begin{aligned} &[b_q,b_q^\dagger]=[N+1]-[N] , \\ &[N,b_q]=-b_q, \quad [N,b_q^\dagger]=b_q^\dagger , \end{aligned} \tag{6}$$

where $N=b^\dagger b$ is the photon number operator, $[x]=(q^x-q^{-x})/(q-q^{-1})$ (throughout this paper, we limit ourselves to the case of $q\in R$), and the deformed operators b_q and $b_q^\dagger$ are connected with the usual operators b and $b^\dagger$ in the following form:

$$b_q=\sqrt{[N]/N}\,b, \quad b_1^\dagger=b^\dagger\sqrt{[N]/N} . \tag{7}$$

The Hopf operations comultiplication, antipode, and counit can be defined explicitly [11]. In the following sections we show that the quantum group $H_q(4)$ plays an important role in the two-photon system.

III. THE GENERALIZED TWO-PHOTON SYSTEM

Let us begin with writing down the free Hamiltonian for the generalized two-photon system

$$\begin{aligned} H_0&=(\Omega+\varepsilon)b_{+,q}^\dagger b_{+,q}+(\Omega-\varepsilon)b_{-,q}^\dagger b_{-,q} \\ &=H_R+H_M \quad (\mathrm{SP}) , \end{aligned} \tag{8}$$

where we have used the notations

$$\begin{aligned} &H_R\equiv\Omega(b_{+,q}^\dagger b_{+,q}+b_{-,q}^\dagger b_{-,q}) \quad (\mathrm{SP}) , \\ &H_M\equiv\varepsilon(b_{+,q}^\dagger b_{+,q}-b_{-,q}^\dagger b_{-,q}) \quad (\mathrm{SP}) , \end{aligned} \tag{9}$$

and SP denotes Schrödinger picture. It is not difficult to check that H_R and H_M are commutative, i.e.,

$$[H_R,H_M]=0 . \tag{10}$$

Here the annihilation operators for the two mode in the Schrödinger picture are denoted by $b_{+,q}$ and $b_{-,q}$ they satisfy the quantum Weyl-Heisenberg group

$$\begin{aligned} &[b_{+,q},b_{-,q}]=[b_{+,q}^\dagger b_{-,q}^\dagger]=0 , \\ &[b_{\pm,q}b_{\pm,q}^\dagger]=[N_\pm+1]-[N_\pm] , \\ &[N_\pm,b_{\pm,q}^\dagger]=b_{\pm,q}, \quad [N_\pm,b_{\pm,q}]=-b_{\pm,q} . \end{aligned} \tag{11}$$

The modulation picture (MP) is an interaction picture in which the free time dependence at the carrier frequency Ω is transferred from the states to the operators, the states retaining free time dependence at modulation frequency ε. Operators in the MP are related with these in the interaction picture (IP) and SP by

$$R_{\mathrm{MP}}\equiv e^{iH_Rt}R_{\mathrm{SP}}e^{-iH_Rt}=e^{iH_Mt}R_{\mathrm{IP}}e^{-iH_MT} . \tag{12}$$

In the SP the quadrature-phase amplitudes are explicitly time-dependent operators defined by

$$\begin{aligned} \alpha_1(t)\equiv&\left[\frac{\Omega+\varepsilon}{2\Omega}\right]^{1/2}b_{+,q}e^{i\Omega t} \\ &+\left[\frac{\Omega-\varepsilon}{2\Omega}\right]^{1/2}b_{-,q}^\dagger e^{-i\Omega t} \quad (\mathrm{SP}) , \\ \alpha_2(t)\equiv&-i\left[\frac{\Omega+\varepsilon}{2\Omega}\right]^{1/2}b_{+,q}e^{i\Omega t} \\ &+i\left[\frac{\Omega-\varepsilon}{2\Omega}\right]^{1/2}b_{-,q}^\dagger e^{-i\Omega t} \quad (\mathrm{SP}) . \end{aligned} \tag{13}$$

In the MP the quadrature-phase amplitudes are constant and are denoted by

$$\begin{aligned}\alpha_1 &\equiv e^{iH_R t}\alpha_1(t)e^{-iH_R t}\\ &=\alpha_1(0)\\ &=\left[\frac{\Omega+\varepsilon}{2\Omega}\right]^{1/2} b_{+,q}+\left[\frac{\Omega-\varepsilon}{2\Omega}\right]^{1/2} b^{\dagger}_{-,q}\ ,\\ \alpha_2 &\equiv e^{iH_R t}\alpha_2(t)e^{-iH_R t}\\ &=\alpha_2(0)\\ &=-i\left[\frac{\Omega+\varepsilon}{2\Omega}\right]^{1/2} b_{+,q}+i\left[\frac{\Omega-\varepsilon}{2\Omega}\right]^{1/2} b^{\dagger}_{-,q}\ .\end{aligned} \tag{14}$$

Introducing the symbols

$$\lambda_{\pm}\equiv\left[\frac{\Omega\pm\varepsilon}{\Omega}\right]^{1/2}, \tag{15}$$

we can rewrite Eqs. (14) in the following compact form:

$$\begin{aligned}\alpha_1 &\equiv \frac{1}{\sqrt{2}}(\lambda_+ b_{+,q}+\lambda_- b^{\dagger}_{-,q})\ ,\\ \alpha_2 &\equiv \frac{1}{\sqrt{2}}(-i\lambda_+ b_{+,q}+i\lambda_- b^{\dagger}_{-,q})\ .\end{aligned} \tag{16}$$

The two-mode quadrature-phase amplitudes obey the following commutation relations:

$$\begin{aligned}[\alpha_1,\alpha_1^{\dagger}]=[\alpha_2,\alpha_2^{\dagger}] &=\tfrac{1}{2}\{\lambda_+^2([N_+ +1]-[N_+])\\ &\quad -\lambda_-^2([N_- +1]-[N_-])\}\ ,\\ [\alpha_1,\alpha_2] &=0\ ,\\ [\alpha_1,\alpha_2^{\dagger}]=[\alpha_1^{\dagger},\alpha_2] &=\frac{i}{2}\{\lambda_+^2([N_+ +1]-[N_+])\\ &\quad +\lambda_-^2([N_- +1]-[N_-])\}\ .\end{aligned} \tag{17}$$

IV. THE MOST GENERAL MINIMUM-UNCERTAINTY STATES

Now we are in a position to discuss the uncertainty principles for the quadrature-phase amplitudes α_1 and α_2. The most important uncertainty principle places a lower limit on the product of the root-mean-square uncertainties in α_1 and α_2. Introduce the Hermitian real and imaginary parts of α_1 and $\alpha_2 e^{i\lambda}$, i.e.,

$$\begin{aligned}\alpha_1 &\equiv \alpha_1^1+i\alpha_1^2\ ,\\ \alpha_2 &\equiv e^{-i\lambda}(\alpha_2^1+i\alpha_2^2)\ ,\end{aligned} \tag{18}$$

where $e^{i\lambda}$ is an arbitrary phase factor. Then we have

$$\begin{aligned}\langle|\Delta\alpha_1|^2\rangle &=\langle(\Delta\alpha_1^1)^2\rangle+\langle(\Delta\alpha_1^2)^2\rangle\ ,\\ \langle|\Delta\alpha_2|^2\rangle &=\langle(\Delta\alpha_2^1)^2\rangle+\langle(\Delta\alpha_2^2)^2\rangle\ ,\end{aligned} \tag{19}$$

where the notation

$$|\Delta\alpha|^2\equiv(\Delta\alpha\Delta\alpha^{\dagger})_{\text{sys}}=\tfrac{1}{2}(\Delta\alpha\Delta\alpha^{\dagger}+\Delta\alpha^{\dagger}\Delta\alpha) \tag{20}$$

has been used. Using Eq. (17), one can derive the following commutators:

$$\begin{aligned}[\alpha_1^1,\alpha_2^1]=[\alpha_1^2,\alpha_2^2] &=\tfrac{1}{4}(e^{-i\lambda}[\alpha_1,\alpha_2^{\dagger}]+e^{i\lambda}[\alpha_1^{\dagger},\alpha_2])\ ,\\ [\alpha_1^1,\alpha_2^2]=-[\alpha_1^2,\alpha_2^1] &=\frac{i}{4}(e^{-i\lambda}[\alpha_1,\alpha_2^{\dagger}]-e^{i\lambda}[\alpha_1^{\dagger},\alpha_2])\ .\end{aligned} \tag{21}$$

Introduce the notation

$$r_j\equiv\langle(\Delta\alpha_1^j)^2\rangle^{1/2}\geq 0\ ,\quad s_j\equiv\langle(\Delta\alpha_2^j)^2\rangle^{1/2}\geq 0\ ,\quad j=1,2\ . \tag{22}$$

The commutators (21) enforce the following uncertainty principles:

$$\begin{aligned}r_1 s_1 &\geq \tfrac{1}{4}c\,|\sin(\delta-\lambda)|\ ,\\ r_2 s_2 &\geq \tfrac{1}{4}c\,|\sin(\delta-\lambda)|\ ,\\ r_1 s_2 &\geq \tfrac{1}{4}c\,|\cos(\delta-\lambda)|\ ,\\ r_2 s_1 &\geq \tfrac{1}{4}c\,|\cos(\delta-\lambda)|\ ,\end{aligned} \tag{23}$$

where we define

$$\langle[\alpha_1,\alpha_2^{\dagger}]\rangle\equiv ce^{i\delta}\ ,\quad c\equiv|\langle[\alpha_1,\alpha_2]\rangle|\ . \tag{24}$$

Hence the problem is to minimize

$$\begin{aligned}\langle|\alpha_1|^2\rangle\langle|\alpha_2|^2\rangle &=(r_1^2+r_2^2)(s_1^2+s_2^2)\\ &=|(r_1+ir_2)(s_1+is_2)|^2\ ,\end{aligned} \tag{25}$$

subject to the constraints (23). It is not difficult to check that the following formulas are satisfied:

$$\begin{aligned}\langle|\Delta\alpha_1|^2\rangle\langle|\Delta\alpha_2|^2\rangle &=(r_1s_1-r_2s_2)^2+(r_1s_2+r_2s_1)^2\\ &\geq \tfrac{1}{4}c^2\cos^2(\delta-\lambda)\ ,\end{aligned} \tag{26}$$

$$\begin{aligned}\langle|\Delta\alpha_1|^2\rangle\langle|\Delta\alpha_2|^2\rangle &=(r_1s_1+r_2s_2)^2+(r_1s_2-r_2s_1)^2\\ &\geq \tfrac{1}{4}c^2\sin^2(\delta-\lambda)\ .\end{aligned} \tag{27}$$

If we choose $\lambda=\delta$ ($\lambda=\delta-\pi/2$), then Eq. (26) [Eq. (27)] implies the uncertainty principle

$$\begin{aligned}\langle|\Delta\alpha_1|^2\rangle^{1/2}\langle|\Delta\alpha_2|^2\rangle^{1/2} &\geq \tfrac{1}{2}|\langle[\alpha_1,\alpha_2^{\dagger}]\rangle|\\ &=\tfrac{1}{4}\langle\{\lambda_+^2([N_+ +1]-[N_+])+\lambda_-^2([N_- +1]-[N_-])\}\rangle\ .\end{aligned} \tag{28}$$

For $\lambda=\delta-\pi/2$, Eq. (27) shows that the equality in Eq. (28) imposes very restrictive conditions on the state $|\psi\rangle$, the equality holds in Eq. (28) if and only if

$$(\Delta\alpha_1^1+i\gamma\Delta\alpha_2^1)|\psi\rangle=0\ , \tag{29}$$

$$(\Delta\alpha_1^2+i\gamma\Delta\alpha_2^2)|\psi\rangle=0\ , \tag{30}$$

where $\gamma\equiv\langle|\Delta\alpha_1|^2\rangle^{1/2}/\langle|\Delta\alpha_2|^2\rangle^{1/2}$. By taking appropriate linear combinations of Eqs. (29) and (30), one can show that equality holds in (28) if and only if

$$\left[\frac{\Delta\alpha_1}{\langle|\Delta\alpha_1|^2\rangle^{1/2}}+i\frac{\Delta\alpha_2}{\langle|\Delta\alpha_2|^2\rangle^{1/2}}\right]|\psi\rangle=0\ ,$$

$$\left[\frac{\Delta\alpha_1^\dagger}{\langle|\Delta\alpha_1|^2\rangle^{1/2}}+i\frac{\Delta\alpha_2^\dagger}{\langle|\Delta\alpha_2|^2\rangle^{1/2}}\right]|\psi\rangle=0\ . \tag{31}$$

In the general case, i.e., $\langle|\Delta\alpha_1|^2\rangle\neq\langle|\Delta\alpha_2|^2\rangle$, we introduce the following notations:

$$\mu=\frac{1}{4}\left[\frac{1}{\langle|\Delta\alpha_1|^2\rangle^{1/2}}+\frac{1}{\langle|\Delta\alpha_2|^2\rangle^{1/2}}\right]\times\langle\lambda_+^2([N_++1]-[N_+])+\lambda_-^2([N_-+1]-[N_-])\rangle\ ,$$

$$\nu=\frac{1}{4}\left[\frac{1}{\langle|\Delta\alpha_1|^2\rangle^{1/2}}-\frac{1}{\langle|\Delta\alpha_2|^2\rangle^{1/2}}\right]\times\langle\lambda_+^2([N_++1]-[N_+])+\lambda_-^2([N_-+1]-[N_-])\rangle\ , \tag{32}$$

$$a_+=\frac{1}{2\sqrt{2}}\left[\frac{\langle\alpha_1\rangle}{\langle|\Delta\alpha_1|^2\rangle^{1/2}}+i\frac{\langle\alpha_2\rangle}{\langle|\Delta\alpha_2|^2\rangle^{1/2}}\right]\ ,$$

$$a_-=\frac{1}{2\sqrt{2}}\left[\frac{\langle\alpha_1^\dagger\rangle}{\langle|\Delta\alpha_1|^2\rangle^{1/2}}+i\frac{\langle\alpha_2^\dagger\rangle}{\langle|\Delta\alpha_2|^2\rangle^{1/2}}\right]\ .$$

Then Eq. (31) can be rewritten into the following form:

$$(\lambda_+\mu b_{+,q}+\lambda_-\nu b_{-,q}^\dagger)|\psi\rangle=a_+|\psi\rangle\ ,$$
$$(\lambda_+\nu b_{+,q}^\dagger+\lambda_-\mu b_{-,q})|\psi\rangle=a_-|\psi\rangle\ . \tag{33}$$

The above formulas define the minimum-uncertainty state as a simultaneous eigenvector of the operators

$$A_+\equiv\lambda_+\mu b_{+,q}+\lambda_-\nu b_{-,q}^\dagger\ ,$$
$$A_-\equiv\lambda_+\nu b_{+,q}^\dagger+\lambda_-\mu b_{-,q}\ . \tag{34}$$

As may be easily verified by a direct computation, the coefficients μ and ν satisfy

$$|\mu|^2-|\nu|^2=1\ . \tag{35}$$

We may parametrize μ and ν by means of hyperbolic functions and, after an irrelevant phase transformation, rewrite Eq. (34) as

$$A_+=\lambda_+\cosh r b_{+,q}+\lambda_-e^{2i\phi}\sinh r b_{-,q}^\dagger\ ,$$
$$A_-=\lambda_+e^{2i\phi}\sinh r b_{+,q}^\dagger+\lambda_-\cosh r b_{-,q}\ . \tag{36}$$

In the case of $\langle|\Delta\alpha_1|^2\rangle=\langle|\Delta\alpha_2|^2\rangle$, i.e., $\nu=0$, and consequently, $r=0$, the general minimum-uncertainty state $|\psi\rangle$ reduces to the deformed two-mode coherent state $|\psi\rangle_{\rm coh}$; plugging in the definitions (16) of α_1 and α_2, one finds that Eqs. (33) reduce to

$$\Delta b_{\pm,q}|\psi\rangle=0\ , \tag{37}$$

as it should be. The two-mode coherent states can be constructed by

$$|\psi\rangle_{\rm coh}\equiv D(b_{+,q},\mu_+)D(b_{-,q},\mu_-)|0\rangle$$
$$=\left[(\exp_q|\mu_+|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{\mu_+^n}{\sqrt{[n]!}}|n\rangle\right]\times\left[(\exp_q|\mu_-|^2)^{-1/2}\sum_{m=0}^{\infty}\frac{\mu_-^m}{\sqrt{[m]!}}|m\rangle\right]\ , \tag{38}$$

where the q analog of the exponential is defined as $\exp_q(x)\equiv\sum_{n=0}^{\infty}x^n/[n]!$ and the Glauber displacement operators $D(b_{+,q},\mu_+)$ and $D(b_{-,q},\mu)$ are of the form

$$D(b_{\pm,q},\mu_\pm)=\exp\left[-\mu_\pm b_{\pm,q}^\dagger\frac{N_\pm+1}{[N_\pm+1]}-\mu_\pm^*\frac{N_\pm+1}{[N_\pm+1]}b_{+,q}\right]\ . \tag{39}$$

Using the fact that $|\psi\rangle_{\rm coh}$ is an eigenstate of $b_{+,q}$ and $b_{-,q}$, we can show, first, that the expectation values of the annihilation operator and the quadrature-phase amplitudes are given by

$$\langle b_{\pm,q}\rangle=\mu_\pm\ ,$$
$$\langle\alpha_1\rangle=\xi_1\equiv\frac{1}{\sqrt{2}}(\lambda_+\mu_++\lambda_-\mu_-^*)\ , \tag{40}$$
$$\langle\alpha_2\rangle=\xi_2\equiv\frac{1}{\sqrt{2}}(-i\lambda_+\mu_++i\lambda_-\mu_-^*)\ ,$$

and, second, that $|\psi\rangle_{\rm coh}$ has TS noise

$$\langle(\Delta b_{\pm,q})^2\rangle=\langle\Delta b_{+,q}\Delta b_{-,q}^\dagger\rangle=\langle\Delta b_{+,q}\Delta b_{-,q}\rangle=0\ , \tag{41}$$

with

$$\langle|\Delta b_{\pm,q}|^2\rangle=\tfrac{1}{2}\langle([N_\pm+1]-[N_\pm])\rangle\ . \tag{42}$$

Turn now to the general case of Eqs. (33), i.e., the two-mode squeezed states. An important subset of the two-mode squeezed states consists of those with $\phi=0$. For this subset, we obtain that

$$b_{\pm,q}=\frac{1}{\lambda_\pm}(A_\pm\cosh r-A_\mp^\dagger\sinh r)\ ,$$
$$b_{\pm,q}^\dagger=\frac{1}{\lambda_\pm}(A_\pm^\dagger\cosh r-A_\mp\sinh r)\ . \tag{43}$$

Using these formulas, we can calculate the following expectation values for the subset of the two-mode squeezed states:

$$\langle b_{\pm,q}\rangle=\frac{1}{\lambda_\pm}(\alpha_\pm\cosh r-\alpha_\mp^*\sinh r)\ ,$$
$$\langle\alpha_1\rangle=\frac{1}{\sqrt{2}}(\cosh r-\sinh r)(\alpha_++\alpha_-^*)\ , \tag{44}$$
$$\langle\alpha_2\rangle=\frac{-i}{\sqrt{2}}(\cosh r+\sinh r)(\alpha_+-\alpha_-^*)\ .$$

In addition, we can also show that $|\psi\rangle$ has the time-stationary quadrature-phase (TSQP) noise (in terms of creation and annihilation operators), i.e.,

$$\langle (\Delta b_{\pm,q})^2 \rangle = 0, \quad \langle \Delta b_{+,q} \Delta b^{\dagger}_{-,q} \rangle = 0 , \qquad (45)$$

with

$$\langle |\Delta b_{\pm,q}|^2 \rangle = \tfrac{1}{2} \cosh(2r) \langle [N_{\pm}+1]-[N_{\pm}] \rangle ,$$

$$\langle \Delta b_{+,q} \Delta b_{-,q} \rangle = \frac{-1}{2\lambda_+ \lambda_-} \sinh(2r) \times \{ \lambda_+^2 (\cosh^2 r) \langle [N_+ +1]-[N_+] \rangle - \lambda_-^2 (\sinh^2 r) \langle [N_- +1]-[N_-] \rangle \} . \qquad (46)$$

We can also write down the variances of the quadrature-phase amplitudes explicitly,

$$\langle |\Delta\alpha_1|^2 \rangle = \tfrac{1}{4} e^{-2r} (\lambda_+^2 \langle [N_+ +1]-[N_+] \rangle + \lambda_-^2 \langle [N_- +1]-[N_-] \rangle) ,$$

$$\langle |\Delta\alpha_2|^2 \rangle = \tfrac{1}{4} e^{2r} (\lambda_+^2 \langle [N_+ +1]-[N_+] \rangle + \lambda_-^2 \langle [N_- +1]-[N_-] \rangle) . \qquad (47)$$

It is apparent that the minimum-uncertainty state $|\psi\rangle$ is squeezed state, the variance of α_1 is below the zero-point level, and the variance of α_2 is above the zero-point level in the general case of $r \neq 0$.

In addition to the uncertainty principle (28), there is a separate uncertainty principle for each quadrature-phase amplitude

$$\langle |\Delta\alpha_m|^2 \rangle \geq \tfrac{1}{2} | \langle [\alpha_m, \alpha_m^{\dagger}] \rangle | , \quad m = 1,2 . \qquad (48)$$

Equality holds in Eq. (48) if and only if the state vector $|\psi\rangle$ is an eigenstate of α_m, i.e.,

$$\Delta\alpha_m |\psi\rangle = 0 . \qquad (49)$$

SInce $\varepsilon \leq \Omega$, it is immediately apparent from Eq. (28) that it is impossible to find a state $|\psi\rangle$ for which both $\langle |\Delta\alpha_1|^2 \rangle$ and $\langle |\Delta\alpha_2|^2 \rangle$ have minimum values. This means that there are not simultaneous eigenstates of α_1 and α_2.

V. CONCLUSION

We have provided a generalization of the two-photon system which possesses quantum-group symmetry. Starting from the analysis of the output of the two-photon system, we have introduced the natural quantum-mechanical operators α_1 and α_2 which are defined in terms of deformed oscillator operators. The commutation relations satisfied by the quadrature-phase amplitudes have been calculated, and we have discussed the uncertainty principles for the quadrature-phase amplitudes. The most general minimum-uncertainty states of the system are simultaneously eigenstates of operators A_+ and A_- with eigenvalues a_+ and a_-, respectively. We have reparametrized the most general minimum-uncertainty states by means of hyperbolic functions. The properties of the minimum-uncertainty states are demonstrated. For an important subset of the minimum-uncertainty states, we have shown that they are squeezed. The coherent states of the two-photon quantum optics have been constructed explicitly by the Glauber displacement operators.

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