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Spin-orbit interaction of a photon in an inhomogeneous medium

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As light propagates in an optically inhomogeneous medium, bending and twisting of the beam cause the rotation of the polarization plane. This is the well-known Rytov-Vladimirsky effect or Berry phase. Considering this effect as the result of an interaction between the spin of the photon (polarization) and its orbital motion, one can expect the reverse effect. In fact, the additional angular shift of a trajectory of the circularly polarized beam (CPB) was recently studied for the particular case of an optical fiber. In this paper the spin-orbit interaction Hamiltonian is obtained both in geometrical optics and in wave optics. We have calculated also the effect of the transverse shift of CPB under refraction on the boundary of two media. The expressions for the angular shift of a trajectory of CPB in optical fibers with two types of refractive-index profiles have been obtained. Geometrical optics expressions can be applied for the typical waveguides only at lengths less than 0.05 cm. However, using the geometric optical picture we can successfully describe statistical properties of speckle patterns of laser radiation as it propagates at a considerable distance.

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I. INTRODUCTION

The general solution of the Maxwell's equations in a homogeneous medium can be readily written in Fourier integral form:

$$\mathbf{E}(\mathbf{r},t)=\int\int\int d^3\mathbf{k}\,\tilde{\mathbf{E}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-i\omega(\mathbf{k})t}\,.$$

For us the most interesting property of that solution is the conservation of polarization state of each plane-wave component $\tilde{\mathbf{E}}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$ under the propagation.

In the special case of light radiated in a small angle near the central direction $\mathbf{k}_0/|\mathbf{k}_0|=\mathbf{e}_z$, important solutions are the totally polarized waves

$$\mathbf{E}(\mathbf{r},t)=(c_1\mathbf{e}_x+c_2\mathbf{e}_y)F(x,y,z,t)e^{ik_0z-i\omega(k_0)t}\,.$$

Such a factorization means that the polarization state has no influence on the propagation process $F(x,y,z,t)$ and vice versa; the propagation does not change the polarization.

In an optically inhomogeneous medium with locally isotropic dielectric constant, the polarization and the spatial structure of radiation can be regarded as mutually independent only approximately. For example, if the beam is deflected due to refraction at an angle about 1 rad, the arbitrary polarization cannot be conserved under propagation because it would contradict the transverse nature of the wave at least. In fact, the well-known Rytov effect of rotation of the polarization plane for the nonplanar ray is the result of that minimal demand: let us try to conserve the polarization vector at the infinitesimal part of a trajectory, or at least let us change it, but only as much as it is necessary to keep it transverse to the propagation direction; see [1,2].

The angle of Rytov rotation for the coincident initial and final beam directions is given by a very simple expression: it is equal to the solid angle subtended by the photon trajectory at the origin of the momentum space. That remarkable result for light was obtained by Vladimirsky in 1941 [3] and for a general quantum-mechanical problem in Berry's works [4] (Berry's phase). It is rather surprising that Rytov-Vladimirsky-Berry rotation was found experimentally only recently [5] in a twisted optical fiber; however, the coincidence of theory and experiment proved to be excellent; see also [6].

Principles such as "for every action there is a reaction" are well known in physics. Quantitatively it manifests itself in an equation

$$\frac{\partial f_1}{\partial x_2}=\frac{\partial f_2}{\partial x_1}\,,$$

where $f_1=-\partial V/\partial x_1$, $f_2=-\partial V/\partial x_2$. Here x_1 and x_2 are independent coordinates and the forces f_1 and f_2 are the corresponding derivatives of the Hamiltonian V. That is a reason to look for the reverse effects of the influence of polarization on the light propagation in a locally isotropic but inhomogeneous medium. Some of those effects are the longitudinal shift of the center of gravity of a beam (different for s and p polarizations) [7] and the transverse shift of opposite sign for right and left circular polarizations [8]. In Ref. [9] we proposed to look for small effects of this type that accumulated from multiple reflections of the beam while it propagated in the fiber. The rotation (angular shift) of the speckle pattern transmitted through the optical fiber under the change of the circular polarization of incident light from the right-handed to the left-handed one was predicted and calculated in [9]. Such an optical "ping-pong effect," reminiscent of the spinning ping-pong ball bouncing off the table or the optical analog of Magnus effect, was found experimentally in a multimode fiber of the length about 1 m in [10], in complete coincidence with the theory from [9]. Calculations in [9] were based on the mode analysis, in which we have shown that polarization corrections to the propagation constants of modes (for the graded-index waveguide with the parabolic profile) are proportional to the product $m\sigma$. Here $\sigma=+1$ for the right circular polarization and $\sigma=-1$ for the left one; i.e., σ is the projection of the photon spin at the fiber axis. The integer parameter m, which defines the angular dependence $\exp(im\phi)$ of modes, is the projection of orbital momentum of the photon at the same direction. Therefore, the optical ping-pong effect can be treated as the manifestation of the spin-orbit interaction of the photon in an inhomogeneous medium.

In this paper we are going to show that both effects—the influence of the trajectory on the polarization as a Rytov-Vladimirskij-Berry-Chiao rotation [1,3–5] and the influence of polarization on the trajectory as a rotation of the speckle pattern [9–11]—can be described by the same spin-orbit interaction Hamiltonian. Expressions of such a Hamiltonian are obtained both in geometrical optics of rays and in wave optics in the first post-paraxial approximation. In addition, we have calculated the effect of the transverse shift of the center of gravity of a circularly polarized beam under refraction on the boundary of two media with almost equal refractive indices. Most of the presented results are of geometrical nature and can be applied, e.g., to transversal acoustic waves as well.

II. TRANSVERSE SHIFT OF THE CIRCULARLY POLARIZED BEAM DUE TO REFRACTION

Let us consider refraction of a light wave on the plane boundary of two homogeneous isotropic media with refractive indices n_1 and n_2, respectively. Snell's law means the conservation of the tangential component of the wave vector $\mathbf{k}=(\omega/c)n\mathbf{s}$:

$$n_1[\mathbf{s}_1-\mathbf{m}(\mathbf{s}_1\cdot\mathbf{m})]=n_2[\mathbf{s}_2-\mathbf{m}(\mathbf{s}_2\cdot\mathbf{m})] \ . \tag{1}$$

Here $\mathbf{m}\equiv\mathbf{e}_z$ is the unit vector normal to the boundary surface and for definiteness we assume that $\mathbf{m}\cdot\mathbf{s}_1>0$, $\mathbf{m}\cdot\mathbf{s}_s>0$; see Fig. 1.

Let us denote also the unit polarization vectors $\mathbf{a}$ and $\mathbf{b}$ for given $\mathbf{s}$ (in both media) in a usual way:

$$\mathbf{a}=\mathbf{e}_\perp=\mathbf{m}\times\mathbf{s}/\sin\vartheta \ , \tag{2a}$$

$$\mathbf{b}=\mathbf{e}_\parallel=\mathbf{s}\times\mathbf{a} \ , \tag{2b}$$

where $\mathbf{m}\cdot\mathbf{s}=\cos\vartheta$. If the propagation direction $\mathbf{s}$ were along the $+z$ axes, our polarization vectors would be $\mathbf{a}=\mathbf{e}_x$, $\mathbf{b}=\mathbf{e}_y$, so that $\mathbf{s}$, $\mathbf{a}$, and $\mathbf{b}$, would be right-handedly oriented system.

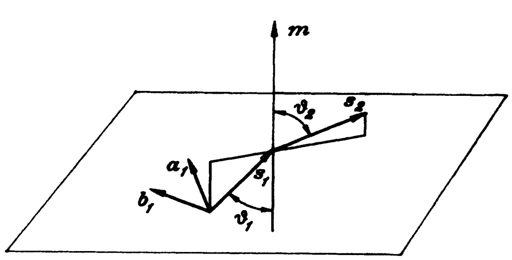

FIG. 1. Diagram for the calculation of the transverse shift under refraction.

Let us consider now the beam that is limited in the transverse direction, i.e., the packet of waves with various wave vectors $\mathbf{k}$. We will reserve the notations $\mathbf{s}$, $\mathbf{a}$, and $\mathbf{b}$ for the central direction in the packet, so that

$$\mathbf{k}(\boldsymbol{\nu})=\frac{\omega}{c}n(\mathbf{s}+\boldsymbol{\nu}) \ , \tag{3}$$

where $\boldsymbol{\nu}$ is the small deviation from the central direction. Then, omitting corrections of the order of $\boldsymbol{\nu}^2$, one can take $\boldsymbol{\nu}\cdot\mathbf{s}=0$ and

$$\boldsymbol{\nu}=\nu_a\mathbf{a}+\nu_b\mathbf{b} \ . \tag{4}$$

Unit vectors of polarization will be (with the same accuracy)

$$\mathbf{e}_\perp(\nu)=\mathbf{a}-\nu_a\mathbf{s}+\nu_a(\cot\vartheta)\mathbf{b} \ , \tag{5a}$$

$$\mathbf{e}_\parallel(\nu)=\mathbf{b}-\nu_b\mathbf{s}-\nu_a(\cot\vartheta)\mathbf{a} \ . \tag{5b}$$

Refraction law gives the relation between ν_1 and ν_2 in two neighboring media:

$$n_1\nu_{1a}=n_2\nu_{2a} \ , \tag{6a}$$

$$n_1(\cos\vartheta_1)\nu_{1b}=n_2(\cos\vartheta_2)\nu_{2b} \ . \tag{6b}$$

Individual plane-wave component $\exp[i(\omega/c)n(\mathbf{s}+\boldsymbol{\nu})\cdot\mathbf{r}]$ has the vector amplitude

$$\tilde{\mathbf{E}}(\nu)=\mathbf{e}_\perp(\nu)\tilde{E}_\perp(\nu)+\mathbf{e}_\parallel(\nu)\tilde{E}_\parallel(\nu) \ . \tag{7}$$

These amplitudes in our two media are related by Fresnel formulas

$$\tilde{E}_{2_\perp}(\nu)=t_\perp(\nu_{1b})\tilde{E}_{1_\perp}(\boldsymbol{\nu}) \ , \tag{8a}$$

$$\tilde{E}_{2_\parallel}(\nu)=t_\parallel(\nu_{1b})\tilde{E}_{1_\parallel}(\boldsymbol{\nu}) \ . \tag{8b}$$

Here we would like to emphasize that, due to the symmetry, the transmitting coefficients $t_\perp$ and $t_\parallel$ depend on the angle of incidence $(\vartheta_1+\nu_{1b})$ and do not depend on the azimuth ν_a.

Suppose that $\tilde{\mathbf{E}}(\boldsymbol{\nu})$ are smooth functions of $\boldsymbol{\nu}$. In that case it is possible to define a single polarization vector for the whole beam as

$$\mathbf{e}_1=|N|[\mathbf{a}\tilde{E}_\perp(\boldsymbol{\nu}=0)+\mathbf{b}\tilde{E}_\parallel(\boldsymbol{\nu}=0)] \ , \tag{9}$$

where the value $|N|$ is used for normalization (so that $|\mathbf{e}_1|=1$). Then, projecting vector amplitudes $\tilde{\mathbf{E}}(\boldsymbol{\nu})$ on the common polarization vector $\mathbf{e}_1\cdot\tilde{\mathbf{E}}(\boldsymbol{\nu})=\tilde{F}(\boldsymbol{\nu})$ we can reduce our problem to the scalar one with the effective field F. Applying the method of a stationary phase to our wave packet

$$F(\mathbf{r})=\int\tilde{F}(\nu)e^{i(\omega/c)n(\mathbf{s}+\nu)\cdot\mathbf{r}}d^2\boldsymbol{\nu} \ , \tag{10}$$

one can calculate the coordinates of the center of gravity of the beam as

$$\mathbf{r}_c=-\frac{c}{\omega n}\frac{\partial}{\partial\boldsymbol{\nu}}\bigg|_{\nu=0}\Phi(F(\boldsymbol{\nu})) \ . \tag{11}$$

Here Φ is the argument of a complex number: $F=|F|\exp(i\Phi)$. Expression (11) can be obtained from the same principles as one uses regarding the propagation of a wave packet in a medium with dispersion in the group-velocity approximation; see, e.g., [12]. Obviously (11) gives only those components of $\mathbf{r}_c$ that are perpendicular to the propagation direction in the corresponding medium.

Now we can explain why all of those complicated notations were introduced. Our intention was to calculate the coordinates $\mathbf{r}_c$ of the incident and the refracted beams using Eq. (11) and find the transverse shift of the beam as the difference $(\mathbf{r}_{c2}-\mathbf{r}_{c1})$. It is important that the mentioned shift depends on the properties of the boundary but not on the properties of the beam itself [because, thanks to relations (8), the main parts of phase dependences for both $F_1(\nu)$ and $F_2(\nu)$ are the same].

Now, to solve the problem, one should substitute the equations in succession (1)–(11). We will present here the expression for the transverse shift $\Delta x_a=\mathbf{a}\cdot(\mathbf{r}_{c2}-\mathbf{r}_{c1})$:

$$\Delta x_a=\frac{2c}{\omega n_1\sin\vartheta_1}\mathrm{Im}\left[\frac{t_\parallel^* E_{1_\parallel}^* t_\perp E_{1_\perp}}{|t_\parallel E_{1_\parallel}|^2+|t_\perp E_{1_\perp}|^2}\cos\vartheta_2-\frac{E_{1_\parallel}^* E_{1_\perp}}{|E_{1_\parallel}|^2+|E_{1_\perp}|^2}\cos\vartheta_1\right]. \tag{12}$$

Further, we will be interested in the case $n_2=n_1+\Delta n$, with $|\Delta n|\ll n$. Then, dropping the corrections of the order of $(\Delta n)^2$, we have $t_\parallel=t_\perp=1$ and $\cos\vartheta_2=\cos\vartheta_1+(\Delta n/n)\sin^2\vartheta_1/\cos\vartheta_1$. Therefore (12) yields

$$\Delta x_a=\frac{c}{\omega n}\frac{\Delta n}{n}(\tan\vartheta_1)\mathrm{Im}\left[\frac{2E_a^* E_b}{|E|_a^2+|E|_b^2}\right]=\sigma\frac{c}{\omega n}\frac{\Delta n}{n}\tan\vartheta_1\ . \tag{13}$$

Here $\sigma=\mathrm{Im}[2E_a^*E_b/(|E_a|^2+|E_b|^2)]$ is equal to $(+1)$ for the right circular polarization of light $(\mathbf{e}_x+i\mathbf{e}_y)\exp(ikz-i\omega t)$ and (-1) for the left one $(\mathbf{e}_x-i\mathbf{e}_y)\exp(ikz-i\omega t)$. In the intermediate case σ can be taken for the degree of circularity of the polarization of the incident wave (and the refracted one if $|\Delta n|\ll n$).

III. DIFFERENTIAL EQUATIONS FOR A TRAJECTORY AND POLARIZATION OF THE RAY: HAMILTONIAN FORM OF THE EQUATIONS

Let us rewrite the obtained above result for the transverse shift in the form

$$\Delta\mathbf{r}_\perp=-\sigma(\Delta n/n)\frac{c}{\omega n}(\mathbf{s}\times\mathbf{m})/\cos\vartheta\ . \tag{14}$$

We are going to apply it to the stratified medium consisting of the pile of layers. If the adjacent layers are at a distance D away from each other we have for one layer $(\Delta n/n)\mathbf{m}=D\nabla\ln n$; ln stands for logarithm. The ray path within one layer is $D/\cos\vartheta$. Hence we have the phenomenological equation

$$\frac{d}{dl}(\Delta\mathbf{r}_\perp)=-\sigma\frac{c}{\omega n}(\mathbf{s}\times\nabla\ln n) \tag{15}$$

for the transverse shift of the ray's trajectory caused by the circularity of polarization. Here l is the length along the trajectory.

It is important to emphasize that the result (15) is based only on the geometrical properties of a polarization (its transverse nature) and on the absence of reflection $(t_\parallel=t_\perp=1)$ for $|\Delta n|\ll n$. Therefore, it will be correct for transverse waves of any physical nature propagating in locally isotropic media.

Let us consider now the ray's trajectory $\mathbf{r}(l)$, in an inhomogeneous medium. The well-known equations of geometrical optics for $\mathbf{r}(l)$ and the unit vector normal to the wave front $\mathbf{s}(l)$ are independent of the polarization and have the form (see, e.g., [2,12,13])

$$\frac{d\mathbf{r}}{dl}=\mathbf{s},\quad \frac{d\mathbf{s}}{dl}=\nabla\ln n-\mathbf{s}(\mathbf{s}\cdot\nabla\ln n)\ . \tag{16}$$

The above-mentioned effect manifests itself in the *shift* of the center of gravity of a beam but not in the change of the propagation direction. Therefore, to take the effect into consideration, we have to modify just the first equation in (16):

$$\frac{d\mathbf{r}}{dl}=\mathbf{s}-\sigma\frac{c}{\omega n}(\mathbf{s}\times\nabla\ln n),\quad \frac{d\mathbf{s}}{dl}=\nabla\ln n-\mathbf{s}(\mathbf{s}\cdot\nabla\ln n)\ . \tag{17}$$

After rather strenuous efforts, we were able to rewrite the equations (17) as canonical equations with some Hamiltonian $\mathcal{H}$. As *coordinates* in $\mathcal{H}$ we chose the instant coordinates of a trajectory $\mathbf{r}$ and the complex three-dimensional (3D) vector $\mathbf{e}$ related to the polarization. As *momenta* in $\mathcal{H}$ we take the *momentum vector of the photon* $\mathbf{p}$ and one more complex 3D vector $\mathbf{u}$. It will be seen from the following consideration that if at the initial moment of time

$$\mathbf{p}=\hbar(\omega/c)n(\mathbf{r}),\quad \mathbf{e}\cdot\mathbf{p}=0,\quad \mathbf{u}\cdot\mathbf{p}=0\ ,$$
$$\mathbf{u}=i\hbar\mathbf{e}^*,\quad \mathbf{e}\cdot\mathbf{e}^*=1,\quad \mathbf{u}\cdot\mathbf{u}^*=\hbar^2\ , \tag{18}$$

these conditions will be satisfied along the whole trajectory as a consequence of the equations of motion. Thus the existence of just *two* complex polarization degrees of freedom and the *transverse* character of the polarization will be provided.

We have taken the Hamiltonian in the form

$$\mathcal{H}(\mathbf{p},\mathbf{u},\mathbf{r},\mathbf{e})=\frac{c}{n(\mathbf{r})}p-\frac{c}{n(\mathbf{r})p}\mathbf{p}\cdot[(\mathbf{e}\times\mathbf{u})\times\nabla\ln n(\mathbf{r})]\ , \tag{19}$$

where $p\equiv(\mathbf{p}\cdot\mathbf{p})^{1/2}$. Canonical Hamilton equations

$$\frac{d\mathbf{r}}{dr}=\frac{\partial\mathcal{H}}{\partial\mathbf{p}},\quad \frac{d\mathbf{p}}{dt}=-\frac{\partial\mathcal{H}}{\partial\mathbf{r}}\ ,$$
$$\frac{d\mathbf{e}}{dt}=\frac{\partial\mathcal{H}}{\partial\mathbf{u}},\quad \frac{d\mathbf{u}}{dt}=-\frac{\partial\mathcal{H}}{\partial\mathbf{e}}\ , \tag{20}$$

for $\mathcal{H}$ from (19) are

$$\frac{d\mathbf{r}}{dt}=\frac{c}{n}\frac{\mathbf{p}}{p}-\frac{c}{np}[(\mathbf{e}\times\mathbf{u})\times\nabla\ln n]+\frac{c}{np^2}\mathbf{p}\{\mathbf{p}\cdot[(\mathbf{e}\times\mathbf{u})\times\nabla\ln n]\}\ , \tag{21}$$

$$\left[\frac{d\mathbf{p}}{dt}\right]_i=\frac{cp}{n}\frac{\partial\ln n}{\partial x_i}+\frac{c}{np}[(\mathbf{e}\times\mathbf{u})\times\mathbf{p}]_k\times\left[\frac{\partial^2\ln n}{\partial x_i\partial x_k}-\frac{\partial\ln n}{\partial x_i}\frac{\partial\ln n}{\partial x_k}\right]\ , \tag{22}$$

$$\frac{d\mathbf{e}}{dt}=-\frac{c}{n}\frac{\mathbf{p}}{p}(\mathbf{e}\cdot\nabla\ln n)+\frac{c}{np}(\mathbf{p}\cdot\mathbf{e})\nabla\ln n\ , \tag{23}$$

$$\frac{d\mathbf{u}}{dt}=-\frac{c}{n}\frac{\mathbf{p}}{p}(\mathbf{u}\cdot\nabla\ln n)+\frac{c}{np}(\mathbf{p}\cdot\mathbf{u})\nabla\ln n\ . \tag{24}$$

Now one can see immediately that the conditions (18) being satisfied at the initial moment of time (in the beginning of a ray's trajectory) will be observed due to (21)–(24) at any t.

The value of $\mathcal{H}(\mathbf{p},\mathbf{u},\mathbf{r},\mathbf{e})$ along the path $\mathbf{p}(t),\mathbf{u}(t),\mathbf{r}(t),\mathbf{e}(t)$ is conserved and equal to $\hbar\omega$. Even more interesting is that the value of spin-orbit correction in the Hamiltonian (19) turned out to be zero along the whole trajectory. This was the main problem in finding the relevant interaction Hamiltonian, and it forced us to use a rather complicated construction of $\mathbf{e}$ and $\mathbf{u}$ vectors, which have formally 12 independent components: $\mathrm{Re}(\mathbf{e})_x,\mathrm{Im}(\mathbf{e})_x,\ldots,\mathrm{Im}(\mathbf{u})_z$. Moreover, the spin-orbit term in the Hamiltonian as a function of its arguments has real and imaginary parts. Fortunately, the final equations proved to be simple and describe both the optical Magnus effect and the Rytov rotation.

Another interesting consequence of Eqs. (23) and (24) is the conservation of the degree of circularity σ, which in new notation is

$$\sigma=(\mathbf{e}\times\mathbf{u})\cdot\mathbf{p}/p\hbar,\quad \frac{d\sigma}{dt}=0\ . \tag{25}$$

To summarize the above-mentioned results, we will rewrite Eqs. (21)–(24) in the form

$$\frac{d\mathbf{r}}{dt}=\frac{c}{n}\left[\mathbf{s}-\sigma\frac{c}{\omega n}(\mathbf{s}\times\nabla\ln n)\right]\ , \tag{26}$$

$$\frac{d\mathbf{s}}{dt}=\frac{c}{n}[\nabla\ln n-\mathbf{s}(\mathbf{s}\cdot\nabla\ln n)]\ , \tag{27}$$

$$\frac{d\mathbf{e}}{dt}=-\frac{c}{n}\mathbf{s}(\mathbf{e}\cdot\nabla\ln n)\ , \tag{28}$$

where $\mathbf{s}=\mathbf{p}/p$, $\sigma=i(\mathbf{e}\times\mathbf{e}^*)\cdot\mathbf{p}/p$.

To transform our equations into the equations of trajectory (17) it is necessary to eliminate time from (26)–(28). Out of the well-known properties of light, it would be natural to have velocity $|d\mathbf{r}/dt|$ equal to $c/n(\mathbf{r})$. To some disappointment, Eq. (26) gives

$$\left|\frac{d\mathbf{r}}{dt}\right|=\frac{c}{n(\mathbf{r})}\left[1+\sigma^2\frac{c^2}{\omega^2n^2}(\nabla\ln n)^2\sin^2\vartheta\right]^{1/2}\ , \tag{29}$$

where ϑ is the angle between $\mathbf{s}$ and ∇n. However, in all of our calculations we took into account only terms of the first order with respect to the small parameter $\lambda|\nabla\ln n|$. That is why we can neglect the difference between the velocity $|d\mathbf{r}/dt|$ and $c/n(\mathbf{r})$. Then we transform Eqs. (26)–(28) into the equation for the time delay $t(l)$ along the trajectory

$$\frac{dt}{dl}=\frac{n(\mathbf{r}(l))}{c} \tag{30}$$

and equations for the trajectory and polarization. Equations for the trajectory just take the form (17). The equation for the unit vector of the polarization $\mathbf{e}$ is

$$\frac{d\mathbf{e}}{dl}=-\mathbf{s}(\mathbf{e}\cdot\nabla\ln n)\ , \tag{31}$$

or, taking into account (17) and (18),

$$\frac{d\mathbf{e}}{dl}=-\mathbf{s}\left[\mathbf{e}\cdot\frac{d\mathbf{s}}{dl}\right]\ . \tag{32}$$

The last equation expresses the original Rytov idea: in that approximation, the polarization alters at the dl path just as much as it is imposed by its transverse nature $(\mathbf{e}\cdot\mathbf{s})=0$.

In our approach, that result comes out as a consequence of the spin-orbit interaction. This means that, in agreement with the conclusions of Sec. II, the spin-orbit interaction has purely kinematical nature if taken into account in the first order with respect to the small parameter $\lambda|\nabla\ln n|$. Therefore, it does not depend on the origin of inhomogeneity of the refractive index $n(\mathbf{r})=[\epsilon(\mathbf{r})\mu(\mathbf{r})]^{1/2}$, whether it is the spatial dependence of scalar dielectric constant $\epsilon(\mathbf{r})$ or magnetic constant $\mu(\mathbf{r})$ or both simultaneously. Moreover, the same Hamiltonian (19) can be applied to the problem of propagation of the transverse acoustic waves in locally isotropic but smoothly inhomogeneous media.

It is not quite clear to us whether the same theory is applicable to waves or particles of various physical nature undergoing refraction due to the strong gravitation field. The problem is that the gravitation field is not scalar and therefore the assumption of local isotropy would be wrong. However, in the first post-Newton approximation, the deflection of light by the gravitation field of the sun does not depend on the polarization: that is a sign of local isotropy.

It is important to note also some specific "nonlinearity" emerging while we reduce Maxwell equations (ME's) into the equations of the geometrical optics (17), and (32). ME's are linear. Therefore we can decompose an incident circularly polarized wave with $\sigma=1$ into the superposition of the two waves with linear polarizations (having the mutual phase shift 90°) with $\sigma=0$ for each of them. But from Eqs. (17) and (32) one can see that in geometrical optics (with spin-orbit interaction) trajectories for $\sigma=1$ and 0 will be somewhat different. In the same way, the wave with linear polarization $(\sigma=0)$ in agreement with ME's can be regarded as the superposition of the two circularly polarized ones (with $\sigma=+1$ and -1). Deviation of the trajectories of the two rays with $\sigma=\pm 1$ at the distance more than the transverse size of one speckle means that initially linearly polarized radiation become depolarized; cf. Refs. [13,14].

Apparently, the most interesting field of applications for our results is the optics of multimode optical fibers. In that case the dependence $n(\mathbf{r})$ usually is axially symmetrical: $n(\mathbf{r})=n(r_\perp)$, where $r_\perp=(x^2+y^2)^{1/2}$. Then the additional term in the Hamiltonian of geometrical optics can be presented in the form

$$\delta\mathcal{H}=-\frac{c^2}{\hbar\omega n^3}\frac{1}{r_\perp}\frac{dn}{dr_\perp}(\mathbf{\Sigma}\cdot\mathbf{M}) \ . \tag{33}$$

Here we have used new notations for the spin of a photon $\mathbf{\Sigma}=\mathbf{e}\times\mathbf{u}$ and the angular momentum of its orbital motion $\mathbf{M}=\mathbf{r}\times\mathbf{p}$; in fact, only the projection M_z at the axis of a fiber is important.

The fact that $\delta\mathcal{H}=0$ along the real trajectories makes our discussion somewhat ephemeral. Nevertheless, we think that (33) allows us to call $\delta\mathcal{H}$ the spin-orbit interaction Hamiltonian.

IV. OPTICAL MAGNUS EFFECT IN A GRADED-INDEX WAVEGUIDE

In [9] we have predicted and calculated the additional angular shift of the ray's trajectory in the graded-index optical waveguide with the axial symmetry. That shift changed its sign if the circular polarization of the incident light was switched from the right-handed to the left-handed one (the optical Magnus effect or the optical ping-pong effect). However, all consideration in [9] was based on wave equations. In the present section we are going to discuss that effect using modified equations of geometrical optics (17).

Let us denote by $\mathbf{r}_\perp=(x,y)$ the transverse and by z the longitudinal coordinates in an optical fiber with the axially symmetrical parabolic profile of the refractive index

$$n(\mathbf{r}_\perp)=n_0(1-Ar_\perp^2) \ , \tag{34}$$

where $r_\perp=(x^2+y^2)^{1/2}$. In real fibers, the smooth dependence (34) stretches only up to some $r_\perp=\rho$, called the radius of the core. Then $A=\Delta n/(n\rho^2)$, where Δn is the difference of refractive indices of the core (in the center of the fiber) and of the cladding. Then Eqs. (17) in the paraxial approximation take the form

$$\begin{aligned}&\frac{dx}{dz}=s_x-2\sigma\frac{c}{\omega n_0}Ay, \quad \frac{dy}{dz}=s_y+2\sigma\frac{c}{\omega n_0}Ax \ , \\ &\frac{ds_x}{dz}=-2Ax, \quad \frac{ds_y}{dz}=-2Ay \ .\end{aligned} \tag{35}$$

They can be transformed into the second-order differential equation for $\mathbf{r}_\perp(z)$:

$$\frac{d^2\mathbf{r}_\perp}{dz^2}=-2A\mathbf{r}_\perp-2\left[\frac{d\mathbf{r}_\perp}{dz}\times\mathbf{\Omega}\right] , \tag{36}$$

where

$$\mathbf{\Omega}(\mathrm{rad/cm})=\mathbf{e}_z\sigma\frac{c}{\omega n_0}A=\sigma\frac{\lambda\Delta n}{2\pi n^2\rho^2}\mathbf{e}_z \ ,$$

λ is the wavelength in the air.

In that form (36) can be interpreted as the equation of motion of the 2D mechanical oscillator with the mass $m=1$ (after the substitution $t\leftrightarrow z$) in the coordinate system rotating with the angular velocity $\mathbf{\Omega}$. Then the last term in (36) is a Coriolis force and we have omitted the "centrifugal force" as an effect of the order of $\mathbf{\Omega}^2$.

Therefore, in the first order of perturbation theory, Ω is the angular velocity of the additional rotation (angular shift) of the trajectory. If we switch the circular polarization of the light from $\sigma=-1$ (left) to $\sigma=+1$ (right), the variation of the ray's azimuth at the length z will be

$$\phi_+-\phi_-=2\Omega_z z=\frac{\lambda\Delta n}{\pi n^2\rho^2}z \ . \tag{37}$$

We would like to mention that at first the expression (37) with exactly the same coefficient was obtained by us just from the wave theory; see also Sec. VIII of this paper. Then we found the proper geometric optical equation (36) and only then Eqs. (17) and the way to deduce them from the first principles expanded in Secs. II and III.

V. OPTICAL PING-PONG EFFECT IN A STEPLIKE INDEX WAVEGUIDE

In an optical fiber with the steplike profile of the refractive index, the ray's trajectory is a broken line with vertices on the boundary between the core and cladding. In those vertices the total reflection (TR) takes place. The transverse shift for a ray undergoing TR was discussed in many papers; see, e.g., Ref. [8]. For the special case of our interest, when $\Delta n/n \ll 1$, which corresponds to rays almost tangent to the boundary, the result can also be obtained from our equations.

Let us consider the linear dependence of $n(x)$ in some interval h instead of the sharp step Δn; see Fig. 2. The

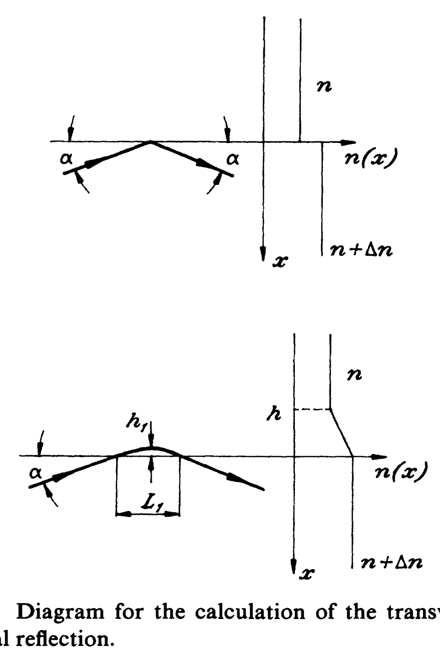

FIG. 2. Diagram for the calculation of the transverse shift under total reflection.

ray's trajectory within the layer with the constant gradient of the refractive index is the part of the parabola. The height of that parabola h_1 depends on the angle α between the ray and the boundary surface in the plane of incidence:

$$h_1 = h(\alpha/\alpha_c)^2 = h\alpha^2 n/(2\Delta n) , \quad (38)$$

where $\alpha_c = (2\Delta n/n)^{1/2}$ is the critical angle of TR for a given $\Delta n/n$. Using the well-known geometrical properties of a parabola, one can easily find that the length of the parabolic part of the ray's path L_1 is

$$L_1 = 4h_1/\tan\alpha \approx \alpha h(2n/\Delta n) . \quad (39)$$

The effect of spin-orbit interaction at that part of trajectory is the transverse shift of the ray

$$\Delta y = \left[\frac{dy}{dl}\right]_{\text{spin-orbit}} L_1 = \sigma(2c\alpha/\omega n) . \quad (40)$$

As was expected, that result does not depend on the formally introduced thickness of the layer h.

In the vector form, the shift (40) can be written as

$$\Delta \mathbf{r}_\perp = \sigma \frac{2c}{\omega n}(\mathbf{s}\times\mathbf{m})(\mathbf{s}\cdot\mathbf{m}) , \quad (41)$$

where $\mathbf{s}$ is the unit vector of the propagation direction of the *incident* ray, and $\mathbf{m}$ is the unit normal to the boundary.

For the propagation of light in the steplike index waveguide, it is convenient to take $\mathbf{s}$ in the form

$$\mathbf{s} \approx \mathbf{e}_z + \boldsymbol{\chi} , \quad (42)$$

where $\boldsymbol{\chi} = (\chi_x, \chi_y)$ is the deviation of $\mathbf{s}$ from the fiber's axis. If we denote the initial transverse coordinates of the ray $\mathbf{r}_0 = (x_0, y_0)$, the trajectory (up to the first reflection) will be

$$\mathbf{r}_\perp(z) = \mathbf{r}_0 + \boldsymbol{\chi} z . \quad (43)$$

The transverse shift at one collision with the boundary transformed into the angular shift of the trajectory is

$$\Delta\phi = |\Delta \mathbf{r}_\perp|/\rho = \sigma \frac{2c}{\omega n \rho^2}[\boldsymbol{\chi}^2(\rho^2 - \mathbf{r}_0^2) + (\mathbf{r}_0\cdot\boldsymbol{\chi})^2]^{1/2} , \quad (44)$$

where ρ is the radius of fiber's core. The number of collisions at the length z is

$$N = \frac{\boldsymbol{\chi}^2 z}{2[\boldsymbol{\chi}^2(\rho^2 - \mathbf{r}_0^2) + (\mathbf{r}_0\cdot\boldsymbol{\chi})^2]^{1/2}} , \quad (45)$$

and hence the spin-orbit correction to the angle of the ray's rotation is

$$\delta\phi(\sigma, \chi) = \sigma \frac{c\boldsymbol{\chi}^2}{\omega n \rho^2} z . \quad (46)$$

Thus we have come to an interesting conclusion: for the optical ping-pong effect in the steplike index fiber, the angular shift is proportional to the second power of the angle $\boldsymbol{\chi}$ between the ray and the fiber's axis. The nature of such a dependence is quite clear: if $\boldsymbol{\chi} \to 0$, both the number of reflections per unit length and the transverse shift in one TR undergo a decrease proportional to $|\boldsymbol{\chi}|$.

Unfortunately, the wave theory in the WKB approximation gives the expression for angular shift smaller by factor of 2 than Eq. (46). We do not yet understand the reason for that discrepancy. However, the functional dependence of $\delta\phi$ on the parameters turns out to be the same for $|\mathbf{r}_0| \to \rho$ in WKB wave theory and in the ray theory of the present section. Therefore, in the rest of this section we shall derive all the consequences of Eq. (46) with the coefficient as written in (46).

If the fiber length of our interest z is long enough to remove evanescent modes from our consideration, the condition of the ray's confinement has a very simple form:

$$\boldsymbol{\chi}^2 = \chi_x^2 + \chi_y^2 < 2\Delta n/n . \quad (47)$$

The averaging of $\boldsymbol{\chi}^2$ over the uniform distribution in the circle $|\boldsymbol{\chi}| < (2\Delta n/n)^{1/2}$ yields $\langle \boldsymbol{\chi}^2 \rangle = \Delta n/n$ and

$$\langle \delta\phi \rangle = \sigma \frac{c\Delta n}{\omega n^2 \rho^2} z , \quad (48)$$

which is equal to $\delta\phi$ in the graded-index fiber with parabolic profile for the same ρ and Δn.

The calculation of the angular shift $\delta\phi$ for an arbitrary profile of the refractive index different from steplike and parabolic indices is a more complicated problem, and we will not deal with it. It is quite clear, however, that for given ρ and Δn the order of magnitude of the effect remains the same.

VI. PARAXIAL APPROXIMATION FOR MAXWELL EQUATIONS

Maxwell equations for the electric field $\mathbf{E}e^{-i\omega t}$ are

$$-\text{rot}\,\text{rot}\mathbf{E} + \frac{\omega^2}{c^2} n^2(\mathbf{r})\mathbf{E} = \mathbf{0} \quad (49a)$$

or

$$\Delta \mathbf{E} + \frac{\omega^2}{c^2} n^2(\mathbf{r}) E = -\nabla[\mathbf{E}\cdot\nabla \ln n^2(\mathbf{r})] , \quad (49b)$$

where $n^2(\mathbf{r}) = \epsilon(\mathbf{r})$ is the dielectric permittivity ($\epsilon = 1$ in vacuum) and we assume that the magnetic permeability μ is identically equal to 1.

Assuming that $n(\mathbf{r})$ is independent of the longitudinal coordinate z, one can rewrite (49b) as the equation for the transverse components of the field E_x and E_y only. Actually, the equation for E_z contains E_x and E_y, but, to the contrary, equations for E_x and E_y are independent of E_z. Such an approach seems to be natural because in the optical waveguide the field is mainly transverse and has just two (not three) polarization degrees of freedom.

We will look for the solution of the form

$$\mathbf{E}(x,y,z) = \exp\left[i\frac{\omega}{c} n_{\text{co}} z\right] \mathbf{E}_\perp(x,y,z) + \mathbf{e}_z E_z(x,y,z) . \quad (50)$$

Here n_{co} is the refractive index of the fiber's core at the axis and we have extracted the fast phase dependence $\exp(i\omega n_{\text{co}} z/c)$ so that $\mathbf{E}_\perp(\mathbf{r})$ is the slow function of its coordinates. Then for the $\mathbf{E}_\perp(\mathbf{r})$ we have the equation of a parabolic type

$$2i\frac{\omega}{c}n_{co}\frac{\partial \mathbf{E}_\perp}{\partial z}=-\Delta_\perp \mathbf{E}_\perp-\frac{\omega^2}{c^2}[n^2(\mathbf{r}_\perp)-n_{co}^2]\mathbf{E}_\perp -\frac{\partial^2 \mathbf{E}_\perp}{\partial z^2}-\nabla_\perp[\mathbf{E}_\perp\cdot\nabla_\perp \ln n^2(\mathbf{r}_\perp)] , \quad (51)$$

where we use the notations $\mathbf{r}=(\mathbf{r}_\perp,z)$ and $\mathbf{r}_\perp=(x,y)$.

If the light propagates in a direction that makes a small angle with the fiber's axis, and variations of the refractive index are small enough, the last two terms in (51) can be neglected, and the difference $[n^2(\mathbf{r}_\perp)-n_{co}^2]$ can be approximately changed by $2n_{co}[n(\mathbf{r}_\perp)-n_{co}]$. Then the wave equation takes the form

$$i\frac{\partial \mathbf{E}_\perp}{\partial z}=-\frac{c}{2\omega n_{co}}\Delta_\perp \mathbf{E}_\perp+\frac{\omega}{c}[n_{co}-n(\mathbf{r}_\perp)]\mathbf{E}_\perp . \quad (52)$$

Apart from the polarization, this equation coincides with the Schrödinger equation (after the substitution $z\leftrightarrow t$) for the particle in the 2D potential proportional to $n_{co}-n(\mathbf{r}_\perp)$.

For us, the most important property of Eq. (52) is that it does not lead to any interaction between Descartes components E_x and E_y even for the arbitrary $n(x,y)$. Thus (52) has factorized solutions, i.e., uniformly polarized waves:

$$\mathbf{E}_\perp(x,y,z)=(c_1\mathbf{e}_x+c_2\mathbf{e}_y)f(x,y,z) , \quad (53)$$

where c_1 and c_2 are the arbitrary complex constants. If we take into account the dropped term proportional to $\nabla_\perp[\mathbf{E}_\perp\cdot\nabla_\perp \ln n^2(\mathbf{r}_\perp)]$, this important property of solutions will be lost. It just corresponds to the spin-orbit interaction.

VII. SPIN-ORBIT CORRECTIONS TO THE PARAXIAL APPROXIMATION: HERMITIAN INTERACTION HAMILTONIAN

Developing the analogy with quantum mechanics, it would be attractive to present the dropped terms [the last two terms in Eq. (51)] as a result of an operation of some perturbation Hamiltonian $\delta\mathcal{H}$ on the vector wave function $\mathbf{E}_\perp(\mathbf{r}_\perp,z)$. An obstacle in this way is that one of the dropped terms contains the second derivative $\partial^2/\partial z^2$. However, iterating the equation (51) and omitting the small corrections of the second order, we can obtain instead of (51)

$$\begin{aligned}i\frac{\partial \mathbf{E}_\perp}{\partial z}=&-\frac{c}{2\omega n_{co}}\Delta_\perp \mathbf{E}_\perp+\frac{\omega}{c}[n_{co}-n(\mathbf{r}_\perp)]\mathbf{E}_\perp\\&+\frac{c^3}{8\omega^3 n_{co}^3}\Delta_\perp\Delta_\perp \mathbf{E}_\perp+\frac{c}{4\omega n_{co}^2}\mathbf{E}_\perp\Delta_\perp n(\mathbf{r}_\perp)\\&+\frac{c}{2\omega n_{co}^2}(\nabla_\perp n\cdot\nabla_\perp)\mathbf{E}_\perp+\frac{c}{2\omega n_{co}^2}[n(\mathbf{r}_\perp)-n_{co}]\Delta_\perp \mathbf{E}_\perp\\&-\frac{c}{2\omega n_{co}^2}\nabla_\perp(\mathbf{E}_\perp\cdot\nabla_\perp \ln n^2) . \end{aligned}\quad (54)$$

In the right-hand side of (54) the first two terms correspond to the "unperturbed Hamiltonian." The third through sixth terms are the perturbations that still do not influence the polarization. The seventh term connects the polarization with the spatial structure of radiation and corresponds to the spin-orbit interaction.

The scalar operator

$$\mathcal{H}_0=-\frac{c}{2\omega n_{co}}\Delta_\perp+\frac{\omega}{c}[n_{co}-n(r_\perp)] \quad (55)$$

in the space of 2D vector functions of two variables $(E_{\perp x}(x,y),E_{\perp y}(x,y))$ is Hermitian with respect to the scalar product

$$(\mathbf{E}_{1_\perp},\mathbf{E}_{2_\perp})=\int\int \mathbf{E}^*_{1_\perp}(x,y)\cdot\mathbf{E}_{2_\perp}(x,y)dx\,dy . \quad (56)$$

Its eigenvalues $(-\Delta\beta_j)$ yield propagation constants of fiber modes $\beta_j=(\omega n_{co})/c+\Delta\beta_j$.

Unfortunately, the most interesting part for us of the perturbation in (54) that corresponds to the polarization corrections is not Hermitian in the sense of the scalar product (56). It is possible, however, to modify the field $\mathbf{E}_\perp(\mathbf{r}_\perp,z)$ in the first order of perturbation theory so that with respect to the new amplitude $\tilde{\mathbf{E}}_\perp(\mathbf{r}_\perp,z)$ the perturbation operator will be Hermitian with respect to the scalar product

$$\int\int \tilde{\mathbf{E}}^*_{1_\perp}(r_\perp,z)\cdot\tilde{\mathbf{E}}_{2_\perp}(\mathbf{r}_\perp,z)d^2\mathbf{r}_\perp .$$

That situation resembles the transformation of Dirac's equation for the bispinor into the Schrödinger-like equation for the two-component spinor with relativistic corrections; see, e.g., [15], Sec. 33.

Using that analogy, we will define the new field $\tilde{\mathbf{E}}_\perp(\mathbf{r})$ so that the scalar product $(\tilde{\mathbf{E}}_\perp,\tilde{\mathbf{E}}_\perp)$ will be equal to the integrated z component of the Pointing vector:

$$\int \tilde{\mathbf{E}}^*_\perp(\mathbf{r}_\perp,z)\cdot\tilde{\mathbf{E}}_\perp(\mathbf{r}_\perp,z)d^2\mathbf{r}_\perp=\text{const}\times\int S_z(\mathbf{r}_\perp)d^2\mathbf{r}_\perp . \quad (57)$$

For the time-averaged Pointing vector

$$\mathbf{S}=(\mathbf{E}\times\mathbf{H}^*)/16\pi+(\mathbf{E}^*\times\mathbf{H})/16\pi , \quad (58)$$

its z component should be expressed by the transverse components of the field $E_{\perp x},E_{\perp y}$ only.

Dropping the intermediate calculations, we would like to show that we have (in the first order of perturbation theory)

$$\int S_z(\mathbf{r}_\perp)d^2\mathbf{r}_\perp=(cn_{co}/8\pi)\int \tilde{\mathbf{E}}^*_\perp(\mathbf{r}_\perp,z)\cdot\tilde{\mathbf{E}}_\perp(\mathbf{r}_\perp,z)d^2\mathbf{r}_\perp \quad (59)$$

if we take

$$\tilde{\mathbf{E}}_\perp=\left[1+[n(\mathbf{r}_\perp)-n_{co}]/2n_{co}+\frac{c^2}{4\omega^2 n_{co}^2}\Delta_\perp\right]E_\perp -\frac{c^2}{2\omega^2 n_{co}^2}\nabla_\perp(\text{div}_\perp \mathbf{E}_\perp) . \quad (60)$$

The Eq. (54) can be modified (with the same accuracy) into the equation for the new field $\tilde{\mathbf{E}}_\perp$:

$$i\frac{\partial \tilde{E}_\alpha}{\partial z}=\left[-\frac{c}{2\omega n_{co}}\Delta_\perp+\frac{\omega}{c}[n_{co}-n(\mathbf{r}_\perp)]\right]E_\alpha$$
$$+\left[\frac{c^3}{8\omega^3 n_{co}^3}\Delta_\perp\Delta_\perp+\frac{c}{2\omega n_{co}^2}[n(\mathbf{r}_\perp)-n_{co}]\Delta_\perp+\frac{c}{4\omega n_{co}}[\Delta_\perp \ln n(\mathbf{r}_\perp)]+\frac{c}{2\omega n_{co}}(\nabla_\perp \ln n)_\beta\frac{\partial}{\partial x_\beta}\right]E_\alpha$$
$$-\frac{c}{2\omega n_{co}}\left[\frac{\partial^2 \ln n}{\partial x_\alpha \partial x_\beta}+\left[\frac{\partial \ln n}{\partial x_\beta}\frac{\partial}{\partial x_\alpha}-\frac{\partial \ln n}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\right]\right]E_\beta \ . \tag{61}$$

Thus we have obtained the modified perturbation operator that is Hermitian in the sense of our scalar product. The third through sixth terms on the right-hand side of Eq. (61) do not involve the polarization. The last three terms in (61) that connect the polarization with the spatial structure of the field through the inhomogeneity of the refractive index $n(x,y)$ constitute the operator of the spin-orbit interaction of a photon in the inhomogeneous medium.

In Ref. [16] the perturbation responsible (in our terms) for the spin-orbit interaction is given just in the non-Hermitian form (54); see Eqs. (32.22) and (32.24) from that reference. Hence, using the arbitrary vector functions for the calculation of the bound modes, one can obtain not just the wrong values for the corrections $\delta\beta$, but even the complex ones, in contradiction with the absence of the damping. Nevertheless, in all specific cases where the proper unperturbed modes of the axially symmetrical waveguide are chosen, the expressions from [16] yield the correct result.

VIII. THE WAVE DESCRIPTION OF THE OPTICAL MAGNUS EFFECT

Let us consider following [9] an optical fiber with a parabolic profile of the refractive index (34). Proper unperturbed modes of such a waveguide are

$$\mathbf{E}_{m,n}^{(\sigma)}=2^{-1/2}(\mathbf{e}_x+i\sigma\mathbf{e}_y)\exp(im\phi)F_{|m|,n}(r_\perp) \ . \tag{62}$$

Here $x=r_\perp\cos\phi$ and $y=r_\perp\sin\phi$;

$$F_{|m|,n}(r_\perp)=(r_\perp/\rho)^{|m|}\exp(-Vr_\perp^2/2\rho^2)\times L_n^{|m|}(Vr_\perp^2/\rho^2) \ . \tag{63}$$

$L_n^{|m|}$ are generalized Laguerre polynomials; $V=(\omega n_{co}/c)\rho(2\Delta n/n)^{1/2}$ is a dimensionless parameter ($V\gg 1$ for a multimode fiber); $\sigma=\pm 1$ is the polarization index (spin); $m=0,\pm 1,\pm 2,\ldots$ is the azimuthal index (orbital angular momentum); $n=0,1,\ldots$ is the radial quantum number; and we shall avoid cases of zeroth full angular momentum ($\sigma=+1$, $m=-1$ and $\sigma=-1$, $m=+1$). In the zeroth order of the perturbation theory the propagation constants of those modes are

$$\beta_{m,n}=(\omega n_{co}/c)-(2\Delta n/n)^{1/2}\rho^{-1}(1+Q) \ , \tag{64}$$

where Q is the principal quantum number of our oscillator.

One can calculate corrections to the propagation constant $\beta_{m,n}$ in the first order of the perturbation theory by averaging the perturbation operator from (61) over the modes (62). We are interested only in terms which depend on the polarization of the mode. That contribution to $\Delta\beta$ have opposite sign for the right ($\sigma=+1$) and left ($\sigma=-1$) circular polarizations:

$$\beta_{m,n}^{\sigma}-\beta_{-m,n}^{\sigma}=2m\sigma\frac{c\Delta n}{\omega n_{co}^2\rho^2}=2Bm\sigma \ . \tag{65}$$

This means that, illuminating the entrance of the fiber by the field

$$\mathbf{E}_\perp(r_\perp,\phi,z=0)=2^{-1/2}(\mathbf{e}_x+i\sigma\mathbf{e}_y)\cos(m\phi)F_{|m|,n}(r_\perp) \ , \tag{66}$$

we will have in the z cross section

$$\mathbf{E}_\perp(r_\perp,\phi,z)=2^{-1/2}(\mathbf{e}_x+i\sigma\mathbf{e}_y)\cos[m(\phi-\sigma Bz)]F_{|m|,n}(r_\perp) \ , \tag{67}$$

i.e., the speckle pattern undergoes a rotation at an angle σB. Switching the circular polarization of the incident light from the left-handed to the right-handed one, we have the mutual angular shift of the speckle patterns

$$\phi_+-\phi_-=2Bz=2\frac{c\Delta n}{\omega n_{co}^2\rho^2}z \ . \tag{68}$$

It is remarkable that, for the optical fiber with the parabolic profile, the angle (68) is the same for all modes. That fact explains the complete coincidence of our results in the wave theory and in the geometrical optics; see Sec. IV.

However, we must recall the discrepancy by a factor of 2 in the relation between ray and WKB wave theory of the ping-pong effect. It is especially surprising that there is no discrepancy between ray and WKB wave theory for the Rytov rotation, neither in a gradient fiber nor in a steplike one; see below.

The wave description of the optical Magnus effect (or the optical ping-pong effect) for the multimode fiber with the steplike index profile was given in [11]. Referring to that paper for the details, we would like to note that the averaged value of the rotation angle for that case proved to be (as a result of numerical calculations) two times less than that for the parabolic profile with the same Δn and ρ. The numerical calculations from [11] proved to be in a good agreement with the results of the experiment on the step-index fiber of the length of $L=96$ cm, with $\rho=100$ μm, $\Delta n=6\times 10^{-3}$, and $n=1.5$ at the wavelength of He-Ne laser $\lambda=0.63$ μm.

The Rytov rotation of the polarization plane for the sagittal ray can be calculated in the same wave picture.

If we regard the mode with the maximal angular momentum m for given radial index n, the rotation of the plane of a linear polarization will be

$$\gamma = z(\beta_{m,n}^{(+)} - \beta_{m,n}^{(-)})/2 = m\frac{c\Delta n}{\omega n_{\rm co}^2\rho^2}z \ . \tag{69}$$

The solid angle subtended by the ray's trajectory at the origin of the momentum space is

$$\Delta\Omega = \pi\chi^2 = 2\pi(\Delta n/n)r^2/\rho^2 \ , \tag{70}$$

where r is the radius of the trajectory. The ray that corresponds to our ultimate mode has $m=(\omega n/c)|\chi|r$, and taking into account that the photon's direction turns through a close circuit at the length $L_0 = 2\pi r/|\chi|$ we have

$$\frac{d\gamma}{dz} = \frac{\Delta\Omega}{L_0} = m\frac{c\Delta n}{\omega n_{\rm co}^2\rho^2}z \ , \tag{71}$$

in complete agreement with the results of Rytov, Vladimirsky, and Berry.

To observe the optical Magnus effect, one should have the rotation angle $\Delta\phi$ being a considerable part of the azimuthal size of the speckle pattern. This yields the condition for the necessary fiber's length

$$z > \pi/mB \ . \tag{72}$$

The circular polarization is conserved under the propagation in the axially symmetrical optical fiber (if we neglect the contribution of the modes with the zeroth value of the full angular momentum $\sigma + m$, see [11], i.e., if we neglect the pure meridional rays). In contrast to what was shown in [14], the degree of linear polarization decreases because of the different sign and value of the Rytov rotation for various sagittal rays. Such a decrease becomes considerable just at the same length π/mB at which the optical Magnus effect for the circularly polarized waves manifests itself.

IX. CONCLUSION

In this paper we have derived the equations of geometrical optics for a trajectory and a polarization of a ray, taking into account their interaction up to the first order of $\lambda|\nabla \ln n|$. The corresponding Hamiltonian can be regarded as the Hamiltonian of the spin-orbit interaction of a photon in an inhomogeneous (but locally isotropic) medium. We explained both the well-known rotation of the polarization plane for a twisted trajectory (according to Rytov, Vladimirsky, Berry, and Chiao) and the optical Magnus effect (or the optical ping-pong effect), which was predicted in [9] and found experimentally in [10]. The wave theory of the spin-orbit interaction of the photon in an inhomogeneous medium is elaborated on as well. We still have a factor-of-2 discrepancy between ray and WKB wave descriptions of the ping-pong effect in a steplike index fiber, which we hope to understand in the future.

The optical effects considered in this paper are of the geometrical nature and should exist for any types of transverse waves in inhomogeneous locally isotropic media.

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