Nonclassical character of states exhibiting no squeezing or sub-Poissonian statistics

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A criterion is evolved for testing the nonclassical character of the field even if it does not exhibit squeezing and sub-Poissonian statistics. An explicit example of a state of the field is given to demonstrate the utility of this criterion. The production of such a state by state-reduction methods is also shown. This criterion also enables us to study the nonclassical character of the Schrödinger "cat" state in regions where it does not exhibit sub-Poissonian statistics.

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Nonclassical light has been characterized in a quantitative way by examining the degree of squeezing [1,2] or sub-Poissonian character [3]. For a single-mode field represented by the annihilation and creation operators a and $a^\dagger$, these two parameters are

$$S=\langle :(ae^{i\theta}+a^\dagger e^{-i\theta})^2: \rangle-\langle (ae^{i\theta}+a^\dagger e^{-i\theta})\rangle^2 \,, \tag{1}$$

$$Q=(\langle a^{\dagger 2}a^2\rangle-\langle a^\dagger a\rangle^2)/\langle a^\dagger a\rangle \,. \tag{2}$$

For squeezing S should be negative. For sub-Poissonian statistics, Q is negative. In the literature a large number of systems have been studied to find the conditions under which S or Q or both are negative [1–5]. Note that if we were to represent the density matrix of the field in terms of diagonal coherent-state representation [6,7] $\rho=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha$, then the negativeness of S and Q implies that the variances of the variables $\alpha^*\alpha$ and $\alpha e^{i\theta}+\alpha^* e^{-i\theta}$ with respect to the distribution $P(\alpha)$ are negative. In other words, negativeness of S and Q implies that $P(\alpha)$ cannot possess all the properties of a classical probability distribution over the whole complex plane.

A question that arises—can the light be still nonclassical even if both S and Q are positive? If yes, then what is the counterpart of Q or S which can quantitatively characterize the nonclassical light? Hong and Mandel [8] introduced the concept of higher-order squeezing, the existence of which also implies nonclassical properties of $P(\alpha)$. A related concept of amplitude-squared squeezing was introduced by Hillery [9]. In what follows we introduce the counter part of Q to characterize nonclassical light for situations where light is nonclassical in spite of the fact that Q is positive. We consider explicitly example of a field given by the density matrix

$$\rho \propto a^{\dagger m}e^{-\beta a^\dagger a}a^m, \quad \beta>0, \quad m=\text{integer} \,, \tag{3}$$

and show that the corresponding P function can be negative even though it is well behaved. Thus the P function possesses nonclassical properties. We exhibit regions of β values in which Q is positive though the P function is negative. We show how the criterion proposed in this paper, can be useful. We further demonstrate how a state like (3) can be produced in the micromaserlike [10] situations by using the process of state reduction [11].

We first obtain an appropriate generalization of the parameter Q. Let us introduce the normally ordered moments

$$m_n=\langle a^{\dagger n}a^n\rangle \equiv \int P(\alpha)|\alpha|^{2n}d^2\alpha \,. \tag{4}$$

Consider the quadratic form constructed from

$$(C_0+C_1|\alpha|^2+C_2|\alpha|^4) \,,$$

where C_0, C_1, and C_2 are arbitrary constants. Clearly, if $P(\alpha)$ behaves like a classical distribution, then the quadratic form $F(\{c\})$

$$F(\{c\})=\sum_{i,j=0}^{2} C_i^* C_j m_{(i+j)} \tag{5}$$

should be positive. In other words, the matrix

$$m^{(3)}=\begin{vmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{vmatrix} \tag{6}$$

should be *positive definite*. More generally, for a classical distribution, the matrices

$$m^{(n)}=\begin{vmatrix} 1 & m_1 & m_2 & \cdots & m_{n-1} \\ m_1 & & & & \\ m_2 & & & & \\ \vdots & & & & \\ m_{n-1} & & & & m_{2n-2} \end{vmatrix} \tag{7}$$

for $n=1,2,3,\ldots$, etc., should be positive definite. For $n=1$, this condition is trivial. For $n=2$ we recover the condition that Q should be positive. For $n>2$ we get a different criterion for the nonclassical fields. Thus for a given field one can test whether the field possesses *phase-insensitive nonclassical properties* or not by computing the matrix $m^{(n)}$ and its eigenvalues and by testing if at least some of the eigenvalues $\lambda^{(n)}$ of the matrix $m^{(n)}$ are negative. Note that $\det m^{(3)}=0$ for a coherent state, whereas for the most nonclassical state, namely the Fock state $\det m^{(3)}$, has the value

$$\det m^{(3)} = -2N^2(N-1) \quad \text{if } \rho = |N\rangle\langle N| \; . \tag{8}$$

It would thus be useful to introduce a normalized quantity so that it is bounded by -1. For this purpose we also introduce the analog of m_n formed from the moments of the number distribution rather than the moments of the P function

$$\mu_n = \langle (a^\dagger a)^n \rangle \tag{9}$$

and we construct the matrix $\mu^{(n)}$ obtained from $m^{(n)}$ by the replacement $m_n \to \mu_n$. Clearly $\mu^{(n)}$'s are positive-definite matrices. Moreover, for a Fock state $\det\mu^{(n)}=0$ if $n>1$. As a measure of the nonclassical property we introduce the quantity

$$A_3 = \frac{\det m^{(3)}}{\det\mu^{(3)} - \det m^{(3)}} \; . \tag{10}$$

Note that A_3 is, respectively, equal to zero and -1 for a coherent state and a Fock state. Note further that in the nonclassical region $\det m^{(3)}<0$, and since $\det\mu^{(3)}>0$, it follows that in the nonclassical region A_3 *lies between zero and* -1.

We next demonstrate the above criterion by considering the state (3) of the field. We first obtain the P function for the state (3). We can rewrite (3) using the diagonal representation of the thermal density matrix as

$$\rho \propto \int e^{-|\alpha|^2/\bar{n}} a^{\dagger m} |\alpha\rangle\langle\alpha| a^m d^2\alpha, \quad \bar{n} = (e^\beta - 1)^{-1} \; . \tag{11}$$

On using the property

$$a^\dagger(|\alpha\rangle e^{|\alpha|^2/2}) = \frac{\partial}{\partial\alpha}(|\alpha\rangle e^{|\alpha|^2/2}) \; ,$$

one can show that (11) can be reduced to

$$\rho \propto \int \frac{1}{\pi\bar{n}} \left[e^{|\alpha|^2} \frac{\partial^{2m}}{\partial\alpha^m \partial\alpha^{*m}} e^{-|\alpha|^2(1+1/\bar{n})} \right] |\alpha\rangle\langle\alpha| d^2\alpha \; . \tag{12}$$

The derivatives in (12) can be expressed in terms of the Laguerre polynomials L_m. The final result for the P function after normalization is found to be

$$P(\alpha) = \frac{1}{\pi\bar{n}} e^{-|\alpha|^2/\bar{n}} L_m \left[\left[1 + \frac{1}{\bar{n}} \right] |\alpha|^2 \right] \Big/ D \; , \tag{13}$$

$$D = \left[\sum_{p=0}^{m} \frac{(-1)^p(\bar{n}+1)^p m!}{(m-p)! p! 2^{2p}} \right] \left[\sum_{q=0}^{p} \frac{2(p-q)! 2q!}{(p-q)!^2 q!^2} \right] \; .$$

The distribution $P(\alpha)$ clearly becomes *negative* as Laguerre polynomials oscillate between positive and negative values. In Fig. 1 we show the nonclassical character of the state (3) by the oscillatory character of (13) for a range of the values of $\bar{n}$ and for fixed m.

The Q function for the state (3) is rather simple,

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle = \frac{|\alpha|^{2m} e^{-|\alpha|^2/(1+\bar{n})}}{\pi(1+\bar{n})^{m+1} m!} \; . \tag{14}$$

This distribution as a function of $|\alpha|^2$ is centered at $m(1+\bar{n})$ and is well behaved as shown by the inset in

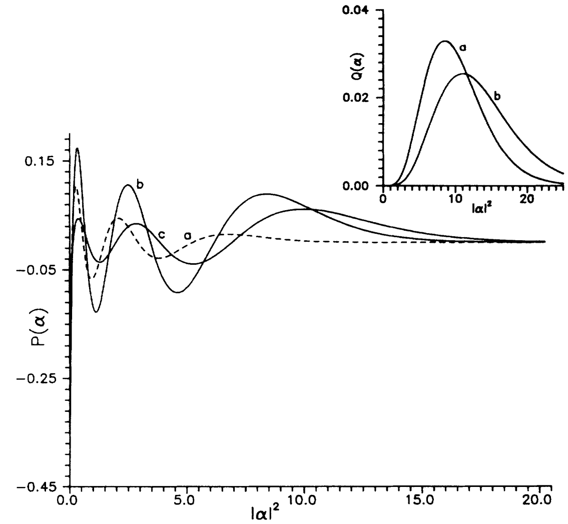

Fig. 1. Distribution $P(\alpha)$ for the state given by Eq. (3) with $m=5$ and average photon number $\bar{n}(a)=0.7$, $(b)=0.95$, and $(c)=1.2$. For the case (a) the plotted values are $\frac{1}{10}$th of the actual values. One can see from (b) and (c) that $P(\alpha)$ becomes negative even if $\bar{n}$ is greater than the critical value $\sqrt{m/(m+1)}$ ($\equiv 0.913$ for $m=5$) given by Eq. (18). The inset corresponds to the plot of $Q(\alpha)$ [Eq. (14)] as a function of $|\alpha|^2$ for $\bar{n}$ $(a)=0.7$ and $(b)=1.2$.

Fig. 1.

We next consider the criterion of the nonclassical nature of the P function based on the sub-Poissonian statistics. The structure of the state (3) is such that the antinormally ordered moments of the intensity are easily computed with the result

$$\langle a^p a^{\dagger p} \rangle = (\bar{n}+1)^p \frac{(m+p)!}{m!} \; . \tag{15}$$

The normally ordered moments can then be obtained by using

$$a^{\dagger p} a^p = (-1)^p \sum_{r=0}^{p} (-1)^r \frac{(p!)^2 a^r a^{\dagger r}}{r!^2 (p-r)!} \; . \tag{16}$$

Using (15) and (16) we find that the parameter Q is

$$Q = \frac{\bar{n}(m+1)\bar{n} - m}{\bar{n}(m+1) + m} \; . \tag{17}$$

Thus the condition for the existence of sub-Poissonian statistics is

$$\bar{n} < \left[\frac{m}{m+1} \right]^{1/2} \; . \tag{18}$$

From Fig. 1 it is clear that $P(\alpha)$ continues to exhibit nonclassical properties even if $\bar{n} > [m/(m+1)]^{1/2}$. Note also that the state (3) has no phase-sensitive properties; i.e., it, for example, exhibits no squeezing $S=0$. Thus for $\bar{n} > [m/(m+1)]^{1/2}$, $Q>0$, $S=0$. In such a case the nonclassical properties can be understood in terms of the criterion developed here. We have examined the positivity of the matrix (6). The results are shown in Fig. 2, where

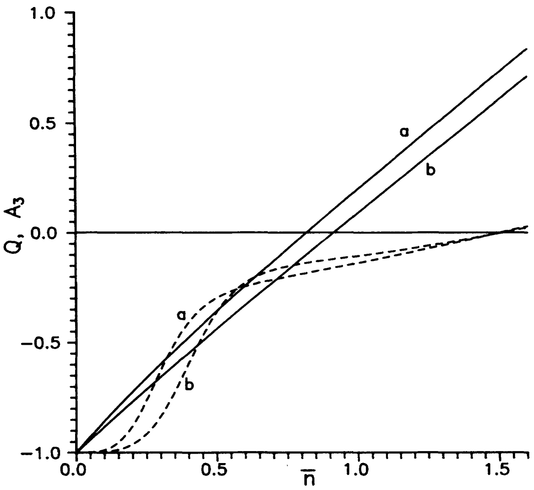

FIG. 2. Parameters Q (solid curves) and A_3 (dashed curves) as functions of average photon number $\bar{n}$ for (a) $m=2$, (b) $m=5$. For $\bar{n}>\sqrt{m/(m+1)}$, the parameter q becomes positive but the parameter A_3 still continues to be negative, implying that the field is nonclassical even if it does not exhibit squeezing as well as sub-Poissonian statistics.

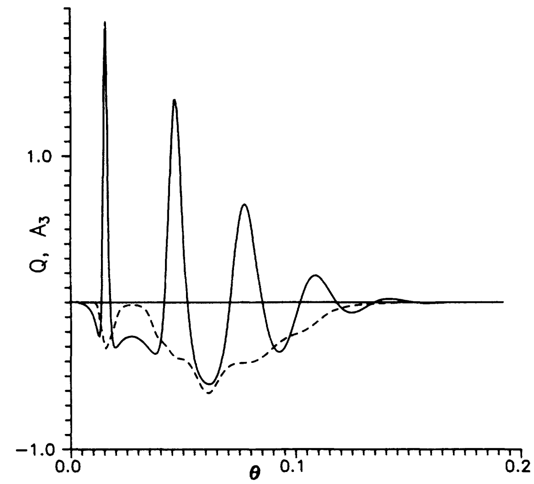

FIG. 3. Parameters Q (solid line) and A_3 (dashed line) as functions of θ for the Schrödinger-cat state given by Eq. (19) for α equal to 8. θ is expressed in units of π. It is interesting to note that A_3 is negative in the range of θ values where Q is positive.

we plot both Q and A_3. This figure clearly demonstrates the utility of the criterion in terms of the positivity of the matrix $m^{(n)}$ when the field state is such that $S>0$, $Q>0$.

We also indicate here how a state like (3) can be produced by the process of state reduction [11]. Consider a single-mode cavity (e.g., a microwave cavity) at a finite temperature T. The field in the cavity is described by $e^{-\beta a^\dagger a}$. Consider now the passage of a beam of well-separated atoms in the excited state through the cavity. Assume that the atoms are so separated that at a given time only one atom is in the cavity. If after the passage the atom is measured to be in the ground state, then the state of the field in the cavity is reduced to (3) with $m=1$. The passage of m atoms will produce the state (3).

As a further illustration of the importance of the criterion developed here, we consider the nonclassical state formed by the superposition of two coherent states [12]

$$|\psi\rangle=\frac{1}{\mathcal{N}}(|\alpha e^{i\theta/2}\rangle+|\alpha e^{-i\theta/2}\rangle)\ , \tag{19}$$

where $\mathcal{N}$ is the normalization constant given by

$$\mathcal{N}^2=\{2+e^{-|\alpha|^2}[\exp(|\alpha|^2e^{i\theta})+\mathrm{c.c.}]\}\ . \tag{20}$$

The Schrödinger-cat-like states have attracted a great deal of attention, as these exhibit very interesting interference effects [12] as well as nonclassical effects [13]. These states can be produced [14] in a number of nonlinear processes or by state-reduction methods. The P-function for the state (19) does not exist; however, the Q function is well behaved. Schleich [13] has studied the sub-Poissonian characteristics of such a state. He showed that there exist regions of θ values for which the state given by Eq. (19) exhibits sub-Poissonian statistics. The question that arises is: does this state have any nonclassical properties for values of θ such that Q is positive? This is where the determinant criterion is useful. We calculate the parameter Q and the parameter A_3 for the state (19). We show in Fig. 3 how the parameters Q and A_3 vary with change in θ for $\alpha=8$. It is seen from Fig. 3 that A_3 becomes negative for the range of θ values where Q is positive.

In conclusion we have developed a quantitative criterion for characterizing the nonclassical properties of the states of the field in regions where say the parameter Q is positive. This criterion is successfully applied to study the nonclassical properties of the Schrödinger-cat-like state and the state obtained by adding photons to a thermal field.

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