Unique Bell state

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A pure quantum state which when split in any way cannot violate Bell's inequality is termed a Bell state. Such a state may be considered as classical a state as possible. We show that for the radiation field the Bell state is unique, and is the Glauber coherent state. The formalism allows the interpretation of a local measurement in an entangled state as a measurement with an extended apparatus on a product state.

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I. INTRODUCTION

In 1966 Aharonov, Falkoff, Lerner, and Pendleton (AFLP) [1] characterized a special quantum state of the radiation field by a classical attribute it possesses. The classical characteristic was the following. Let two observers, C and D, receive two beams. According to classical radiation theory the observers cannot ascertain by any of their (local) measurements (including correlating their observations) whether the two beams emanated from one source that was subsequently split—or they came from two independent sources. The reason for this is as follows (they refer pictorially to radio signals). "In classical physics the stochastic variations of the radio frequency oscillators and other components contributing to the signals can be imagined to be as small as we desire." Therefore, in classical physics we can suppose that any prescribed nonstochastic signal can be reproduced as accurately as desired. Hence the beams at C and D that were split from one source can be simulated by two beams from independent sources. AFLP proved that only one quantum state possesses this classical attribute: Glauber's coherent state [2].

Bell introduced his basic inequality in 1964 [3]. However, its clear experimental implications were given by Clauser, Horne, Shimony, and Holt [4] in 1969. [We shall refer to their inequality as Bell's inequality in the sequel; it is given below—in Eq. (5)]. This inequality clearly distinguishes quantum idiosyncrasies—some quantum states violate Bell's inequality. It is thus natural to inquire: what is the state of the radiation field that when split into two daughter states, the latter cannot violate Bell's inequality? This could sharpen the AFLP meaning of "classical quantum state." Indeed we shall show below that the unique state of the radiation field that fulfills the above requirement is indeed the Glauber coherent state (CS). We call a Bell state a quantum state such that upon splitting will not violate Bell's inequality (i.e., it is a state that possesses the basic attribute of classical physics) and we shall show that (a) a state that factorizes upon splitting never violates the inequality, while (b) a state that gives rise to an entangled state [i.e., a state involving a sum; see Eq. (8) below] can always, by a suitable choice of local apparatus, be made to violate Bell's inequality. Then we use the AFLP result, which proved that the only state that factorizes upon splitting is Glauber's coherent state, to deduce that Bell's state is unique. The splitting we have in mind could be, e.g., that obtained by a half-silvered mirror. Finally, we discuss the problem from a "Heisenberg" point of view, which shows that local measurements in an entangled state may be interpreted as nonlocal measurements in a product state.

II. SPLIT BEAM

In this section we formulate the problem mathematically. Thus consider a state of the radiation field ($|\rangle$ denotes the vacuum state)

$$|R\rangle = f(a_A^\dagger)|\rangle \ . \tag{1}$$

Here $a_A^\dagger$ is the creation operator for the mode A (A could specify the wave number and polarization). Upon splitting, this state becomes [1,5]

$$f(a_A^\dagger)|\rangle = f(\mu a_C^\dagger + \nu a_D^\dagger)|\rangle \ , \tag{2}$$

where μ is the probability amplitude that the photon will leave the splitter in the mode C, while ν is the probability amplitude for mode D. Now if the state $|R\rangle$ is such that upon splitting it factorizes, i.e.,

$$f(\mu a_C^\dagger + \nu a_D^\dagger)|\rangle = f_1(a_C^\dagger) f_2(a_D^\dagger)|\rangle \ , \tag{3}$$

then following AFLP [1], f, f_1, and f_2 generate CS, which are given by (including normalization)

$$f(a_A^\dagger)=\exp(\alpha a_A^\dagger-\alpha a_A)\ , \quad (4a)$$

$$f_1(a_C^\dagger)=\exp(\alpha_1 a_C^\dagger-\alpha_1^* a_C)\ , \quad (4b)$$

$$f_2(a_D^\dagger)=\exp(\alpha_2 a_D^\dagger-\alpha_2^* a_D)\ , \quad (4c)$$

with

$$\alpha_1=\mu\alpha,\quad \alpha_2=\nu\alpha\ . \quad (4d)$$

Note that Eq. (4) represents the unique solution of Eq. (3) [1]. To see that this state trivially satisfies Bell's inequality, we consider any operators $\hat{C}(\rho)$ and $\hat{D}(\sigma)$ with the following properties. $\hat{C}$ acts only on the C mode (it represents the apparatus of observer C). It is such that in the normalized state $f_1(a_C^\dagger)|\rangle$, depending on its parameter ρ, its eigenvalues are ± 1. The same applies to $\hat{D}(\sigma)$. Now Bell's inequality is [4,6]

$$|\langle\Psi|\hat{C}(\rho)\hat{D}(\sigma)+\hat{C}(\rho)\hat{D}(\sigma')+\hat{C}(\rho')\hat{D}(\sigma) -\hat{C}(\rho')\hat{D}(\sigma')|\Psi\rangle|\le 2\ . \quad (5)$$

In our case we have a factorized state,

$$|\Psi\rangle=|f_C\rangle|f_D\rangle\ , \quad (6)$$

with $|f_C\rangle$ the normalized state $f_1(a_C^\dagger)|\rangle$, with similar meaning for $|f_D\rangle$. Now because $|\Psi\rangle$ is factorized, we have

$$\langle\Psi|\hat{C}(\rho)\hat{D}(\sigma)|\Psi\rangle=\langle f_C|\hat{C}(\rho)|f_C\rangle\langle f_D|\hat{D}(\sigma)|f_D\rangle\ . \quad (7)$$

Recalling that $|\langle f_C|\hat{C}(\rho)|f_C\rangle|\le 1$ [and a similar inequality for $\langle f_D|\hat{D}(\sigma)|f_D\rangle$], we have that the left-hand side of Eq. (5) is necessarily less than or equal to 2, i.e., Bell's inequality is not violated. We now show that whenever the split state is an entangled state, a violation of Bell's inequality is possible.

III. VIOLATION OF BELL'S INEQUALITY FOR NONPRODUCT STATES

Gisin [7] showed that Bell's inequality is always violated by entangled states (i.e., nonproduct states). Our problem, i.e., that an entangled state that results from splitting always violates Bell's inequality, is essentially the same. Our method of proof involves a generalization of that of Ref. [6], which considers equal weight states and a slight generalization of Ref. [7], which considers states of the same phase. We use the language and techniques of Ref. [6], which seems to us more transparent.

Starting with the Schmidt decomposition [7,8], we can always write the state emerging from the splitter in the following form:

$$f(\mu a_C^\dagger+\nu a_D^\dagger)=f_1(a_C^\dagger)g_1(a_D^\dagger)+f_2(a_C^\dagger)g_2(a_D^\dagger)+\omega\ . \quad (8)$$

Here ω includes any other operator functions, which acting on $|\rangle$ generate states orthogonal to those generated by the first two, and we may assume, without loss of generality, that

$$\langle|f_1^*(a_C)f_2(a_C^\dagger)|\rangle=\langle|g_1^*(a_D)g_2(a_D^\dagger)|\rangle=0\ . \quad (9)$$

The assumption of entanglement implies that at least the first two terms in Eq. (8) are nonvanishing. We shall consider operators $\hat{C}$ and $\hat{D}$, which give 0 when acting on ω and study their expectation values in the 2×2 subspace spanned by $f_i|\rangle, g_i|\rangle, i=1,2$. Thus we take the state under study as normalized, i.e., we are interested in

$$|\Psi\rangle=\bar{\mu}|f_1\rangle|g_1\rangle+\bar{\nu}|f_2\rangle|g_2\rangle \quad (10)$$

with

$$|f_i\rangle\propto f_i(a_C^\dagger)|\rangle\quad i=1,2\ , \quad (11a)$$

$$|g_i\rangle\propto g_i(a_D^\dagger)|\rangle\quad i=1,2\ , \quad (11b)$$

and

$$\langle f_i|f_j\rangle=\delta_{ij}=\langle g_i|g_j\rangle\ , \quad (11c)$$

$$|\bar{\mu}|^2+|\bar{\nu}|^2=1\ . \quad (12)$$

Our problems is to show that, provided $\bar{\mu}\bar{\nu}\neq 0$ (i.e., the state is entangled), we can always violate Bell's inequality for some "orientation" of the apparatus. By analogy with Ref. [6], we now define our operators

$$\hat{C}(\lambda,\phi)=\cos\lambda[|f_1\rangle\langle f_1|-|f_2\rangle\langle f_2|] +\sin\lambda[e^{i\phi}|f_1\rangle\langle f_2|+e^{-i\phi}|f_2\rangle\langle f_1|]\ , \quad (13)$$

$$\hat{D}(\delta,\gamma)=\cos\delta[|g_1\rangle\langle g_1|-|g_2\rangle\langle g_2|] +\sin\delta[e^{i\gamma}|g_1\rangle\langle g_2|+e^{-i\gamma}|g_2\rangle\langle g_1|]\ . \quad (14)$$

These operators clearly give 0 when acting on states orthogonal to our subspace, as they were required to. They are Hermitian. It is easily checked that [the relation between $(\bar{\alpha},\bar{\beta})$ and (λ,ϕ) is given below]

$$\hat{C}(\lambda,\phi)|\psi(\bar{\alpha},\bar{\beta})\rangle=|\psi(\bar{\alpha},\bar{\beta})\rangle\ , \quad (15)$$

$$\hat{C}(\lambda,\phi)|\bar{\psi}(\bar{\alpha},\bar{\beta})\rangle=-|\bar{\psi}(\bar{\alpha},\bar{\beta})\rangle\ , \quad (16)$$

where ($|\bar{\alpha}|^2+|\bar{\beta}|^2=1$)

$$\psi(\bar{\alpha},\bar{\beta})=\bar{\alpha}|f_1\rangle+\bar{\beta}|f_2\rangle\ , \quad (17)$$

$$\bar{\psi}(\bar{\alpha},\bar{\beta})\equiv\psi(-\bar{\beta}^*,\bar{\alpha}^*)=-\bar{\beta}^*|f_1\rangle+\bar{\alpha}^*|f_2\rangle\ , \quad (18)$$

and with the proviso that λ here is such that

$$\cos\lambda=|\bar{\alpha}|^2-|\bar{\beta}|^2\ , \quad (19)$$

and

$$\phi=\phi_\alpha-\phi_\beta\ , \quad (20)$$

where

$$\bar{\alpha}=|\bar{\alpha}|e^{i\phi_\alpha}\ , \quad (21a)$$

$$\bar{\beta}=|\bar{\beta}|e^{i\phi_\beta}\ . \quad (21b)$$

We have, trivially,

$$\langle\bar{\psi}(\bar{\alpha},\bar{\beta})|\psi(\bar{\alpha},\bar{\beta})\rangle=0\ ,$$

i.e., $|\psi(\bar{\alpha},\bar{\beta})\rangle$ and $|\bar{\psi}(\bar{\alpha},\bar{\beta})\rangle$ span the subspace of $|f_1\rangle$ and $|f_2\rangle$. The above implies that for an arbitrary state $|\psi(\bar{\alpha}',\bar{\beta}')\rangle$ in this subspace

$$|\langle\psi(\bar{\alpha}',\bar{\beta}')|F(\lambda,\phi)|\psi(\alpha',\bar{\beta}')\rangle|\le 1\ . \quad (22)$$

Here F denotes $\hat{C}$ or $\hat{D}$, or their product, i.e., this expectation value corresponds to r_α of Ref. [6]. Thence violation of Bell's inequality is tantamount to finding parameters for the operators $\hat{C}(\lambda,\phi)$ and $\hat{D}(\delta,\gamma)$, for which (cf. Ref. [6])

$$|\langle\Psi|\hat{C}(\lambda,\phi)\hat{D}(\delta,\gamma)+\hat{C}(\lambda,\phi)\hat{D}(\delta',\gamma')+\hat{C}(\lambda',\phi')\hat{D}(\delta,\gamma) -\hat{C}(\lambda',\phi')\hat{D}(\delta',\gamma')|\Psi\rangle|>2 . \quad (23)$$

Here $|\Psi\rangle$ is our entangled state [Eq. (10)]. To this end we evaluate

$$\langle\Psi|\hat{C}(\lambda,\phi)\hat{D}(\delta,\gamma)|\Psi\rangle =\cos\lambda\cos\delta+2|\bar{\mu}\bar{\nu}|\sin\lambda\sin\delta\cos(\phi+\gamma-\bar{\phi}) . \quad (24)$$

Here

$$\bar{\phi}=\phi_\mu-\phi_\nu , \quad (25)$$

$$\bar{\mu}=|\bar{\mu}|e^{i\phi_\mu} , \quad (26a)$$

$$\bar{\nu}=|\bar{\nu}|e^{i\phi_\nu} . \quad (26b)$$

Note that in Ref. [7] $\bar{\phi}$ was assumed zero, while in Ref. [6] $\bar{\phi}$ was assumed equal to π, and $2|\bar{\mu}\bar{\nu}|$ was assumed equal to 1. Also, our γ is minus γ of Ref. [6]. If we now choose our "apparatus orientation," i.e., our parameters, as $\lambda=0,\delta=-\delta',\phi+\gamma-\bar{\phi}=\phi'+\gamma'-\bar{\phi}=0,\lambda'=\pi/2$, then the left-hand side of Eq. (23) becomes (with $c\equiv2|\bar{\mu}\bar{\nu}|$)

$$2(\cos\delta+c\sin\delta)=2(1+c^2)^{1/2}\cos(\delta-\chi) , \quad (27)$$

where

$$\cos\chi=\frac{1}{(1+c^2)^{1/2}} . \quad (28)$$

We are still free to choose δ, which we take equal to χ, to get finally for the left-hand side of Eq. (23)

$$2(1+c^2)^{1/2} . \quad (29)$$

This is manifestly greater than 2, i.e., for $c=2|\bar{\mu}\bar{\nu}|\neq0$ we have a violation of Bell's inequality, QED.

We should note here that our operators $\hat{C}$ and $\hat{D}$, while being Hermitian and therefore measurable in principle, may not be easily implemented in actual experiments, as they are not easy to express in terms of ordinary experimental setups, e.g., in terms of photon-counting experiments.

IV. OPERATORS LEADING TO VIOLATION OF BELL'S INEQUALITY

Inspection of Eq. (23) and (24) reveals that the violation of Bell's inequality is due to the particular dependence of the expectation value, Eq. (24), on the parameters of the entanglements ($\bar{\mu}$ and $\bar{\nu}$). On the other hand, we can see [Eq. (8)] that the entangled state can be related to a nonentangled state by a unitary transformation

$$f(\mu a_C^\dagger+\nu a_D^\dagger)=Uf(a_C^\dagger)U^{-1} . \quad (30)$$

For example, for μ and ν real, e.g., $\mu=\cos\theta$, $\nu=\sin\theta$,

$$U=\exp[\theta(a_C a_D^\dagger-a_D a_C^\dagger] , \quad (31)$$

clearly gives

$$Ua_C^\dagger U^{-1}=\cos\theta a_C^\dagger+\sin\theta a_D^\dagger , \quad (32)$$

$$Ua_C U^{-1}=\cos\theta a_C+\sin\theta a_D , \quad (33)$$

$$Ua_D^\dagger U^{-1}=-\sin\theta a_C^\dagger+\cos\theta a_D^\dagger , \quad (34)$$

$$Ua_D U^{-1}=-\sin\theta a_C+\cos a_D . \quad (35)$$

Thus for these particular parameters, U of Eq. (31) satisfies Eq. (30). Since the vacuum state is unaffected by U, we have

$$f(\mu a_C^\dagger+\nu a_D^\dagger)|\rangle=Uf(a_C^\dagger)|\rangle . \quad (36)$$

Returning to Eq. (24) we can rewrite the expectation value as

$$\langle|f^*(a_C)[\hat{C}'(\lambda,\phi)\hat{D}'(\delta,\gamma)]f(a_C^\dagger)|\rangle . \quad (37)$$

Here,

$$\hat{C}'(\lambda,\phi)=U^{-1}\hat{C}(\lambda,\phi)U , \quad (38)$$

$$\hat{D}'(\delta,\gamma)=U^{-1}\hat{D}(\delta,\gamma)U . \quad (39)$$

Clearly the eigenvalues of the primed operators are still ±1; however, the primed operators now act on both subspaces. On the other hand, the expectation value Eq. (37) is now for a *nonentangled* state, i.e., we have the (perhaps expected) result that upon making a "nonlocal measurement"—"apparatus" $\hat{C}'$ and $\hat{D}'$ are nonlocal—we get a violation of Bell's inequality. We illustrate this new viewpoint with the following example. Let

$$f(a_A^\dagger)=a_A^\dagger . \quad (40)$$

In this case we get for the entangled state, via Eq. (8) with $\mu=\cos\theta$, $\nu=\sin\theta$,

$$a_A^\dagger|\rangle=(\mu a_C^\dagger+\nu a_D^\dagger)|\rangle=Ua_C^\dagger|\rangle , \quad U=\exp\theta(a_C a_D^\dagger-a_D a_C^\dagger) . \quad (41)$$

Written explicitly [cf. Eq. (10)],

$$|f_1\rangle=a_C^\dagger|\rangle_C,\ \ g_1=|\rangle_D,\ \ f_2=|\rangle_C,\ \ g_2=a_D^\dagger|\rangle . \quad (42)$$

Here the operators $\hat{C}'$ and $\hat{D}'$ can be evaluated. Using Eq. (13) and (14) in conjunction with (38), (39), and (31) we get, e.g., for $\hat{C}'(\lambda,\phi)$:

$$\begin{aligned}\hat{C}'(\lambda,\phi)=\cos\lambda\{&(\mu\nu)^2(a_C^{\dagger2}|\rangle\langle|a_C^2+a_D^{\dagger2}|\rangle\langle|a_D^2-a_C^{\dagger2}|\rangle\langle|a_D^2-a_D^{\dagger2}|\rangle\langle|a_C^2)\\&+(\mu^2-\nu^2)(a_C^\dagger a_D^\dagger|\rangle\langle|a_C a_D)+\mu\nu(\mu^2-\nu^2)[a_C^\dagger a_D^\dagger|\rangle\langle|(a_C^2-a_D^2)+\text{H.c.}]-|\rangle\langle|\}\\+\sin\lambda\{&e^{i\phi}[\mu\nu(-\nu a_C^{\dagger2}|\rangle\langle|a_C-\mu a_C^{\dagger2}|\rangle\langle|a_D+\mu a_D^{\dagger2}|\rangle\langle|a_D-\nu a_D^{\dagger2}|\rangle\langle|a_C)\\&-(\mu^2-\nu^2)(\nu a_C^\dagger a_D^\dagger|\rangle\langle|a_C-\mu a_C^\dagger a_D^\dagger|\rangle\langle|a_D)+(\mu a_C^\dagger+\nu a_D^\dagger)|\rangle\langle|]+\text{H.c.}\} .\end{aligned}$$

A similar expression is obtained for $\hat{D}'$. These operators (whose eigenvalues are ± 1) are seen to be complicated and each involves both channels C and D, i.e., they represent "nonlocal apparatus."

The reason for the relatively simple expectation values Eq. (24) is due to the simple *nonentangled* wave function that we need deal with now, $a_C^\dagger|\rangle$. We see then that the conceptual simplicity that this Heisenberg-like description allows, viz., we are not surprised to get nonlocal results from nonlocal instruments, is paid for by the complexity of the apparatus.

V. SUMMARY

A state of the radiation field, when subjected to a splitter, e.g., a half-silvered mirror, can either be factorized in its two daughter states or it can form an entangled state (i.e., a state involving a sum). A Bell state is a state which, when subjected to a splitter, leads to a classical-like daughters' state in the sense that the resultant state cannot violate Bell's inequality. We showed that for the radiation field Bell's state is unique and is Glauber's coherent state, i.e., we showed that only Glauber's coherent state, which factorizes upon splitting, abides by Bell's inequality. Any other state that is, necessarily, an entangled state can violate Bell's inequality for a proper choice of experiment. This classical-like feature of the coherent state may be related to its being the unique pure state possessing a non-negative, and no more singular than a δ function, Glauber-Sudarshan P representation [9]. A Heisenberg-like approach to the expectation value allows us to view local measurements in an entangled state as nonlocal measurements in a product state.

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