Bell inequalities with a magnitude of violation that grows exponentially with the number of particles

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A simple Bell inequality for an n-particle system in a Greenberger-Horne-Zeilinger state is derived. Quantum mechanics violates this inequality by an exponentially large factor of $2^{(n-2)/2}$ for n odd or $2^{(n-1)/2}$ for n even.

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In 1935, Einstein, Podolsky, and Rosen (EPR) [1] used certain plausible propositions about locality and realism as premises of an argument to show that even at the quantum level, there must exist definable elements or dynamical variables that precisely determine the results of individual measurements. Since these variables are not included in the formalism of quantum mechanics, EPR concluded that quantum-mechanical states do not provide a complete description of physical reality. Bell's [2] extraordinary contribution was to show that the premises of EPR lead to the validity of an inequality that is sometimes grossly violated by the statistical predictions of quantum mechanics. Bell's theorem is of paramount importance for understanding the conceptual foundation of quantum mechanics because it rigorously formulates the premises of EPR and shows that these premises, in general, are incompatible with the quantitative predictions of quantum theory. Over the years, Bell's theorem has been generalized for values of spin other than $\frac{1}{2}$ [3], and for number of axes larger than 3 [4]. Recently an even more provocative demonstration of the incompatibility of the premises of EPR with quantum mechanics was discovered by Greenberger, Horne, and Zeilinger (GHZ) [5–7]. They showed that the premises of EPR cannot even handle the perfect correlations of quantum mechanics for systems of three or more particles. The GHZ argument is stronger than the arguments for the two-particle systems where no contradiction arises at the level of perfect correlations.

Motivated by the GHZ argument, Mermin [8] recently derived an n-particle Bell inequality. Quantum mechanics violates his inequality by an amount that grows exponentially with n. Mermin's result provides the first spectacular demonstration of the fact that there is no limit to the amount by which quantum-mechanical correlations can exceed the limits imposed by the premises of EPR.

In this paper, we derive a simple Bell inequality for an n-particle system in a GHZ state [9]. Similar to Mermin's analysis, we assume that the measurements are imperfect so that the measured correlations, in general, do not attain their extreme values. We shall then show that quantum mechanics violates Bell's inequality by an exponentially large factor of $2^{(n-2)/2}$ for n odd or $2^{(n-1)/2}$ for n even (in contrast, quantum mechanics violates Mermin's inequality by a factor of $2^{(n-1)/2}$ for n odd or $2^{(n-2)/2}$ for n even).

Consider an n-particle GHZ state Φ [10,11]:

$$|\Phi\rangle=\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\cdots\uparrow\rangle-|\downarrow\downarrow\downarrow\cdots\downarrow\rangle)\,, \tag{1}$$

where $\uparrow$ or $\downarrow$ in the jth position corresponds to the spin of the jth particle along the z axis. We take the z axis for each particle to be along its direction of flight, and the x and the y axes along any two orthogonal directions perpendicular to the z axis. After the particles are well separated, we measure the spin of particles $1, 2, \ldots, n-1$ along the x or y axis, but we measure the spin of the nth particle along axis **a**, which is the xy plane and makes 45° with the x axis, or along axis **b**, which is also in the xy plane but makes 135° with the x axis.

We now note that we can learn in advance the result of measuring of the spin of any particle k along any axis **u**, i.e., m_u^k, by far-away measurements of the spin of the other particles. EPR account for this by insisting that the result of measuring the spin of any particle must have already been specified prior to any of the measurements. This assumption is quite natural, since the particles are spatially separated so that the orientation of the analyzer used for the measurement of the spin of any particle should not influence the measurements carried out on the other particles. This assumption, which generally is known as local realism, essentially means that any system in the state Φ must be characterized by a $2n$-axis probability distribution function $P_{x_1y_1\cdots a_nb_n}(m_1\ldots,m_n)$. Quantum mechanics, however, vehemently denies that such a distribution function has any meaning, since it assigns simultaneous values to noncommuting spin operators. In this paper, we shall show that the existence of such a distribution function is numerically incompatible with the quantitative predictions of quantum theory.

Consider operator A defined as $A=A_1+A_2$, where

$$\begin{aligned}A_1=(&-\sigma_x^1\sigma_x^2\sigma_x^3\cdots\sigma_x^{n-1}+\sigma_y^1\sigma_y^2\sigma_x^3\cdots\sigma_x^{n-1}+\cdots\\&-\sigma_y^1\sigma_y^2\sigma_y^3\sigma_y^4\sigma_x^5\cdots\sigma_x^{n-1}-\cdots\\&+\sigma_y^1\cdots\sigma_y^6\sigma_x^7\cdots\sigma_x^{n-1}+\cdots-\cdots)(\sigma_a^n-\sigma_b^n)\,,\end{aligned} \tag{2}$$

and

$$A_2=(\sigma_y^1\sigma_x^2\cdots\sigma_n^{n-1}+\cdots-\sigma_y^1\sigma_y^2\sigma_y^3\sigma_x^4\cdots\sigma_x^{n-1}-\cdots+\sigma_y^1\cdots\sigma_y^5\sigma_x^6\cdots\sigma_x^{n-1}+\cdots-\sigma_y^1\cdots\sigma_y^7\sigma_x^8\cdots\sigma_x^{n-1}-\cdots+\cdots)(\sigma_a^n+\sigma_b^n)\ . \tag{3}$$

Each term in (2) and (3) contains all distinct permutations of the subscripts that give distinct products. In the following, we shall calculate the expected value of A according to the standard rules of quantum mechanics, and according to the requirement of local realism. We shall then show that quantum mechanics violates Bell's inequality by an amount that grows exponentially with n.

First we calculate the expected value of A according to the assumption of local realism. We note that if local realism holds, that is, if the $2n$-axis probability distributions function exists, then the expected value of A, which we denote by the function F, is defined as

$$F=\sum P_{x_1y_1\cdots a_nb_n}(m_1,\ldots,m_n)(M_1+M_2)\ , \tag{4}$$

where

$$M_1=(-m_x^1m_x^2m_x^3\cdots m_x^{n-1}+m_y^1m_y^2m_x^3\cdots m_x^{n-1}+\cdots-m_y^1m_y^2m_y^3m_y^4m_x^5\cdots m_x^{n-1}-\cdots+m_y^1\cdots m_y^6m_x^7\cdots m_x^{n-1}+\cdots-\cdots)\times(m_a^n-m_b^n)\ , \tag{5}$$

or

$$M_1=-\operatorname{Re}\left[\prod_{j=1}^{j=n-1}(m_x^j+im_y^j)\right](m_a^n-m_b^n)\ . \tag{6}$$

Similarly

$$M_2=(m_y^1m_x^2\cdots m_x^{n-1}+\cdots-m_y^1m_y^2m_y^3m_x^4\cdots m_x^{n-1}-\cdots+m_y^1\cdots m_y^5m_x^5\cdots m_x^{n-1}+\cdots-m_y^1\cdots m_y^7m_x^8\cdots m_x^{n-1}-\cdots+\cdots)\times(m_a^n+m_b^n)\ , \tag{7}$$

or

$$M_2=\operatorname{Im}\left[\prod_{j=1}^{j=n-1}(m_x^j+im_y^j)\right](m_a^n+m_b^n)\ . \tag{8}$$

Thus we can write the function F as

$$F=\sum P_{x_1y_1\cdots a_nb_n}(m_1,\ldots,m_n)\times\left[-\operatorname{Re}\left[\prod_{j=1}^{j=n-1}(m_x^j+im_y^j)\right](m_a^n-m_b^n)+\operatorname{Im}\left[\prod_{j=1}^{j=n-1}(m_x^j+im_y^j)\right](m_a^n+m_b^n)\right]\ . \tag{9}$$

We now use the generalized Clauser-Horne-Shimony-Holt [12] lemma to obtain an upper bound on the function F.

Lemma: If u, u', v, and v' are random variables having a probability distribution function $P(u,v,u',v')$, then the following elementary relation always holds:

$$\sum P(u,u',v,v')[u(v-v')+u'(v+v')]\le 2\max\{|u|,|u'|\}\max\{|v|,|v'|\}\ . \tag{10}$$

Using inequality (10) together with the fact that m_a^n and m_b^n are constrained to lie between 1 and -1, we can immediately conclude that

$$F\le 2\max\left[\left|-\operatorname{Re}\left[\prod_{j=1}^{j=n-1}(m_x^j+im_y^j)\right]\right|,\left|\operatorname{Im}\left[\prod_{j=1}^{j=n-1}(m_x^j+im_y^j)\right]\right|\right]\ . \tag{11}$$

Let

$$F_1=\max\left|-\operatorname{Re}\left[\prod_{j=1}^{j=n-1}(m_x^j+im_y^j)\right]\right| \tag{12}$$

and

$$F_2=\max\left|\operatorname{Im}\left[\prod_{j=1}^{j=n-1}(m_x^j+im_y^j)\right]\right|\ . \tag{13}$$

Obviously

$$F\le 2\max\{F_1,F_2\}\ . \tag{14}$$

We now use Mermin's elegant technique to calculate the maximum of the function F_1. We note that F_1 is bounded by a product of complex numbers, each of which has a maximum magnitude of $\sqrt{2}$ and a phase of $\pm45°$ or $\pm135°$. When $n-1$ is even, the product can lie along the real axis and can attain a maximum value of $2^{(n-1)/2}$; when $n-1$ is odd, the product must lie along an axis at 45° to the real or the imaginary axis and can only attain a maximum value of $2^{(n-2)/2}$. Thus

$$F_1\le\begin{cases}2^{(n-2)/2}, & n\text{ even}\\ 2^{(n-1)/2}, & n\text{ odd}\ ,\end{cases} \tag{15}$$

and similarly [8]

$$F_2\le\begin{cases}2^{(n-2)/2}, & n\text{ even}\\ 2^{(n-1)/2}, & n\text{ odd}\ .\end{cases} \tag{16}$$

Therefore the function F, which is defined as the expected value of A according to the assumption of local realism, is bounded by

$$F\le\begin{cases}2^{n/2}, & n\text{ even}\\ 2^{(n+1)/2}, & n\text{ odd}\ .\end{cases} \tag{17}$$

Having obtained an upper bound on the expected value of A according to the requirement of local realism, we now proceed to calculate the expected value of A according to the standard rules of quantum mechanics. First we calculate $\langle\Phi|A_1|\Phi\rangle$:

$$\langle\Phi|A_1|\Phi\rangle=-\langle\Phi|\sigma_x^1\cdots\sigma_x^{n-1}\sigma_a^n|\Phi\rangle+\langle\Phi|\sigma_x^1\cdots\sigma_x^{n-1}\sigma_b^n|\Phi\rangle$$
$$+\langle\Phi|\sigma_y^1\sigma_y^2\sigma_x^3\cdots\sigma_x^{n-1}\sigma_a^n|\Phi\rangle-\langle\Phi|\sigma_y^1\sigma_y^2\sigma_x^3\cdots\sigma_x^{n-1}\sigma_b^n|\Phi\rangle+\cdots$$
$$-\langle\Phi|\sigma_y^1\cdots\sigma_y^4\sigma_x^5\cdots\sigma_x^{n-1}\sigma_a^n|\Phi\rangle+\langle\Phi|\sigma_y^1\cdots\sigma_y^4\sigma_x^5\cdots\sigma_x^{n-1}\sigma_b^n|\Phi\rangle-\cdots+\cdots \quad (18)$$

or [13]

$$\langle\Phi|A_1|\Phi\rangle=\cos(\varphi_x^1+\cdots+\varphi_x^{n-1}+\varphi_a^n)-\cos(\varphi_x^1+\cdots+\varphi_x^{n-1}+\varphi_b^n)$$
$$-\cos(\varphi_y^1+\varphi_y^2+\varphi_x^3+\cdots+\varphi_x^{n-1}+\varphi_a^n)+\cos(\varphi_y^1+\varphi_y^2+\varphi_x^3+\cdots+\varphi_x^{n-1}+\varphi_b^n)-\cdots$$
$$+\cos(\varphi_y^1+\cdots+\varphi_y^4+\varphi_x^5+\cdots+\varphi_x^{n-1}+\varphi_a^n)$$
$$-\cos(\varphi_y^1+\cdots+\varphi_y^4+\varphi_x^5+\cdots+\varphi_x^{n-1}+\varphi_b^n)+\cdots-\cdots\ , \quad (19)$$

where $\varphi_x^k=0°$, $\varphi_y^k=90°$, $(k\in\{1,2,\ldots,n-1\}),\varphi_a^n=45°$, and $\varphi_b^n=135°$. Substituting these values, it can easily be checked that each term in (19) is equal to $\sqrt{2}/2$.

We now calculate $\langle\Phi|A_2|\Phi\rangle$:

$$\langle\Phi|A_2|\Phi\rangle=\langle\Phi|\sigma_y^1\sigma_x^2\cdots\sigma_x^{n-1}\sigma_a^n|\Phi\rangle+\langle\Phi|\sigma_y^1\sigma_x^2\cdots\sigma_x^{n-1}\sigma_b^n|\Phi\rangle+\cdots$$
$$-\langle\Phi|\sigma_y^1\sigma_y^2\sigma_y^3\sigma_x^4\cdots\sigma_x^{n-1}\sigma_a^n|\Phi\rangle-\langle\Phi|\sigma_y^1\sigma_y^2\sigma_y^3\sigma_x^4\cdots\sigma_x^{n-1}\sigma_b^n|\Phi\rangle-\cdots$$
$$+\langle\Phi|\sigma_y^1\cdots\sigma_y^5\sigma_x^6\cdots\sigma_x^{n-1}\sigma_a^n|\Phi\rangle+\langle\Phi|\sigma_y^1\cdots\sigma_y^5\sigma_x^6\cdots\sigma_x^{n-1}\sigma_b^n|\Phi\rangle+\cdots-\cdots\ , \quad (20)$$

or

$$\langle\Phi|A_2|\Phi\rangle=-\cos(\varphi_y^1+\varphi_x^2+\cdots+\varphi_x^{n-1}+\varphi_a^n)-\cos(\varphi_y^1+\varphi_x^2+\cdots+\varphi_x^{n-1}+\varphi_b^n)-\cdots$$
$$+\cos(\varphi_y^1+\varphi_y^2+\varphi_y^3+\varphi_x^4+\cdots+\varphi_x^{n-1}+\varphi_a^n)+\cos(\varphi_y^1+\varphi_y^2+\varphi_y^3+\varphi_x^4+\cdots+\varphi_x^{n-1}+\varphi_b^n)+\cdots$$
$$-\cos(\varphi_y^1+\cdots+\varphi_y^5+\varphi_x^6+\cdots+\varphi_x^{n-1}+\varphi_a^n)$$
$$-\cos(\varphi_y^1+\cdots+\varphi_y^5+\varphi_x^6+\cdots+\varphi_x^{n-1}+\varphi_b^n)-\cdots+\cdots\ . \quad (21)$$

Again it can easily be checked that each term in (21) is equal to $\sqrt{2}/2$. Since the total number of terms in (19) and (21) is 2^n, we can immediately conclude that

$$\langle\Phi|A|\Phi\rangle=2^{n-(1/2)}\ . \quad (22)$$

The quantum theoretic value (22) exceeds the limit imposed by the premises of EPR (17) by an exponentially large amount of $2^{(n-1)/2}$ for n even or $2^{(n-2)/2}$ for n odd.

Note that for odd n (even n) the magnitude of violation of Mermin's inequality is larger (smaller) than the magnitude of violation of the inequality derived here. We thus conclude that for odd n, perfect correlations lead to the largest magnitude of violation, whereas for even n, statistical correlations lead to the largest violation.

A final comment is in order about the experimental implications. Recently Greenberger *et al.* [6] have proposed an experiment to test the violation of GHZ correlation for a set of four particles. The entangled state that they have proposed can also be used to test the violation of Bell inequalities derived here. Quantum mechanics violates their inequality (i.e., GHZ correlation) by a factor of 2, but it violates the inequality derived here by a factor of $2\sqrt{2}$.

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[9] Since the inequality derived here is based on the GHZ entangled state, any experiment designed to test the violation of GHZ correlations can also be used to test the violation of the inequalities derived in this paper.
[10] The inequalities derived here also work if we choose the state $|\Phi\rangle$ to be

$$|\Phi\rangle=(1/\sqrt{2})(|\downarrow\uparrow\uparrow\uparrow\cdots\uparrow\rangle\pm|\uparrow\downarrow\downarrow\downarrow\cdots\downarrow\rangle)$$
$$\text{or } |\Phi\rangle=(1\sqrt{2})(|\downarrow\downarrow\uparrow\uparrow\cdots\uparrow\rangle\pm|\uparrow\uparrow\downarrow\downarrow\cdots\downarrow\rangle)\ ,$$
$$\text{or } |\Phi\rangle=(1/\sqrt{2})(|\downarrow\downarrow\downarrow\uparrow\cdots\uparrow\rangle\pm|\uparrow\uparrow\uparrow\downarrow\cdots\downarrow\rangle)\ ,$$

or any other state containing distinct permutations of ↑ and ↓.

[11] The inequalities derived here also work if we choose Mermin's state $|\Phi\rangle=(1/\sqrt{2})(|\downarrow\uparrow\uparrow\cdots\uparrow\rangle \pm i|\uparrow\downarrow\downarrow\cdots\downarrow\rangle)$, or any other state containing distinct permutation of ↑ and ↓.

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