Two-field theory of incompressible-fluid turbulence

W. D. McComb* and A. G. Watt
Department of Physics, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, Scotland
(Received 18 February 1992)

The turbulent velocity field in wave-number space is decomposed into two distinct fields. One is a purely *chaotic field;* while the other is a *correction field,* and carries all the phase information. Application of this decomposition to a thin shell of wave numbers in the dissipation range allows the elimination of modes in that shell, with the usual mode-coupling problems being circumvented by the use of a conditional average. The (conditional) mean effect of the eliminated modes appears as an increment to the viscosity, with terms of order λ^2 being neglected, where λ is a dimensionless measure of bandwidth thickness, such that $0 \leq \lambda \leq 1$. An iteration (with appropriate rescaling) to successively lower shells reaches a fixed point, corresponding to a renormalized turbulent viscosity. As previously reported [W. D. McComb and A. G. Watt, Phys. Rev. Lett. **65**, 3281 (1990)], the spectrum of the purely chaotic field is found to take the Kolmogorov $-5/3$ power-law form, with a value for the Kolmogorov spectral constant of $\alpha = 1.6$, independent of λ over the range of bandwidths for which the theory is valid.

PACS number(s): 47.10.+g, 47.25.Cg

I. INTRODUCTION

In this paper we present a method of eliminating turbulent modes. This reduces the computation required for the numerical simulation of the Navier-Stokes equations in wave-number space. Our method is based on the use of a conditional average to distinguish between amplitude and phase correlation effects. It has its roots in the method of iterative averaging, which was developed over a number of years as a possible method of applying the renormalization-group (RG) approach to real fluid turbulence [1]. However, it also incorporates a certain amount of new thinking, which not only provides a rational basis for our calculations, but also seems to fit in with other developments in the subject, particularly those having to do with intermittency and ideas of phase correlation. Some details of the calculation have already been given [2], while the formalism underlying our conditional average has recently been explained elsewhere [3].

However, before going further, we should first emphasize that what we are presenting here is a theory of real fluid turbulence as governed by the usual equations of Newtonian fluid motion. We are *not* putting forward any version of a renormalization-group theory of stirred hydrodynamics (we shall expand on this point in the next section). In complete contrast to such approaches, we begin our calculation in the dissipation range of wave numbers, where our formal procedures for separating amplitude and phase effects allow us to exploit what might be called (in the language of fluid dynamics) a boundary-layer-type of approximation. That is, we can treat the various scales (i.e., velocity, time, length) of the retained modes as "standard magnitudes," compared with which the corresponding scales in the shell being eliminated may be regarded as small. All such approximations made by us will be justified clearly and unambiguously (so far as we are able) in terms of the normal rules of mathematical analysis and the known physics of fluid turbulence. We shall show that exponents for the purely chaotic field are determined by power counting and that amplitudes at the fixed point are determined by the numerical integration of a recursion relation involving small but finite blocks of modes.

In the next section, we shall state the problem in a formal way. At the same time, we shall try to give a particularly clear explanation of the way in which mode-mode coupling (a phenomenon that is of the essence in turbulence) puts difficulties in the way of a straightforward perturbative application of the renormalization-group method to the Navier-Stokes equations. This may seem naive or obvious to some readers, but we believe that certain quite crucial aspects of this are not well understood and that it is necessary to understand them clearly in order to appreciate the differences between approaches, and indeed what our present theory has to offer.

In the rest of the paper, we show in Sec. III how the conditional average is used to eliminate modes from the first shell; in Sec. IV, we employ the two-field decomposition; in Sec. V we derive the recursion relationship for the elimination of subsequent shells, and present the renormalized equations for the velocity field, the energy spectrum, and the dissipation rate, along with some details of the numerical calculation. Section VI concludes the paper with an assessment of the approximations made, a consideration of their relationship to other work, especially that involving ideas of phase coherence, and a brief discussion of the possible applications of our theory.

II. FORMULATION AND STATEMENT OF THE PROBLEM

We consider incompressible-fluid turbulence as governed by the solenoidal Navier-Stokes equation in wave-number ($\mathbf{k}$) space. This takes the form [4]

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)\ , \tag{1}$$

where ν_0 is the kinematic viscosity of the fluid and the inertial transfer operator $M_{\alpha\beta\gamma}(\mathbf{k})$ is given by

$$M_{\alpha\beta\gamma}(\mathbf{k})=(2i)^{-1}[k_\beta D_{\alpha\gamma}(\mathbf{k})+k_\gamma D_{\alpha\beta}(\mathbf{k})]\ , \tag{2}$$

while the projector $D_{\alpha\beta}(\mathbf{k})$ is expressed in terms of the Kronecker delta $\delta_{\alpha\beta}$ as

$$D_{\alpha\beta}(\mathbf{k})=\delta_{\alpha\beta}-k_\alpha k_\beta|\mathbf{k}|^{-2}\ . \tag{3}$$

In order to introduce a statistical treatment, we denote the operation of taking an ensemble average by Dirac brackets, thus $\langle\ \rangle$. We further restrict our attention to incompressible fluids subject to turbulent velocity fields with zero mean, and consider only such fields that are, in addition, homogeneous, isotropic, and stationary in time. As a result, the second-order moment takes the form

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\rangle=Q(k,t-t')D_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\ , \tag{4}$$

where $\alpha,\beta=1$, 2, or 3, and $Q(k,t-t')$ is the spectral density. The energy spectrum is introduced by taking $t=t'$, summing over $\alpha=\beta$ and setting $Q(k,0)=Q(k)$, thus

$$E(k)=4\pi k^2 Q(k)\ , \tag{5}$$

Now consider the turbulent velocity field in wave-number space $u_\alpha(\mathbf{k},t)$, on the interval $0\le k\le k_0$, with k_0 being defined through the dissipation integral

$$\varepsilon=\int_0^\infty 2\nu_0 k^2 E(k)dk\simeq\int_0^{k_0}2\nu_0 k^2 E(k)dk\ , \tag{6}$$

where ε is the dissipation rate and ν_0 is the kinematic viscosity. This definition ensures that k_0 is of the same order of magnitude as the Kolmogorov dissipation wave number. For turbulence at high Reynolds number, k_0 is large and so the number of modes to be resolved in a computational simulation becomes very large indeed, in even the simplest case. Accordingly, there is both fundamental and practical interest in the problem of how one reduces the number of modes that it is necessary to calculate in order to achieve an accurate representation of a real (i.e., laboratory or environmental) turbulent flow. In view of its success in other problems involving many degrees of freedom, it is natural to turn to the renormalization group for the answer to this question and this we now do.

A. Renormalization group applied to turbulence

In order to introduce the renormalization-group approach to turbulence, we divide up the velocity field at $k=k_1$ in the following way:

$$u_\alpha(\mathbf{k},t)=\begin{cases}u_\alpha^-(\mathbf{k},t) & \text{for } 0\le k\le k_1\\ u_\alpha^+(\mathbf{k},t) & \text{for } k_1\le k\le k_0\ ,\end{cases} \tag{7}$$

where k_1 is defined by

$$k_1=(1-\lambda)k_0\ , \tag{8}$$

with the bandwidth parameter λ satisfying the condition $0\le\lambda\le 1$. In principle, the renormalization-group approach now involves two stages.

(i) Solve the Navier-Stokes equation (NSE) on $k_1\le k\le k_0$. Substitute that solution for the mean effect of the high-k modes into the NSE on $0\le k\le k_1$. This results in an increment to the viscosity $\nu_0\to\nu_1=\nu_0+\delta\nu_0$.

(ii) Rescale the basic variables, so that the NSE on $0\le k\le k_1$ looks like the original Navier-Stokes equation on $0\le k\le k_0$.

This procedure is appealingly simple and has a clear physical interpretation. But, as is well known, it has not proved easy to put into practice in the turbulence problem. In order to see why this has been so, we shall examine the first stage in the program (i.e., the elimination of the first shell of modes) in greater detail.

We begin by introducing the unit step functions

$$\Theta^-(k)=\begin{cases}1 & \text{if } 0\le k\le k_1\\ 0 & \text{if } k_1\le k\le k_0\ ,\end{cases} \tag{9}$$

$$\Theta^+(k)=\begin{cases}0 & \text{if } 0\le k\le k_1\\ 1 & \text{if } k_1\le k\le k_0\ ,\end{cases} \tag{10}$$

so that we can define filtered forms of the velocity field and inertial transfer operator, as follows:

$$\begin{aligned}&u_\alpha^-(\mathbf{k},t)=\Theta^-(k)u_\alpha(\mathbf{k},t)\ ,\\ &u_\alpha^+(\mathbf{k},t)=\Theta^+(k)u_\alpha(\mathbf{k},t)\ ,\\ &M_{\alpha\beta\gamma}^-(\mathbf{k})=\Theta^-(k)M_{\alpha\beta\gamma}(\mathbf{k})\ ,\\ &M_{\alpha\beta\gamma}^+(\mathbf{k})=\Theta^+(k)M_{\alpha\beta\gamma}(\mathbf{k})\ .\end{aligned} \tag{11}$$

Then, using (11), we can decompose Eq. (1) into separate filtered forms for the low-wave-number and high-wave-number components, viz.,

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^-(\mathbf{k},t)\\ &\quad=M_{\alpha\beta\gamma}^-(\mathbf{k})\int d^3j\{\,u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t)\\ &\qquad\qquad+2u_\beta^-(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\\ &\qquad\qquad+u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\}\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^+(\mathbf{k},t)\\ &\quad=M_{\alpha\beta\gamma}^+(\mathbf{k})\int d^3j\{\,u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t)\\ &\qquad\qquad+2u_\beta^-(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\\ &\qquad\qquad+u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\}\ .\end{aligned} \tag{13}$$

The main problems that arise due to mode coupling are now obvious by inspection. Equation (12) for $\mathbf{u}^-$ contains terms in $\mathbf{u}^+$, and evidently these are the terms that we wish to eliminate by solving Eq. (13). However, Eq. (13) for $\mathbf{u}^+$ contains terms in $\mathbf{u}^-$. Therefore, substitution for $\mathbf{u}^+$ on the right-hand side of Eq. (12) introduces a tri-

ple nonlinearity in the $\mathbf{u}^-$, thus immediately violating form invariance and preventing the implementation of the second stage of the renormalization-group program.

This is the fundamental problem that prevents the standard application of perturbation theory in this area, but if we go ahead anyway, then we encounter the further problem that averaging out the high-wave-number modes requires the property

$$\langle \mathbf{u}^- \mathbf{u}^+ \mathbf{u}^+ \rangle = \mathbf{u}^- \langle \mathbf{u}^+ \mathbf{u}^+ \rangle \ .$$

In general this cannot be true, as $\mathbf{u}^-$ and $\mathbf{u}^+$ are just parts of the same velocity field and are not statistically independent. In the next two sections, we shall give a brief account of the two main lines of approach to these problems, viz., *stirred hydrodynamics* and *iterative averaging.*

B. Stirred hydrodynamics

The theoretical study of stationary isotropic turbulence has the strict status of a *gedanken* experiment and requires the introduction of hypothetical stirring forces in order to sustain the turbulent against viscous dissipation. Accordingly, for many years, attempts to renormalize perturbation theory [4] have been based on the iterative solution of (1) with a random force added to the right-hand side, thus

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha(\mathbf{k},t) = \lambda_0 M_{\alpha\beta\gamma}(\mathbf{k}) \times \int d^3 j \, u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) + f_\alpha(\mathbf{k},t) \ , \tag{14}$$

where a bookkeeping parameter λ_0 $(=1)$ has been placed in front of the nonlinear term. Evidently, the random force must (like the velocity) satisfy the continuity equation and, in order to make it suitable for perturbation theory, it is necessary to choose it to have a distribution that is multivariate normal and highly uncorrelated in time. Then it is possible to set up a perturbation series in the bookkeeping parameter, thus

$$u_\alpha(\mathbf{k},t) = u_\alpha^{(0)}(\mathbf{k},t) + \lambda_0 u_\alpha^{(1)}(\mathbf{k},t) + \lambda_0^2 u_\alpha^{(2)}(\mathbf{k},t) + \cdots \ , \tag{15}$$

where the Navier-Stokes equation has to be solved iteratively for the coefficients $u_\alpha^{(1)}(\mathbf{k},t), u_\alpha^{(2)}(\mathbf{k},t)$ and so on, in terms of the zero-order field, as given by

$$u_\alpha^0(\mathbf{k},t) = D_{\alpha\beta}(\mathbf{k}) \int dt' G_0(k;t,t') f_\beta(\mathbf{k},t') \ , \tag{16}$$

where G_0 is the Green's function representing the purely viscous response of the fluid, and which satisfies

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] G_0(k;t,t') = \delta(t-t') \ . \tag{17}$$

The specification of the random stirring forces is completed by the choice of correlation. For a well-posed problem, it is important that correlations observed between the velocities should not be due to correlations between the arbitrarily chosen stirring forces, and accordingly it is usual to choose the force-force covariance to satisfy

$$\langle f_\alpha(\mathbf{k},t) f_\beta(-\mathbf{k},t') \rangle = D_{\alpha\beta}(\mathbf{k}) W(k) \delta(t-t') \ . \tag{18}$$

It is then a simple matter to show [4] that stationarity requires that the rate at which the stirring forces do work is the same as the rate of viscous dissipation, or

$$\int_0^\infty 4\pi k^2 W(k) dk = \varepsilon \ , \tag{19}$$

where ε is given by Eq. (6).

Since the late 1950s, the aim of renormalized perturbation theories has been to find a way of summing certain classes of the primitive perturbation series to all orders, having used the Gaussian properties of the zero field to evaluate moments. In 1976, Forster, Nelson, and Stephen (FNS) [5] used this formalism in an ingenious way to evade some of the problems that arise due to mode-mode coupling. However, they did not claim that the result was a theory of turbulence. Instead, they claimed only to have established the long-wavelength properties of a randomly stirred fluid. The result was a new class of models, now referred to generically as "stirred hydrodynamics."

The approach by FNS was motivated by the application of renormalization-group methods to dynamical critical phenomena, and could be regarded as a way of tailoring the turbulence problem to make it resemble a form of the Ising model, complete with an upper critical dimension of $d=4$. They took the turbulent modes to be restricted to the range $0 \le k \le \Lambda$, where the cutoff wave number Λ was chosen to be small enough to exclude inertial effects (which would mean that it would be orders of magnitude smaller than the Kolmogorov dissipation wave number). The specification of their class of models was completed by the choice of a power law for the force correlation function, as defined by Eq. (18), thus

$$W(k) = k^{-y} \ , \tag{20}$$

so that the choice of a value for y amounted to a choice of model.

In order to give a brief account of their way of implementing the renormalization group, we shall make a temporary change of notation. Let us for the moment replace our decomposition of the velocity field by

$$u_\alpha(\mathbf{k},t) = \begin{cases} u_\alpha^<(\mathbf{k},t) & \text{for } 0 \le k \le b\Lambda \\ u_\alpha^>(\mathbf{k},\mathbf{t}) & \text{for } b\Lambda \le k \le \Lambda \ , \end{cases} \tag{21}$$

where the bandwidth parameter b satisfies $0 \le b \le 1$. Then, using an abbreviated notation, the FNS adaptation of the perturbation theory discussed above may be summarized as follows. We seek to eliminate the $u^>$ from (12) by solving Eq. (13) in terms of the arbitrary stirring forces. Symbolically we write the zero-order term as

$$u_0^> = G_0(k) f^>(k) \ , \tag{22}$$

where the stirring force has been decomposed into $f^<$ and $f^>$, in the same way as the velocity field. The general solution for $u^>$ may then be written as

$$u^>(k) = u_{(0)}^> + \lambda_0 u_{(1)}^> + \lambda^2 u_{(2)}^> + \cdots \ . \tag{23}$$

Averages over the high-k modes present no problem now.

As the stirring forces are multivariate normal, it follows that so also are the zero-order terms in the expansion for the velocity. The calculation then proceeds to the renormalization of the viscosity, the stirring forces, and the bookkeeping parameter. An important feature of the analysis [5] is that the triple nonlinearity in the $u^<$, mentioned as a stumbling block in the previous section, turns out to be an irrelevant variable. That is to say, it becomes negligible as the iteration proceeds to a fixed point. There are various restrictions on this result, but the principle one is that it is only valid asymptotically, as k tends to zero.

A full discussion of the FNS theory will be found either in the original reference ([5]) or in Ref. [4], but we shall not pursue it here. From our present point of view, it is sufficient to make the point that this is not a theory of turbulence and does not pretend to be. However, recently these ideas have been given fresh life, most notably by the *correspondence principle* of Yakhot and Orszag [6], which is a hypothesis that, if the stirring forces are chosen to give the same energy spectrum as in a real turbulent flow, then the associated numerics will also be correct for that flow. This is a bold assumption, and so far its main justification has been that it produces some good values for representative constants when compared with experiments and computer simulations. However, it remains to be seen whether or not some more fundamental justification can be found. Another approach is the work recently put forward by Avellaneda and Majda [7], who have adapted FNS theory to develop "exactly renormalizable" models of stirred hydrodynamics. It seems a reasonable hope that this development could play the same role in turbulence theory as the Ising model does in studies of critical phenomena.

C. Iterative averaging

The FNS theory may be seen as a way of applying the ϵ expansion of critical phenomena (and, originally, quantum field theory) to stirred hydrodynamics. That is, FNS derive differential relations for the renormalized quantities and seek a fixed point in the coupling-constant space. In contrast, the method of iterative averaging is based upon the derivation of a recurrence relation, which eliminates finite blocks of modes while maintaining the form invariance of the dynamical equation (i.e., the NSE, in this case). The end result is a fixed point that corresponds to a renormalized viscosity.

The difference between the two approaches may be underlined by noting that in the first method, FNS ensure that the nonlinear coupling is small by restricting their attention to a particular region (small wave numbers) where the viscous and stirring terms are in approximate detail balance, whereas in the second method, we begin our elimination scheme in a region (large wave numbers) where the viscous effects are dominant and the nonlinearity is (by definition) inherently small. This seems the more natural way to apply the renormalization-group program to turbulence, a fact that seems to have been first recognized by Rose [8], who employed a straightforward iterative perturbation method to the explicitly linear problem of passive scalar convection. However, we shall discuss Rose's type of approach as applied to the velocity field rather than the passive scalar case, as in this way we can make it clear just where the difficulties lie when we tackle turbulence directly.

Let us make some plausible physical assumptions about the modes that we wish to eliminate. If we take k_1 to be of the same order as the dissipation wave number, then we can assume that the u^+ are small and that they are rapidly varying on the time scales of the u^-, so that we may solve (13) to first order by neglecting the time derivative on the left-hand side. We then substitute the result into Eq. (12), which is then averaged in such a way that the u^- are held constant, but the average of u^+ vanishes. Denoting the conditional average by $\langle\ \rangle_c$, the result, as given by McComb and Shanmugasundaram [9], is

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^-(\mathbf{k},t)= M^-_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t)$$
$$+2M^-_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\int d^3p\, M^+_{\gamma\rho\delta}(\mathbf{j})(\nu_0 j^2)^{-1}\{\langle u_\gamma^-(\mathbf{k}-\mathbf{j},t)u_\rho^-(\mathbf{p},t)u_\delta^-(\mathbf{j}-\mathbf{p},t)\rangle_c$$
$$+u_\gamma^-(\mathbf{k}-\mathbf{j},t)\langle u_\rho^+(\mathbf{p},t)u_\delta^+(\mathbf{j}-\mathbf{p},t)\rangle_c\}\ . \qquad (24)$$

The two problems mentioned in our preliminary discussion are now quite obvious. First, as pointed out previously]9], the equation for $\mathbf{u}^-$ now contains a triple nonlinearity $\langle u^-u^-u^-\rangle_c$, which breaks the form invariance of the equation of motion under renormalization-group (RG) transformation (and which will only be the lowest order of such terms that arise in subsequent iterations). Second, it should be noted that the conditional average $\langle u^+u^+\rangle_c$ is not the same as the ensemble average $\langle u^+u^+\rangle$, unless there is a spectral gap between the u^- and the u^+ modes.

As Rose [8] studied the passive convection of a scalar contaminant (ϕ, say), he did not encounter the first of these two problems in precisely the form just discussed. Instead, he was faced by an analogous term $\langle\phi^-u^-u^-\rangle_c$, which he treated as part of a new scalar diffusion equation. This new equation then exhibits form invariance under RG transformation, hence reaching a fixed point for the renormalized diffusivity. But of course, his procedure still involves an arbitrary truncation, which has to be repeated in every cycle of iteration in order to preserve the form invariance. Evidently the problem with the triple nonlinearity is a serious obstacle in the way of pursuing a straightforward perturbation theory.

A way around this difficulty was given by McComb [1],

who introduced a variant of Reynolds averaging in which the equation of motion is first conditionally averaged term by term and then subtracted from the unaveraged equation. The result is an equation for the high-k modes that does not contain a term nonlinear in the low-k modes, and hence the problem of the triple nonlinearity, as outlined above, does not arise. This method was called *iterative averaging* and gave some reasonable results, in that the recursion relation for the effective viscosity reached a fixed point and the theory gave an acceptable value for the Kolmogorov spectral constant. However, over the years it became clear that there was something amiss with the method. In particular, calculations revealed the unphysical feature that the values obtained depended on the rate at which the modes were eliminated (a feature it shared with Rose's method [8]). This behavior was attributed to an inadequate treatment of the relation between the conditional average and the ensemble average [10], and it was later found that this was indeed part of the explanation [2]. We shall not develop these points any further here, as in subsequent sections we shall be presenting the correct form of our theory with full mathematical rigor.

We shall close this section with a few general remarks about the work that has been discussed and its relationship to other more mainstream work in fluid turbulence. It should be appreciated that research in turbulence exists as a discipline in its own right (as a branch of fluid mechanics). Yet, in the area that is the subject of this paper, we have the situation that the impetus originally came from various branches of theoretical physics, including critical phenomena and quantum field theory, where theorists have been interested in possible new applications for techniques that have proved successful in their own often overcrowded fields. One result of this is that beginning with the pioneering paper by FNS, publications in the general area of RG field-theoretic methods and mode-coupling formalisms as applied to turbulence have almost invariably been couched in specialized and arcane jargon, to such an extent that they are not readily accessible to theorists working in the field of turbulence.

In contrast, the paper by Rose [8] is a readable and intelligible account of how renormalization group might *in principle* be applied to the subgrid modeling problem in turbulence. It could also be argued that having eschewed the artificial simplicities of stirred hydrodynamics, Rose was forced to introduce concepts such as conditional averages and Markovian approximations in order to make progress with his analysis.

We are not, of course, saying that these concepts were new to turbulence research at that time: such tools had been in use since the early 1960s in research on intermittency and coherent structures and also later in certain phenomenological theories (see, in particular, the paper by Tchen [11]). Nevertheless, their introduction in the context of renormalization group has undoubtedly proved helpful to later workers in the field. Accordingly, it is arguable that Rose's paper is of interest because it has made some of the basic ideas of RG accessible to a wider audience; it has also, even if indirectly, drawn attention to some of the fundamental problems.

III. MODE ELIMINATION USING A CONDITIONAL AVERAGE

Our purpose in this section is to formulate the operation of taking a conditional average and then to use this average in such a way that we can eliminate the triple nonlinearity referred to in the previous section. A full treatment of the formalism has been given elsewhere [3], and we shall concentrate here on the application of the method.

The basic idea is quite simple. Putting it at its most basic level, we select from the full ensemble of turbulence realizations a subensemble, the members of which have their low-k modes equal to $u_\alpha^-(\mathbf{k},t)$. Then we perform averages of functions of the $u_\alpha^+(\mathbf{k},t)$ over this subensemble.

However, there is more to it than this. If $\mathbf{u}(\mathbf{k},t)$ is the solution of the Navier Stokes equation, corresponding to prescribed boundary conditions, then we are faced with (in principle) a deterministic process and to prescribe $\mathbf{u}^-$ is to prescribe $\mathbf{u}^+$. That is to say, if $\mathbf{u}^-$ is invariant under our conditional average, then so also is $\mathbf{u}^+$. In order to get around this problem, we invoke the defining characteristic of deterministic chaos. This is to the effect that any uncertainty in the specification of the system will be amplified exponentially, so that as time goes on the difference between almost identical solutions will increase to the point of unpredictability.

In the present case, we replace the concept of time going on by the number of steps of the cascade in wave number. That is, our ideas about the turbulent cascade, and particularly ideas about localness of energy transfer, suggest that if we prescribe conditions at wave number k_1, then $\mathbf{u}^+(\mathbf{k}_0,t)$ will be unaffected provided that k_0 is much larger than k_1. In other words, as the bandwidth becomes large ($\lambda\to 1$), the conditional average of u^+ becomes free of constraint and we can expect that

$$\langle u^+(\mathbf{k}_0,t)\rangle_c \to \langle u^+(\mathbf{k}_0,t)\rangle \ .$$

On the other hand, for $k_0\to k_1$, it is intuitively clear that the conditional average must tend to become effectively deterministic, with

$$\langle u^+(\mathbf{k}_0,t)\rangle_c \to u^+(\mathbf{k}_0,t) \ .$$

We shall return to this at a later stage, but for the present it should be appreciated that the point at issue here is one of phase correlation. If two realizations are slightly out of phase at wave number k_1, then we can (given the nature of turbulence) expect that this phase difference will amplify exponentially so that phase correlation will decline throughout the band as one moves from k_1 to k_0. Hence, for $u^+(\mathbf{k}_0,t)$ to be chaotic, despite the prescription $u^+(\mathbf{k}_1,t)=u^-(\mathbf{k}_1,t)$, it is a requirement that the bandwidth be large enough. This imposes a lower bound on possible values of the bandwidth parameter λ.

We may take account of these various aspects by introducing a fuzzy criterion for our conditional average. We now choose as our subensemble the subset of realizations for which the low-k modes differ from $\mathbf{u}^-(\mathbf{k},t)$ by a small

amount $\Phi^-(\mathbf{k},t)$. Obviously Φ^- is an arbitrary criterion (apart from the constraint that $u^- + \Phi^-$ must be a possible solution of the Navier-Stokes equation) and should be chosen to satisfy the conditions that we wish to impose upon our average. We shall introduce these formally in the next section. But here we should lay emphasis on two points. First, the function Φ^- is arbitrarily chosen and so is independent of u^-; hence we have

$$\langle u^- \Phi^- \rangle_c = \langle u^- \rangle_c \langle \Phi^- \rangle_c = 0 \ . \tag{25}$$

Second, this does not imply a similar independence between Φ^- and u^+, and in general we must have

$$\langle u_\alpha^-(\mathbf{k},t) u_\beta^+(\mathbf{k}',t') \rangle_c = u_\alpha^-(\mathbf{k},t) \langle u_\beta^+(\mathbf{k}',t') \rangle_c + \langle \Phi_\alpha^-(\mathbf{k},t) u_\beta^+(\mathbf{k}',t) \rangle_c \ . \tag{26}$$

A. Conditionally averaged equations for high and low wave numbers

Let us now denote the operation of taking a conditional average over the modes in the band $k_1 \le k \le k_0$ by angle brackets, with a subscript 0. This change of notation permits the subsequent generalization to subscripts $1,2,\ldots,n$, as we remove shells of wave numbers progressively. Then we list the ideal defining properties of the conditional average as

$$\langle \mathbf{u}^-(\mathbf{k},t) \rangle_0 = \mathbf{u}^-(\mathbf{k},t) \ , \tag{27}$$

$$\langle \mathbf{u}^-(\mathbf{k},t) \mathbf{u}^-(\mathbf{k}',t) \rangle_0 = \mathbf{u}^-(\mathbf{k},t) \mathbf{u}^-(\mathbf{k}',t) \ , \tag{28}$$

and so on, for products of the low-wave-number modes of any order.

In order to satisfy Eq. (27), we choose $\Phi_\alpha^-(\mathbf{k},t)$ such that

$$\langle \Phi_\alpha^-(\mathbf{k},t) \rangle_0 = 0 \ , \tag{29}$$

but Eq. (28) can only be satisfied as an approximation, and requires $\Phi_\alpha^-(\mathbf{k},t)$ to be small, in the sense that we can write

$$\langle \Phi_\alpha^-(\mathbf{k},t) \Phi_\beta^-(\mathbf{k}',t) \rangle_0 \simeq 0 \ . \tag{30}$$

Thus invariance of products of low-wave-number modes under conditional averaging holds only to the second order of small quantities, and in practice Eq. (28) must be rewritten as

$$\langle \mathbf{u}^-(\mathbf{k},t) \mathbf{u}^-(\mathbf{k}',t) \rangle_0 = \mathbf{u}^-(\mathbf{k},t) \mathbf{u}^-(\mathbf{k}',t) + O(\langle \mathbf{\Phi}^- \mathbf{\Phi}^- \rangle_0) \ . \tag{31}$$

We now conditionally average both Eqs. (12) and (13), which are, respectively, the low-k and high-k filtered Navier-Stokes equations.

First, we obtain the conditionally averaged NSE on the interval $0 \le k \le k_1$ by replacing each $\mathbf{u}^-(\mathbf{k},t)$ with $\mathbf{u}^-(\mathbf{k},t) + \mathbf{\Phi}^-(\mathbf{k},t)$ in Eq. (12), and average according to Eqs. (27) and (31) to obtain

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^-(\mathbf{k},t) = M_{\alpha\beta\gamma}^-(\mathbf{k}) \int d^3j \{ u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) + 2u_\beta^-(\mathbf{j},t) \langle u_\gamma^+(\mathbf{k}-\mathbf{j},t) \rangle_0 + 2\langle \Phi_\beta^-(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \rangle_0 + \langle u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \rangle_0 + O(\langle \Phi^- \Phi^- \rangle_0) \} \ . \tag{32}$$

We now repeat the steps just taken, but this time we apply them to Eq. (13) for the high-k modes. Thus replacing $\mathbf{u}^-$ by $\mathbf{u}^- + \mathbf{\Phi}^-$ in Eq. (13), we get

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^+(\mathbf{k},t) = M_{\alpha\beta\gamma}^+(\mathbf{k}) \int d^3j \{ u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) + 2u_\beta^-(\mathbf{j},t) \Phi_\gamma^-(\mathbf{k}-\mathbf{j},t) + \Phi_\beta^-(\mathbf{j},t) \Phi_\gamma^-(\mathbf{k}-\mathbf{j},t) + 2u_\beta^-(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) + 2\Phi_\beta^-(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) + u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \} \ , \tag{33}$$

and taking the conditional average of each term according to Eqs. (27) and (31) gives

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] \langle u_\alpha^+(\mathbf{k},t) \rangle_0 = M_{\alpha\beta\gamma}^+(\mathbf{k}) \int d^3j \{ u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) + \langle \Phi_\beta^-(\mathbf{j},t) \Phi_\gamma^-(\mathbf{k}-\mathbf{j},t) \rangle_0 + 2u_\beta^-(\mathbf{j},t) \langle u_\gamma^+(\mathbf{k}-\mathbf{j},t) \rangle_0 + 2\langle \Phi_\beta^-(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \rangle_0 + \langle u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \rangle_0 \} \ . \tag{34}$$

This may seem a rather laborious proceeding, but in fact we shall need both of these high-wave-number equations in the next section.

B. Elimination of the triple nonlinearity

We now address the first of the mode-coupling problems that were outlined in Sec. II: see, in particular, Eq. (24). This arises from the term quadratic in the low-k modes, which occurs in Eq. (33). We rearrange (34), which is the conditionally averaged form of (33), to obtain an expression for $M_{\alpha\beta\gamma}^+(\mathbf{k}) \int d^3j \, u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t)$, thus

$$M^{+}_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u^{-}_{\beta}(\mathbf{j},t)u^{-}_{\gamma}(\mathbf{k}-\mathbf{j},t)=\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]\langle u^{+}_{\alpha}(\mathbf{k},t)\rangle_0$$
$$-M^{+}_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\{\langle\Phi^{-}_{\beta}(\mathbf{j},t)\Phi^{-}_{\gamma}(\mathbf{k}-\mathbf{j},t)\rangle_0+2u^{-}_{\beta}(\mathbf{j},t)\langle u^{+}_{\gamma}(\mathbf{k}-\mathbf{j},t)\rangle_0$$
$$+2\langle\Phi^{-}_{\beta}(\mathbf{j},t)u^{+}_{\gamma}(\mathbf{k}-\mathbf{j},t)\rangle_0+\langle u^{+}_{\beta}(\mathbf{j},t)u^{+}_{\gamma}(\mathbf{k}-\mathbf{j},t)\rangle_0\}\ . \tag{35}$$

We then substitute this back into Eq. (33) to get

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u^{+}_{\alpha}(\mathbf{k},t)=M^{+}_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\{2u^{-}_{\beta}(\mathbf{j},t)u^{+}_{\gamma}(\mathbf{k}-\mathbf{j},t)+u^{+}_{\beta}(\mathbf{j},t)u^{+}_{\gamma}(\mathbf{k}-\mathbf{j},t)\}+H_{\alpha}(\mathbf{k},t)\ , \tag{36}$$

where $H_{\alpha}(\mathbf{k},t)$ is given by

$$H_{\alpha}(\mathbf{k},t)=M^{+}_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\{-2u^{-}_{\beta}(\mathbf{j},t)\langle u^{+}_{\gamma}(\mathbf{k}-\mathbf{j},t)\rangle_0-\langle u^{+}_{\beta}(\mathbf{j},t)u^{+}_{\gamma}(\mathbf{k}-\mathbf{j},t)\rangle_0+2u^{-}_{\beta}(\mathbf{j},t)\Phi^{-}_{\gamma}(\mathbf{k}-\mathbf{j},t)$$
$$+2\Phi^{-}_{\beta}(\mathbf{j},t)u^{+}_{\gamma}(\mathbf{k}-\mathbf{j},t)-2\langle\Phi^{-}_{\beta}(\mathbf{j},t)u^{+}_{\gamma}(\mathbf{k}-\mathbf{j},t)\rangle_0+\Phi^{-}_{\beta}(\mathbf{j},t)\Phi^{-}_{\gamma}(\mathbf{k}-\mathbf{j},t)$$
$$-\langle\Phi^{-}_{\beta}(\mathbf{j},t)\Phi^{-}_{\gamma}(\mathbf{k}-\mathbf{j},t)\rangle_0\}+\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]\langle u^{+}(\mathbf{k},t)\rangle_0\ . \tag{37}$$

We shall show in the next section that the contribution from $H_{\alpha}(\mathbf{k},t)$ is at the second order of small quantities. We conclude here by noting that the corresponding procedure used over the years in the method of iterative averaging [1,10] could be criticized as being heuristic in the sense that the Reynolds-style decomposition used would really require one to Fourier transform to physical space time, carry out the derivation, and then transform back. In fact this was how it was done in the original reference, but the present method can claim full mathematical rigor.

IV. FIRST-SHELL ELIMINATION USING THE TWO-FIELD DECOMPOSITION

Our objective now is to solve Eq. (36) for the conditional average $\langle u^{+}_{\beta}(\mathbf{j},t)u^{+}_{\gamma}(\mathbf{k}-\mathbf{j},t)\rangle_0$ and substitute the result back into (32) in order to have a closed equation for the low-wave-number modes. In order to do this, we shall ultimately have to reckon with the need to relate conditional averages to full ensemble averages. Accordingly, we begin this section with the two-field decomposition that is our basis for this procedure.

Let us write the exact decomposition

$$u^{+}_{\alpha}(\mathbf{k},t)=v^{+}_{\alpha}(\mathbf{k},t)+\Delta^{+}_{\alpha}(\mathbf{k},t)\ , \tag{38}$$

where $v^{+}_{\alpha}(\mathbf{k},t)$ is any other realization of our turbulent ensemble. In other words, $v^{+}_{\alpha}(\mathbf{k},t)$ has exactly the same statistical properties as $u^{+}_{\alpha}(\mathbf{k},t)$, but has no phase relationship to $u^{-}_{\alpha}(\mathbf{k},t)$. It follows by definition that $\Delta^{+}_{\alpha}(\mathbf{k},t)$ is simply the phase difference (in the band of modes to be eliminated) between the two realizations. It also follows that from the point of view of the realization under study, $v^{+}_{\alpha}(\mathbf{k},t)$ is the purely chaotic part of the field and $\Delta_{\alpha}(\mathbf{k},t)$ is the correction field that carries all the phase information.

We are now in a position to write down an expression relating the conditional average of the high-wave-number part of the velocity field to its full ensemble average. Taking the conditional average of both sides of Eq. (38), it may be shown [3] or it is intuitively obvious that

$$\langle u^{+}_{\alpha}(\mathbf{k},t)\rangle_0=\langle v^{+}_{\alpha}(\mathbf{k},t)\rangle+\langle\Delta^{+}_{\alpha}(\mathbf{k},t)\rangle_0\ . \tag{39}$$

Now we seek a relationship between $\mathbf{v}^{+}$ and $\mathbf{u}^{+}$, which is such that the conditionally averaged correction term $\langle\Delta^{+}\rangle_0$ may be neglected as small. In other words, we need an ansatz for the correction term and naturally this will depend on the physical nature of the system that we are studying.

In the case of macroscopic fluid turbulence, we are guided by the well-established idea that turbulent energy transfer in wave number takes the form of a cascade and is therefore to some extent local in wave number. In terms of our present approach, we take this to mean that in any particular realization, the mode-mode coupling is short range in k space. Thus on a statistical picture, based on many such realizations, modes that are widely separated may be taken to be independent of each other. Hence, providing that the bandwidth parameter λ as defined by Eq. (8) is not too small, we can assume that $\mathbf{u}^{+}(\mathbf{k}_0,t)$ is independent of $\mathbf{u}^{+}(\mathbf{k}_1,t)$ in the sense that we can write

$$\langle\mathbf{u}^{+}(\mathbf{k}_0,t)\rangle_0=\langle\mathbf{u}^{+}(\mathbf{k}_0,t)\rangle=\langle\mathbf{v}^{+}(\mathbf{k}_0,t)\rangle\ , \tag{40}$$

where the last step follows from the definition of v^{+}, as another realization of the turbulence ensemble with the same statistical properties as u^{+}, but with no phase relationship to u^{-}.

This now leads us toward a natural ansatz for the relationship between $\mathbf{v}^{+}$ and $\mathbf{u}^{+}$. Relying on the fact that we are dealing with a problem in continuum mechanics, we take $v^{+}_{\alpha}(\mathbf{k},t)$ to be given by a first-order truncation of the expansion of $u^{+}_{\alpha}(\mathbf{k},t)$ in Taylor series about $\mathbf{k}=\mathbf{k}_0$ thus

$$v^{+}_{\alpha}(\mathbf{k},t)=u^{+}_{\alpha}(\mathbf{k}_0,t)+(\mathbf{k}-\mathbf{k}_0)\cdot\nabla_k u^{+}_{\alpha}(\mathbf{k},t)|_{k=k_0}+O(\lambda^2)\ . \tag{41}$$

Note that we conclude that terms of order λ^2 have been neglected because the maximum value of $|\mathbf{k}-\mathbf{k}_0|$ is λk_0.

Hence from Eqs. (39) and (41), it follows that we have

$$\langle \Delta^+(\mathbf{k},t)\rangle_0 = O(\lambda^2) \ . \tag{42}$$

It also follows, therefore, that we are simultaneously imposing both upper and lower bounds on acceptable values of λ. On the one hand, λ must be large enough for us to assume that $u^+(\mathbf{k}_0,t)$ is independent of $u^+(\mathbf{k}_1,t)$, while on the other hand, λ must be small enough for us to neglect terms that are of order λ^2 in Eq. (41).

A. Two-field decomposition of the low- and high-wave-number equations

Let us begin with the high-wave-number equation. We substitute $u^+ = v^+ + \Delta^+$, as in (38), into Eq. (32)

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^-(\mathbf{k},t) = M^-_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \{ u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) + 2u_\beta^-(\mathbf{j},t)\langle \Delta_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$$
$$+2\langle \Phi_\beta^-(\mathbf{j},t)\Delta_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 + \langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 + O(\langle \Phi^-\Phi^-\rangle_0)\} \ . \tag{43}$$

From (42) and (30) the terms $\langle \Delta^+\rangle_0$ and $\langle \Phi^-\Delta^+\rangle_0$ are assumed $O(\lambda^2)$ at lowest order, and are discarded, leaving

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^-(\mathbf{k},t) = M^-_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \{ u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) + \langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$$
$$+O(\langle \Delta^+\rangle_0, \langle \Phi^-\Phi^-\rangle_0, \langle \Phi^-\Delta^+\rangle_0)\} \ . \tag{44}$$

We should note that we are now left with an equation that differs from the Navier-Stokes equation (to order λ) only by the presence of the nontrivial mode-coupling term. We may eliminate this by using Eq. (36) to form an equation for $M^-_{\alpha\beta\gamma}(\mathbf{k})\int d^3j \langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$. To do this we take the following steps.

(i) Rewrite Eq. (36) for $[(\partial/\partial t)+\nu_0 j^2]u_\beta^+(\mathbf{j},t)$ on the left-hand side and multiply it through by $u_\gamma^+(\mathbf{k}-\mathbf{j},t)$.

(ii) Rewrite Eq. (36) for $[(\partial/\partial t)+\nu_0|\mathbf{k}-\mathbf{j}|^2]u_\gamma^+(\mathbf{k}-\mathbf{j},t)$ and multiply it by $u_\beta^+(\mathbf{j},t)$.

(iii) Add the two equations formed by steps 1 and 2 and take the conditional average. The result is a first-order differential equation in the time variable for the quantity $\langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$.

(iv) Solve the differential equation using an integrating factor.

(v) Integrate with respect to j and multiply by $M^-_{\alpha\beta\gamma}(\mathbf{k})$.

(vi) Use the symmetry properties of $M^-_{\alpha\beta\gamma}(\mathbf{k})$ under $\beta\leftrightarrow\gamma$ and symmetry of the integral under $\mathbf{j}\leftrightarrow\mathbf{k}-\mathbf{j}$ to regroup terms.

This procedure yields

$$M^-_{\alpha\beta\gamma}(\mathbf{k})\int d^3j \langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$$
$$= 2M^-_{\alpha\beta\gamma}(\mathbf{k})\int d^3j \int_{-\infty}^{t} dt' \exp[-(\nu_0 j^2+\nu_0|\mathbf{k}-\mathbf{j}|^2)(t-t')] M^+_{\beta\delta\epsilon}(\mathbf{j})$$
$$\times \int d^3p \{2u_\delta^-(\mathbf{p},t')\langle u_\epsilon^+(\mathbf{j}-\mathbf{p},t')u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0 + \langle u_\delta^+(\mathbf{p},t')u_\epsilon^+(\mathbf{j}-\mathbf{p},t')u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0\}$$
$$+2M^-_{\alpha\beta\gamma}(\mathbf{k})\int d^3j \int_{-\infty}^{t} dt' \exp[-(\nu_0 j^2+\nu_0|\mathbf{k}-\mathbf{j}|^2)(t-t')]\langle H_\beta(\mathbf{j},t')u_\gamma^+(\mathbf{k}-\mathbf{j},t')\rangle_0 \ . \tag{45}$$

The term $\langle H_\beta(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$ can be shown to be $O(\langle \Delta^+\rangle_0)=O(\lambda^2)$ at lowest order, from (42). Recalling from Eq. (37) the form of $H_\alpha(\mathbf{k},t)$, we obtain

$$\langle H_\beta(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 = M^+_{\beta\delta\epsilon}\int d^3p \{ -2u_\delta^-(\mathbf{p},t)\langle u_\epsilon^+(\mathbf{j}-\mathbf{p},t)\rangle_0\langle u_\gamma^+(\mathbf{k}-\mathbf{j},t\rangle_0$$
$$-\langle u_\delta^+(\mathbf{p},t)u_\epsilon^+(\mathbf{j}-\mathbf{p},t)\rangle_0\langle u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$$
$$+2u_\delta^-(\mathbf{p},t)\langle \Phi_\epsilon^-(\mathbf{j}-\mathbf{p},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$$
$$+2\langle \Phi_\delta^-(\mathbf{p},t)u_\epsilon^+(\mathbf{j}-\mathbf{p},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$$
$$+2\langle \Phi_\delta^-(\mathbf{p},t)u_\epsilon^+(\mathbf{j}-\mathbf{p},t)\rangle_0\langle u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$$
$$+\langle \Phi_\delta^-(\mathbf{p},t)\Phi_\epsilon^-(\mathbf{j}-\mathbf{p},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0$$
$$+\langle \Phi_\delta^-(\mathbf{p},t)\Phi_\epsilon^-(\mathbf{j}-\mathbf{p},t)\rangle_0\langle u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0\}$$
$$+\langle u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0\left[\frac{\partial}{\partial t}+\nu_0 j^2\right]\langle u_\beta^+(\mathbf{j},t)\rangle_0 \ . \tag{46}$$

We again use the decomposition of the high-wave-number modes given in Eq. (38) and the properties of the $\mathbf{v}^+$ field that indicate that $\langle \mathbf{u}^+ \rangle_0 = \langle \Delta^+ \rangle_0$ and that $\langle \Delta^+ \rangle_0 = O(\lambda^2)$ is small; we also know that Φ^- is small from Eq. (30), so we can conclude that

$$\langle H_\beta(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \rangle_0 = O(\lambda^2) \tag{47}$$

at lowest order.

Using Eq. (38) once more, we decompose the term $\langle u_\epsilon^+ \mathbf{u}_\gamma^+ \rangle_0$ and retain only the leading-order term, discarding the terms of $O(\lambda^2)$. In this way, we end up with Eq. (46) in the form

$$\begin{aligned} M^-_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \langle u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \rangle_0 \\ = 2M^-_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \int_{-\infty}^{t} dt' \exp[-(\nu_0 j^2 + \nu_0 |\mathbf{k}-\mathbf{j}|^2)(t-t')] M^+_{\beta\delta\epsilon}(\mathbf{j}) \\ \times \int d^3p \{ 2u_\delta^-(\mathbf{p},t') \langle v_\epsilon^+(\mathbf{j}-\mathbf{p},t') v_\gamma^+(\mathbf{k}-\mathbf{j},t') \rangle + \langle u_\delta^+(\mathbf{p},t') u_\epsilon^+(\mathbf{j}-\mathbf{p},t') u_\gamma^+(\mathbf{k}-\mathbf{j},t') \rangle_0 \} \\ + O(\lambda^2) . \end{aligned} \tag{48}$$

Equation (44) for the explicit scales (to order λ^2) and Eq. (48) for the term representing mode coupling between the retained or explicit modes and the modes in the band or shell being eliminated (again, to order λ^2) jointly represent our first-shell elimination. In the next section, we develop a calculation in which the (conditional) mean effect of the eliminated modes is found to be an increased viscosity. In this way, turbulent collective interactions may be said to renormalize the bare molecular viscosity.

V. EFFECTIVE VISCOSITY AND THE RECURSION PROCEDURE

In carrying the work further, we have to make two approximations that seem to be justified on physical grounds (i.e., in terms of what we know about the properties of turbulence). These are what we have earlier described as "boundary-layer"-style approximations. That is to say, it may be helpful to draw a loose analogy between the relationship of the thin viscous boundary layer on (say) a flat plate to the free stream, and the relationship borne by the narrow band of modes in the viscous dissipation region to the remainder of the turbulent modes.

In the present case, the two essential features upon which to base such an approximation may be identified as follows. (i) In general, the velocity components in the band are very much smaller than the velocity components of the retained modes, and (ii) in general, the velocity modes in the band will evolve very much more rapidly (i.e., on shorter timescales) than the retained modes.

Of course, these statements become less true as one approaches (from above) the wave number dividing the retained modes from the explicit modes. However, bearing in mind that these approximations will appear under integral signs, it may be appropriate to state the basis for the proposed boundary-layer approximations as

$$\int [u_\alpha^+(\mathbf{k},t)]^2 d^3k \ll \int [u_\alpha^-(\mathbf{k},t)]^2 d^3k \tag{49}$$

and

$$\frac{\partial}{\partial t} \int u_\alpha^+(\mathbf{k},t) d^3k \gg \frac{\partial}{\partial t} \int u_\alpha^-(\mathbf{k},t) d^3k . \tag{50}$$

Now let us consider how to apply these ideas to the solution of Eq. (48). First, we note the presence of the triple moment of the form $\langle u^+ u^+ u^+ \rangle_0$. Application of the two-field decomposition to this term would give zero (due to homogeneity) at lowest order, and it is necessary to calculate this term iteratively. This applies to all orders of conditional moments in the band, and it is clear that the moment closure problem is still with us. However, in view of our first boundary-layer approximation, it seems reasonable on physical grounds to neglect $u^+ u^+ u^+$ in comparison to $u^- u^+ u^+$, and this we shall do.

Accordingly, we write Eq. (48) as

$$\begin{aligned} M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \langle u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \rangle_0 \\ = 4M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \int_{-\infty}^{t} dt' \exp[-\omega_2(j,l)(t-t')] M_{\beta\delta\epsilon}(\mathbf{j}) \int d^3p \, u_\delta^-(\mathbf{p},t') \langle v_\epsilon^+(\mathbf{j}-\mathbf{p},t') v_\gamma^+(\mathbf{k}-\mathbf{j},t') \rangle , \end{aligned} \tag{51}$$

where $\omega_2(j,l) = \nu_0(j^2+l^2)$, and $l = |\mathbf{k}-\mathbf{j}|$. Noting that the $\mathbf{v}^+$ field is homogeneous, isotropic, and stationary, we can also write

$$\begin{aligned} \langle v_\epsilon^+(\mathbf{j}-\mathbf{p},t) v_\gamma^+(\mathbf{k}-\mathbf{j},t) \rangle \\ = Q_v^+(\mathbf{k}-\mathbf{j}) D_{\epsilon\gamma}(\mathbf{k}-\mathbf{j}) \delta(\mathbf{k}-\mathbf{p}) , \end{aligned} \tag{52}$$

where $D_{\alpha\beta}(\mathbf{k}-\mathbf{j})$ is the projection operator defined by Eq. (3); this form is just an extension of Eq. (4) to the v^+ field, as indicated by the subscript on $Q_v^+(\mathbf{k}-\mathbf{j})$. Substituting from Eq. (52) into (51) then gives us

$$\begin{aligned} M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \langle u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \rangle_0 \\ = 4M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j M_{\beta\delta\epsilon}(\mathbf{j}) D_{\epsilon\gamma}(\mathbf{k}-\mathbf{j}) Q_v^+(|\mathbf{k}-\mathbf{j}|) \\ \times \int_0^\infty \exp[-\omega_2(j,l)\tau] u_\delta^-(\mathbf{k},t-\tau) d\tau . \end{aligned} \tag{53}$$

Now we make our second boundary-layer-type approximation, which is based on the assumption that the u^- are slowly varying on the time scales of the u^+. We do this by expanding $u_\delta^-(\mathbf{k},t-\tau)$ in a Taylor series in τ about $\tau=0$, and truncating at zero order, thus

$$\int_0^\infty \exp[-\omega_2(j,l)\tau]u_\delta^-(\mathbf{k},t-\tau)d\tau = \int_0^\infty \exp[-\omega_2(j,l)\tau][u_\delta^-(\mathbf{k},t)+O(\tau)]d\tau \ . \quad (54)$$

Thus with these two approximations, Eq. (48) for the mode-coupling term may be written in the form

$$M^-_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\langle u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\rangle_0 = 4M^-_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\frac{1}{\nu_0 j^2+\nu_0|\mathbf{k}-\mathbf{j}|^2}M^+_{\beta\delta\epsilon}(\mathbf{j}) \times\int d^3p u_\delta^-(\mathbf{p},t)Q_v(\mathbf{k}-\mathbf{j}) \times D_{\epsilon\gamma}(\mathbf{k}-\mathbf{j})\delta(\mathbf{k}-\mathbf{p}) \ . \quad (55)$$

The important thing to note about this result is that it is linear in the retained modes $\mathbf{u}^-$ and so may be interpreted in terms of an increment to the viscosity. We may make this interpretation as follows. Substitute the above result into Eq. (44) and write the equation for the retained modes as

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^-(\mathbf{k},t)-T_{\alpha\beta}(\mathbf{k})u_\beta^-(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t)+f_\alpha(\mathbf{k},t) \ , \quad (56)$$

where the form of the increment

$$T_{\alpha\beta}(k)u_\beta^-(k) \quad (57)$$

can be deduced by comparison with Eq. (55). For isotropic fields, it may be shown [12] that

$$T_{\alpha\beta}(k)u_\beta^-(k)=B(k)D_{\alpha\beta}(k)u_\beta^-(k) =B(k)u_\alpha^-(k) \quad (58)$$

with

$$B(k)=\frac{1}{d-1}\mathrm{Tr}T_{\alpha\beta}(k) \ , \quad (59)$$

for a d-dimensional system. Thus in $d=3$, we may make the replacement

$$T_{\alpha\beta}(\mathbf{k})u_\beta^-(\mathbf{k},t)=\tfrac{1}{2}T_{\beta\beta}(\mathbf{k})u_\alpha^-(\mathbf{k},t) \ , \quad (60)$$

and Eq. (56) may be written as

$$\left[\frac{\partial}{\partial t}+(\nu_0+\delta\nu_0)k^2\right]u_\alpha^-(\mathbf{k},t) =M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t)+f_\alpha(\mathbf{k},t) \ , \quad (61)$$

where the increment to the viscosity is given by

$$\delta\nu_0(k)=\frac{1}{k^2}\int d^3j\frac{L(\mathbf{k},\mathbf{j})Q_v^+(|\mathbf{k}-\mathbf{j}|)}{\nu_0 j^2+\nu_0|\mathbf{k}-\mathbf{j}|^2} \quad (62)$$

with

$$0\le k\le k_1; k_1\le j, |\mathbf{k}-\mathbf{j}|\le k_0 \ ,$$

and the coefficient $L(\mathbf{k},\mathbf{j})$ takes the form

$$L(\mathbf{k},\mathbf{j})=-2M_{\rho\beta\gamma}(\mathbf{k})M_{\beta\rho\delta}(\mathbf{j})D_{\delta\gamma}(\mathbf{k}-\mathbf{j}) =-\frac{[\mu(k^2+j^2)-kj(1+2\mu^2)](1-\mu^2)kj}{k^2+j^2-2kj\mu} \ , \quad (63)$$

where μ is the cosine of the angle between the vectors $\mathbf{k}$ and $\mathbf{j}$.

Alternatively, we can reexpress this result in terms of an effective viscosity, viz.,

$$\nu_1=\nu_0+\delta\nu_0 \ , \quad (64)$$

which acts on the left-hand side of Eq. (61).

Lastly, we may truncate the series for $\mathbf{v}^+$, as given by Eq. (41), at zero order,

$$v_\alpha^+(\mathbf{k},t)=u_\alpha^+(\mathbf{k}_0,t) \ , \quad (65)$$

or, at first order, thus

$$v_\alpha^+(\mathbf{k},t)=u_\alpha^+(\mathbf{k}_0,t)+(\mathbf{k}-\mathbf{k}_0)\cdot\nabla u_\alpha^+(\mathbf{k},t)|_{k=k_0} \ . \quad (66)$$

We shall mainly concentrate on the first order in this work.

A. Inductive treatment of the nth shell

We may extend the above procedure to further shells, as follows.

(i) Set $u_\alpha^-(\mathbf{k},t)=u_\alpha(\mathbf{k},t)$ in the equation for the explicit modes, so that we now have a new NSE with effective viscosity $\nu_1(k)$ for Fourier modes on the interval $0<k<k_1$.

(ii) Make the decomposition into retained modes and modes to be eliminated, but this time at $k=k_2$ such that $u_\alpha^+(\mathbf{k},t)$ is now defined in the band $k_2\le k\le k_1$.

(iii) Repeat the procedures used to eliminate the first shell of modes in order now to eliminate modes in the band $k_2\le k\le k_1$.

We define the nth shell in this procedure by

$$k_n=(1-\lambda)^n k_0, \quad 0\le\lambda\le 1 \ , \quad (67)$$

which is just a generalization of Eq. (8). Then by induction, relation (64) for the first-shell effective viscosity generalizes to

$$\nu_{n+1}(k)=\nu_n(k)+\delta\nu_n(k) \ . \quad (68)$$

Now, for the first shell, Eq. (62) for the viscosity increment evaluated to first order according to Eq. (66) is

$$\delta\nu_0(k)=\frac{1}{k^2}\int d^3j\frac{L(\mathbf{k},\mathbf{j})\left\{Q(l)|_{l=k_0}+(l-k_0)\dfrac{\partial Q(l)}{\partial l}\bigg|_{l=k_0}\right\}}{\nu_0 j^2+\nu_0|\mathbf{k}-\mathbf{j}|^2} \ , \quad (69)$$

where $l=|\mathbf{k}-\mathbf{j}|$. By induction, this is extended to the nth shell as

$$\delta\nu_n(k)=\frac{1}{k^2}\int d^3j\frac{L(\mathbf{k},\mathbf{j})\left\{Q(l)|_{l=k_n}+(l-k_n)\dfrac{\partial Q(l)}{\partial l}\bigg|_{l=k_n}\right\}}{\nu_n(j)j^2+\nu_n(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2}\ . \tag{70}$$

B. Conservation equations for the retained modes

With the progressive elimination of modes in the range $k_{n-1}\le k\le k_0$, Eq. (44) is iterated to the new form

$$\left[\frac{\partial}{\partial t}+\nu_n(k)k^2\right]u_\alpha^-(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\times\int d^3j\,u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t)+f_\alpha(\mathbf{k},t) \tag{71}$$

for wave numbers

$$0\le k,j,|\mathbf{k}-\mathbf{j}|\le k_n\ ,$$

where we again include a random force, as in Eq. (14). However, this time, the stirring force is not intended to act as a basis for perturbation theory. Rather, it is added purely to sustain the turbulence against viscous dissipation, so that we may study a stationary system. As it stands, Eq. (71) expresses conservation of momentum per unit mass of fluid for the retained modes.

We can then obtain the equation expressing conservation of turbulent kinetic energy in the retained modes per unit mass of fluid in the following way. Multiply each term in Eq. (71) by $u_\alpha^-(-\mathbf{k},t)$, and average over the full turbulent ensemble. Then sum over α, multiply through by $2\pi k^2$, and invoke (5) for the energy spectrum to obtain

$$\left[\frac{\partial}{\partial t}+2\nu_n(k)k^2\right]E(k,t)=2\pi k^2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\langle u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t)u_\alpha^-(-\mathbf{k},t)\rangle+2\pi k^2\langle f_\alpha(\mathbf{k},t)u_\alpha^-(-\mathbf{k},t)\rangle\ . \tag{72}$$

The second term on the right-hand side represents the rate at which the stirring forces do work on the fluid in sustaining the velocity field. It can be expressed in terms of the autocorrelation function $W(k)$ of the stirring force, as defined by Eq. (18), to yield the form [4]

$$\left[\frac{\partial}{\partial t}+2\nu_n(k)k^2\right]E(k,t)=2\pi k^2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\langle u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t)u_\alpha^-(-\mathbf{k},t)\rangle+4\pi k^2W(k)\ . \tag{73}$$

The choice of force spectrum is arbitrary, apart from the restrictions on its statistics, as specified in Sec. II. In particular, in order to have a stationary velocity field, the integral of $W(k)$ over all k must be equal to the dissipation rate: see Eq. (19). This is in fact our only interest in the stirring forces: so far as we are concerned, their sole purpose is to maintain the turbulence against the viscous dissipation. Accordingly, if we integrate each term in the above equation with respect to k and, for stationarity, put the time derivative equal to zero, we obtain

$$\int_0^{k_n}2\nu_n(k)k^2E(k)dk=\varepsilon\ . \tag{74}$$

This result follows from the symmetry property of the nonlinear term (which is necessary in order to conserve energy during inertial transfer). That is, when integrated with respect to k over the same range as the integration with respect to j, the inertial transfer term vanishes. It should also be noted that (74) is the renormalized version of the usual dissipation integral, as given in (6). The reduced upper limit on the integral is compensated for by the augmented turbulent effective viscosity.

C. Scaling relationships

We now assume that the energy spectrum in the band is given by a power law of the form

$$E(k)=\alpha\varepsilon'^r k^s\ , \tag{75}$$

and make the scaling transformation

$$k=k_{n+1}k'\ , \tag{76}$$

where k' is nondimensional,

$$k_{n+1}=hk_n\ , \tag{77}$$

and, for compactness, we define $h=(1-\lambda)$. This amounts to a choice of power-law behavior for the effective viscosity as well; and we shall shortly see that the power-law exponents for the spectrum and effective viscosity can be fixed by a combination of dimensional analysis and power counting.

We impose the obvious consistency requirement that $\nu_n(k)$ and $\delta\nu_n(k)$ should scale in the same way. Then the effective viscosity may be written in terms of the nondi-

mensional function $\tilde{\nu}_n$ from the equations for the nth-cycle increment and the nth-cycle NSE as

$$\nu_n(k_n k')=\alpha^{1/2}\varepsilon^{r/2}k_n^{(s-1)/2}\tilde{\nu}_n(k') . \tag{78}$$

Substitution of this equation into the renormalized dissipation relation as given by (74) fixes the exponents as $r=\frac{2}{3}$ and $s=-\frac{5}{3}$: the well-known Kolmogorov spectrum. With these results, we can scale all the relevant equations. First, the scaled effective viscosity (78) now takes the explicit form

$$\nu_n(k_n k')=\alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(k') . \tag{79}$$

Then we scale the recursion relation and the increment to viscosity in the following way. We start from the relation $\nu_{n+1}(k)=\nu_n(k)+\delta\nu_n(k)$, where $\delta\nu_n(k)$ is given by Eq. (70) and is restricted to $0\le k\le k_{n+1}$.

Now, Eq. (79) for the nth cycle can be extended to the $(n+1)$th cycle, thus

$$\nu_{n+1}(k)=\alpha^{1/2}\varepsilon^{1/3}k_{n+1}^{-4/3}\tilde{\nu}_{n+1}(k') , \tag{80}$$

and this now defines

$$k'=k_{n+1}^{-1}k . \tag{81}$$

Similarly, the increment for the nth cycle has the similarity solution

$$\begin{aligned}\nu_n(k)&=\alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(k/k_n)\\&=\alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(hk') .\end{aligned} \tag{82}$$

Scaling the viscosity increment, as given by (70), we find

$$\delta\nu_n(k)=\frac{1}{(k_{n+1}k')^2}\int d^3j'\frac{k_{n+1}^3}{4\pi}\frac{[k_{n+1}^2L(\mathbf{k}',\mathbf{j}')]\alpha\varepsilon^{2/3}k_{n+1}^{-11/3}Q'}{\alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}[\tilde{\nu}_n(hj')(k_{n+1}^2j'^2)+\tilde{\nu}_n(hl')(k_{n+1}^2l'^2)]} , \tag{83}$$

where $j=k_{n+1}j'$ and $l=|\mathbf{k}-\mathbf{j}|=k_{n+1}l'$, and where

$$Q'=h^{11/3}-\tfrac{11}{3}h^{14/3}(l'-h^{-1})+\text{higher-order terms} , \tag{84}$$

for the first-order approximation.

This gives an explicit form for the scaled increment as

$$\delta\nu_n(k_{n+1}k')=\alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}h^{-8/3}\delta\tilde{\nu}_n(k') , \tag{85}$$

where

$$\delta\tilde{\nu}_n(k')=\frac{1}{4\pi k'^2}\int d^3j'\frac{L(\mathbf{k}',\mathbf{j}')Q'}{\tilde{\nu}_n(hj')j'^2+\tilde{\nu}_n(hl')l'^2} \tag{86}$$

for the wave-number bands

$$0\le k'\le 1;1\le j',l'\le h^{-1} .$$

Now, with the substitution of (80), (82), and (83) into (68), the unscaled recursion relation becomes

$$\begin{aligned}&\alpha^{1/2}\varepsilon^{1/3}k_{n+1}^{-4/3}\tilde{\nu}_{n+1}(k')\\&\quad=\alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}\tilde{\nu}_n(hk')\\&\qquad+\alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}h^{-8/3}\delta\tilde{\nu}_n(k') ,\end{aligned} \tag{87}$$

and since, from (77), $k_{n+1}=hk_n$, we have

$$h^{-4/3}\tilde{\nu}_{n+1}(k')=\tilde{\nu}_n(hk')+h^{-8/3}\delta\tilde{\nu}_n(k') . \tag{88}$$

It follows that the scaled recursion relation can be written

$$\tilde{\nu}_{n+1}(k')=h^{4/3}\tilde{\nu}_n(hk')+h^{-4/3}\delta\tilde{\nu}_n(k') . \tag{89}$$

Note that this is different from the earlier form, as used in iterative averaging [1,9].

D. Calculations and results

The basic calculation is the iteration of the recursion relation, as given by Eq. (89), with the increment to the viscosity given by (86). The fixed point is defined by the condition

$$\tilde{\nu}_{n+1}(k')=\tilde{\nu}_n(k')\equiv\tilde{\nu}_N(k') . \tag{90}$$

At the fixed point, Eq. (74) for the renormalized dissipation integral becomes

$$\varepsilon=\int_0^{k_N}2\nu_N(k)k^2E(k)dk . \tag{91}$$

Hence using the scaled form of the effective viscosity and the Kolmogorov form for the energy spectrum, we obtain a formula for the Kolmogorov spectral constant [10], thus

$$\alpha=\left\{2\int_0^1\tilde{\nu}_N(k')k'^{1/3}dk'\right\}^{-2/3} . \tag{92}$$

In fact, the calculation of the fixed point was done using $\omega_n(k)=\nu_n(k)k^2$ instead of $\nu_n(k)$. This cancels the difficult factor of $1/k^2$ in Eq. (70), which causes large rounding errors in the low-k' range of the integration. It avoids the introduction of an asymptotic formula for the integral that was used previously [9].

It is a simple matter to modify Eqs. (82), (89), and (86) to yield the scaling relation for $\omega_n(k)$ as

$$\omega_n(k_n k')=\alpha^{1/2}\epsilon^{1/3}k_n^{2/3}\tilde{\omega}_n(k') , \tag{93}$$

the recursion relation as

$$\tilde{\omega}_{n+1}(k')=h^{-2/3}\tilde{\omega}_n(hk')+h^{2/3}\delta\tilde{\omega}_n(k') , \tag{94}$$

and the scaled increment

$$\delta\tilde{\omega}_n(k')=\frac{1}{4\pi}\int d^3j'\frac{L(\mathbf{k}',\mathbf{j}')Q'}{\tilde{\omega}_n(hj')+\tilde{\omega}_n(hl')} \tag{95}$$

with the wave numbers confined in the usual way,

$$0 \le k' \le 1; 1 \le j', l' \le h^{-1},$$

where $l' = |\mathbf{k}' - \mathbf{j}'|$.

In order to carry out the numerical calculation, we made the k', j', and μ ranges discrete and calculated the increment to $\tilde{\omega}_n(k')$ by quadrature. Then we calculated $\tilde{\omega}_{n+1}(k')$ from the recursion relation, and checked for convergence of $\tilde{\omega}_n(k')$ to a fixed point. The chosen test for convergence was to check each value of $\tilde{\omega}_n(k')$ at each k' array point and only accept convergence to the fixed point when all values of $\tilde{\omega}_{n+1}(k')$ were within a tolerance of 0.1% of the corresponding $\tilde{\omega}_n(k')$ values.

As previously reported [2], our calculations (as tested by our theoretical value of the Kolmogorov constant α), showed some dependence on the bandwidth parameter for both small and large values of λ with a "plateau" in between, where α, as calculated from (92), was insensitive to the chosen bandwidth. Here we show some of the underlying details of the calculation. We begin with Fig. 1, which shows the iteration reaching a fixed point for a range of starting values of the molecular viscosity ($\nu_0 = 0.1$, 1, and 2). The results are given in terms of the scaled effective viscosity, as defined by Eq. (78), and the calculations are based on the first-order approximation [as in Eq. (66)] for v^+, at a plateau value of the bandwidth $\lambda = 0.36$. This illustrates the principle of universality, where the answers are independent of details of the system. The results depend only on the general dynamics of the inertial range. This fixed point is valid for any Reynolds number provided only that the Reynolds number is high enough for there to be an inertial range. This is to be expected, since the theory implicitly assumes that viscous effects play no part in the description of the eliminated modes, and these results confirm that view.

In Fig. 2, the development of the real (unscaled) effective viscosity is shown under the iteration from an initial value of 1. The results presented are again for the first order in λ and for a value of $\lambda = 0.36$, and thus lie in the plateau region. The final value of n given is where the scaled effective viscosity reaches the fixed point.

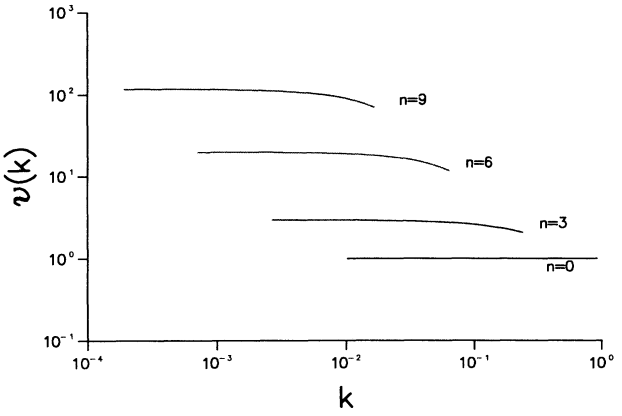

FIG. 2. Development of the total (unscaled) effective viscosity during the iteration.

As more modes are eliminated, so the effective viscosity rises. Also, it is clear that the effective viscosity shows the characteristic asymptotic trend to a constant value as k becomes small compared to the cutoff wave number. This behavior reflects, of course, the increasing validity of the concept of an eddy viscosity as the retained and eliminated scales become more widely separated. It should be noted that this graph employs a set of units in which both ν_0 and k_0 are equal to 1.

In Fig. 3, we return to the scaled eddy viscosity in order to illustrate the effect of varying the bandwidth parameter. Clearly the two extreme values of λ, for which results are shown here, are outside the range of validity of our theory. However, the different ways in which they behave is of interest. For wide bands ($\lambda = 0.74$), we know that our first-order truncation of the Taylor series in wave number will be less accurate. But the weaker dependence on the wave number observed in this case reflects the greater scale separation possible with a wide band, whereas for narrow bands ($\lambda = 0.1$) the dependence on wave number is markedly increased over a much wider range of k values, reflecting the fact that the conditional average is much more deterministic in this case.

It should be noted that the effective viscosity is depen-

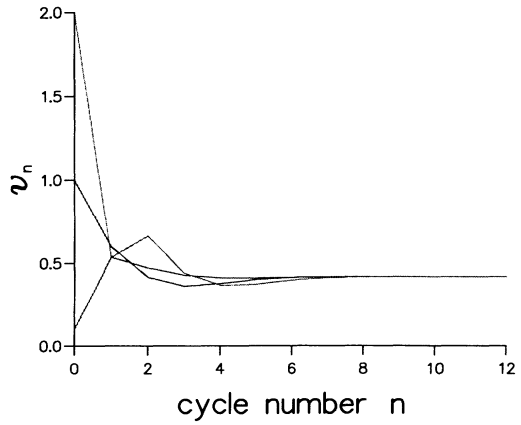

FIG. 1. Convergence of the scaled effective viscosity to the fixed point for several values of the initial molecular viscosity (value of scaled wave number: $k' = 0.01$).

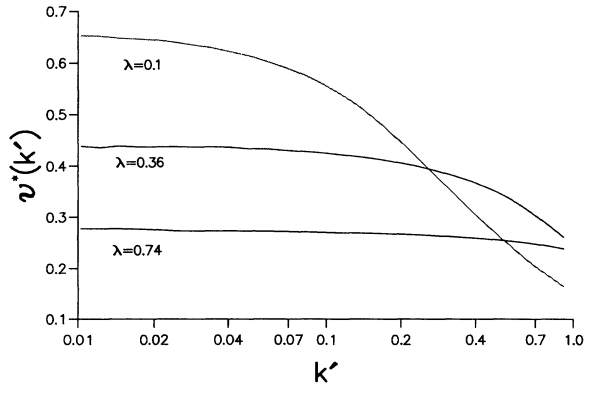

FIG. 3. Dependence of the scaled effective viscosity on wave number, showing the effect of varying the bandwidth parameter λ.

dent on λ over all λ values, and thus it is not possible to argue for the existence of a special value of the parameter from this data. However, as reported previously [2], our calculation of the Kolmogorov constant clearly indicates such a special region.

In Fig. 4, we show both the zeroth- and first-order approximations to our calculation of the spectral constant α. There is a region on both curves where the value of α is independent of the bandwidth. The first order shows a range of λ values for which the Kolmogorov constant is independent of λ much larger than that for the zeroth order, where the breakdown of the Taylor series affects the results at much smaller values of λ. The graphs show a tendency to converge as $\lambda \rightarrow 0$. A precise convergence would be expected if the Taylor series were the only approximation, but, the breakdown in our assumption of pure chaos affects the results in this region.

The arguments above lead to the selection of the value of the Kolmogorov constant from the plateau region. The first order yields a value of 1.60 ± 0.01 in the range $\lambda = 0.25-0.45$. The zeroth order yields a result of 1.62 ± 0.01 in the range $\lambda = 0.14-0.25$. These results lie in the generally accepted range of values for α, which is

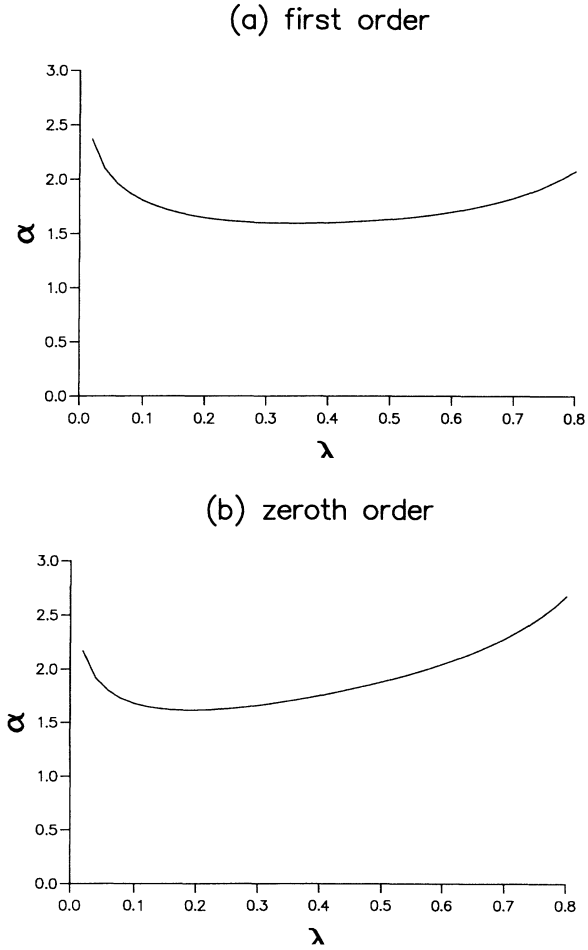

FIG. 4. Calculated values of the Kolmogorov spectral constant α, showing the dependence on the bandwidth parameter λ for (a) a first-order calculation, and (b) a zero-order calculation.

1.4–1.8. Clearly, the value of α does not represent a definitive test of a theory, but it seems fair to conclude that our theory is in good agreement with the experimental values for α.

Lastly, we note that the α value is not *exactly* independent of λ in our "plateau region," as the graph is the result of a numerical calculation. The results of the α calculation depend on the fixed point of the effective viscosity, which is only calculated to 0.1%. The error in the α value reflects this, and we feel that we are justified in calling this region a plateau to an accuracy of about 1%.

VI. DISCUSSION

The calculation presented here was based on the iteration of the scaled recursion relation as given by Eq. (89) along with Eq. (86) for the scaled increment. As we have seen, the iteration reaches a fixed point in the sense that the scaled effective viscosity $\tilde{\nu}_N(k')$ is unchanging for values of the iteration cycle number n greater than N. This procedure corresponds to the elimination of modes on $k_N \leq k \leq k_0$. The evolution equation for the retained modes is given by Eq. (71) with $n = N$, while the augmented viscosity $\nu_N(k)$ can be obtained from the scaled form $\tilde{\nu}_N(k')$ by means of Eq. (79).

The fact that the conditional mean effect of the eliminated modes has been shown to be linearly proportional to the retained modes justifies its representation in terms of an increment to the viscosity. (Indeed, this feature holds to all orders of approximation [10,12].) In turn, the renormalization-group calculation leads to the coefficient of effective turbulent viscosity. It can also be seen from our results that the interpretation in terms of a coefficient of viscosity holds good for wave numbers k much less than k_N. That is, $\nu_N(k)$ tends to a constant as $k \rightarrow 0$. However, as $k \rightarrow k_N$, we see that the effective viscosity depends to some extent on k, tending to roll off towards k_N.

Of course, there is nothing new about this. It has been known since the phenomenological theory of Heisenberg in the 1940s (see, for instance, [4] for a discussion) that the concept of an effective coefficient of viscosity must break down near the boundary between the retained and eliminated modes. This can be understood in terms of the analogy between the randomizing effect of molecular motions on the hydrodynamic modes (which are normally many orders of magnitude larger) and the randomizing effect of small turbulent eddies on larger turbulent eddies. Obviously, such an analogy must break down as $k \rightarrow k_N$, when the scale separation of the molecular case is entirely absent in its turbulent analog. The implications for $\mathbf{x}$ space are readily deduced from a consideration of the Fourier transform of the renormalized viscous term in Eq. (71), taken in conjunction with the convolution theorem. Evidently, the turbulent "constitutive relationship" cannot be Newtonian, except as an approximation, but this is not a surprise.

If we turn to qualitative assessment of our theory, then we face the situation that the art of numerical simulation of turbulence has not quite advanced to the point of providing a definitive form of the effective viscosity. Accord-

ingly, we have invoked the renormalized dissipation relation in order to calculate the Kolmogorov spectral constant, which we find to be [based on our first-order approximation in Eq. (86)] $\alpha=1.60\pm0.01$ for a range of bandwidths characterized by $0.25\leq\lambda\leq0.45$. This is a reasonable result, and indicates that the various error terms in our approximation scheme give only a small contribution to the dissipation integral. It is also perhaps worth emphasizing again that λ is not exactly a free parameter, as the theory must break down for either very small or very large values of the bandwidth: we shall expand on this shortly. However, within the range of validity, we note that the calculated value of α is insensitive to the value of λ to within 1%. We shall conclude the paper by discussing three aspects of the work, viz., the nature of the approximations made, the relationship of our theory to the current "two-fluid" picture of turbulence, and the potential of the work for practical calculations.

A. Approximations made

There is nothing mysterious or obscure about the various approximations made in this work. They are *all* rationally based on physical or mathematical grounds and all have been clearly stated. Nevertheless, it is probably worth recapitulating the underlying ideas here.

Let us begin by reminding ourselves that our approximations come in two categories. First, there are what we have called the boundary-layer-type approximations. These are to the effect that (i) the modes in the thin shell being eliminated give a small contribution to integrals, compared to the retained modes, and (ii) the Markovian approximation based on the idea that the shell modes are, on average, fully evolved on the time scales of the retained modes. They are set out in detail in Sec. V, in Eqs. (49) and (50), and are used to truncate the conditional moment expansion in the $\mathbf{u}^+$, as in the transition from Eq. (48) to Eq. (51), and to justify the truncation of the Taylor series for the time dependence of the retained modes in (54).

In terms of what we know about the rapid decay of turbulence at high wave numbers, these approximations seem likely to be good. Also, they are amenable to improvement by truncation at higher order [10,12]. However, the question of just how accurate they are is really a matter to be settled in conjunction with the application of our method to numerical simulation of the Navier-Stokes equations. Accordingly, this point will be deferred until we treat that aspect in a further paper.

As we showed in Sec. II, the above approximations are not accessible to straightforward perturbation theory. It is only our introduction of the combination of a conditional average and the two-field decomposition that allows us to invoke these boundary-layer-type approximations in the work presented here. In particular, we should note that the crisis that occurs at the cutoff wave number where $\mathbf{u}^-(\mathbf{k}_1,t)=\mathbf{u}^+(\mathbf{k}_1,t)$ is avoided because in our method all leading-order terms are evaluated in terms of the $\mathbf{v}^+$, and in general we have $\mathbf{u}^-(\mathbf{k}_1,t)\neq\mathbf{v}^+(\mathbf{k}_1,t)$. This brings us to our second type of approximation: when we attempt to relate conditional averages to unconditional ensemble averages. This arises with the introduction of the ansatz relating the $\mathbf{v}^+$ field to the $\mathbf{u}^+$ field by means of the Taylor series given in Eq. (41). The consequences of this approximation are restrictions on the permitted values of the bandwidth of the modes being eliminated. We shall discuss this in terms of the parameter λ and take the upper and lower bounds in turn.

The existence of an upper bound $\lambda_{\max}$, say, is easily seen. As we pointed out in connection with Eq. (41), truncation of the Taylor series at first order in wave number is equivalent (after rescaling) to neglecting terms of order λ^2 in the integral for the viscosity increment. The effect of this may be seen in Fig. 4, where we plot the calculated value of the Kolmogorov spectral constant against the bandwidth parameter λ for both zero-order and first-order approximations. It should also be noted that these two cases do not become identical as $\lambda\rightarrow0$. This is because of the existence of a lower cutoff $\lambda_{\min}$.

The existence of the lower limit on the bandwidth is rather less obvious, but at the same time has implications that are rather more profound. It arises, as we said earlier in Sec. III, because the band of modes being eliminated must be wide enough for the condition $\langle\mathbf{u}^+(\mathbf{k}_0,t)\rangle_0=\langle\mathbf{u}^+(\mathbf{k}_0,t)\rangle$ to be satisfied for small values of the arbitrary uncertainty Φ^-. This is necessary so that we can neglect terms of order $(\Phi^-)^2$ in, for example, Eq. (32). Evidently $\lambda_{\min}$ may be seen as a measure of the distance in wave-number space, over which phase correlations die away. Again, Fig. 4 shows clear evidence of the breakdown of the theory as the bandwidth tends to zero.

B. Relationship to the "two-fluid" picture of turbulence

The idea of a two-fluid model or picture has been a recurrent one in turbulence. Recently it has taken on new importance as part of the growing recognition of the need to bridge the gap between two of the main topics of turbulence research; viz., coherent structures on the one hand, and the energy spectrum on the other. A full discussion of these topics has been given elsewhere [4], but it will be of interest to make a few remarks here.

Research in coherent structures has its roots in observations ca. 1950 of quasiregular patterns in turbulent shear flows. The discovery of the remarkably regular roll vortices of the mixing layer, in the late 1960s, was the seed about which the subject crystallized; it is now a major activity in which the emphasis is on order rather than chaos. Its progress has largely relied on flow visualization, and this predisposes the subject toward a view based on the single realization in $\mathbf{x}$ space with a natural concentration on the concepts of *phase* and *phase coherence*. It is also the case that many turbulent coherent structures resemble the vortex forms of classical (inviscid) hydrodynamics, and this gives a growing bias towards a description of turbulence in terms of vorticity fields and, in particular, their topological properties. Indeed, the evolution equation for the vorticity (which is obtained by taking the curl of each term in the Navier-Stokes equation) provides a natural framework for a qualitative description of the energy cascade in terms of vortex stretching, with the transfer of energy to small eddies be-

ing accompanied by the formation of extended vortex filaments. Such vortex filaments are the coherent structures of isotropic homogeneous turbulence, and it has been known for decades that they are responsible for spatial intermittency or spottiness of the turbulence.

At the same time, from a different viewpoint, researchers have studied the energy spectrum of turbulence, where the emphasis is on a statistical view, based (in principle) on an ensemble of many realizations. The measurement of the spectrum has been the subject of much ingenuity; its presentation and analysis have emphasized its universality (at least, at large enough wave numbers), and it has been the theoretical target for some, ranging from engineers to quantum theorists. It has led, quite naturally, to a picture of turbulence as a random phenomenon, characterized by the transfer of energy through many degrees of freedom. Naturally, concepts of order and coherence never arise. As is well known, the energy spectrum is dependent only on the amplitudes of the harmonics of the velocity field, and not on their phases. That is, the combination of Fourier transformation of the two-point pair correlation of velocities and full ensemble averaging suppresses all phase information. Accordingly, such a description of the field is unable to distinguish between pure chaos (Gaussian) and coherent motion (non-Gaussian or intermittent).

However, it is when one considers higher-order moments that the spectral approach has to acknowledge the existence of coherent structures and, in particular, intermittency. The whole question of intermittency corrections to the power laws for spectra has formed an important interface between the two topics [4], although probably it is true to say that attempts to predict such corrections have suffered from their inability to take phase correlations into account.

At the present time, there seems to be a growing awareness of the need to take both phase and amplitude into account in spectral treatments of turbulence. As recent examples, one may instance the analytical demonstration that the presence of intermittency implies phase coherency [13], or the numerical demonstration that an artificial reduction of phase coherence in the spectral representation is equivalent to the destruction of the vortex filament stretching mechanism, which is the **x**-space picture of the energy cascade [14]. Results of this kind offer support to the view that turbulence must be seen in terms of the interplay between order (or coherent structures) and disorder (pure chaos). It is an entirely tenable point of view that neither aspect can be profitably studied any longer in complete isolation from the other.

Unfortunately, there are, as we have pointed out earlier, difficulties in the way of taking phase coherence into account in analytical spectral theories, and it is in just this area that our present work can offer an alternative methodology. If we consider the basic vortex stretching mechanism, then it is clear that there must be some degree of phase coherence between neighboring wave numbers. However, if we try to find out the extent of such a relationship in **k** space by considering the pair correlation, then Eq. (4) conveys the message that the ensemble average of such coherence is zero, which is of no help. Clearly (as we said earlier), the pair correlation in **k** space is not a suitable measure of this effect. In contrast, our *conditionally* averaged pair correlation of the $\mathbf{u}^+$ does not satisfy a relationship like (4). In fact, the requirement that this procedure must satisfy the constraints embodied in Eqs. (27) and (28) ensures that phase correlation is of the essence in this procedure. Our formalism then defines phase difference as the difference between realizations in the turbulent ensemble. Thereafter we have to find an ansatz in order to express our conditionally averaged pair correlation in terms of the ensemble-averaged form that satisfies (4). Our choice is the hypothesis that phase correlations die away quite quickly in wave number (a property that must be related in some way to the idea of "localness," which is associated with the energy cascade), but there may well be better ones.

C. Applications to real turbulent simulations

Ultimately any turbulent theory will be judged by its ability to assist with real problems involving the transport of scalar contaminants such as heat and mass, or involving flows that are anisotropic and inhomogeneous. Our present theory has been extended to the case of passive scalar convection in isotropic turbulence [12]. With appropriate restrictions, the spectrum in the inertial-convective range of wave numbers is found to take the usual Kolmogorov form with a calculated Corsin-Oboukhov constant of 1.02 ± 0.01 for the range $0.17 \leq \lambda \leq 0.33$. The theory has also been applied to the general case of shear flows, with appropriate generalizations to inhomogeneity and anisotropy. A full discussion of this additional work will be presented in a further paper.

*Electronic mail address: w.d.mccomb@uk.ac.ed

[1] W. D. McComb, Phys. Rev. A **26**, 1078 (1982).
[2] W. D. McComb and A. G. Watt, Phys. Rev. Lett. **65**, 3281 (1990).
[3] W. D. McComb, W. Roberts, and A. G. Watt, Phys. Rev. A **45**, 3507 (1992).
[4] W. D. McComb, *The Physics of Fluid Turbulence* (Clarendon, Oxford, 1990).
[5] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. Lett. **36**, 867 (1976).
[6] V. Yakhot and S. A. Orszag, J. Sci. Comput. **1**, 3 (1987).
[7] Marco Avellaneda and Andrew J. Majda, Commun. Math Phys. **131**, 381 (1990).
[8] H. A. Rose, J. Fluid Mech. **81**, 719 (1977).
[9] W. D. McComb and V. Shanmugasundaram, Phys. Rev. A **28**, 2588 (1983).
[10] W. D. McComb, in *Direct and Large Eddy Simulation of Turbulence,* Vol. 15 of *Notes on Numerical Fluid Mechanics*, edited by U. Schumann and R. Friedrich (Vieweg, Braunschweig, 1986), Vol. 15.
[11] C. M. Tchen, Phys. Fluids **16**, 13 (1973).
[12] A. G. Watt, Ph.D. thesis, University of Edinburgh, 1991.
[13] Tsutomu Sanada, Prog. Theor. Phys. **84**, 12 (1990).
[14] E. Levich, L. Shtilman, and A. V. Tur, Physica A **176**, 241 (1991).