Effects of additive noise at the onset of Rayleigh-Bénard convection

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The effects of additive Gaussian white noise on the onset of Rayleigh-Bénard convection are studied by means of a phenomenological model, the stochastic Swift-Hohenberg equation. The strength of the noise term arising from thermal fluctuations is given for both free-slip and rigid horizontal boundaries. As was already pointed out by previous authors this term contains the small parameter $k_B T/\rho d \nu^2$, where ρ is the mass density, d the plate separation, and ν the kinematic viscosity. For typical liquids this parameter is of order 10^{-9}. Experiments involving fluctuation effects may be interpreted in terms of this model if the noise strength is treated as an adjustable parameter, which turns out to be larger than the typical thermal value by four orders of magnitude. The effects of fluctuations on the bifurcation of an infinite system are studied, and the earlier arguments of the present authors leading to a first-order transition are reviewed [Swift and Hohenberg, Phys. Rev. A **15**, 319 (1977)]. The conditions under which the multimode model can be approximated by a single-mode stochastic amplitude equation are investigated, and an earlier analytic approximation scheme for calculating the response to a time-dependent Rayleigh number is applied to the multimode model. A comparison with available experimental and numerical simulation data is presented.

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I. INTRODUCTION

Macroscopic pattern formation in hydrodynamics, e.g., Rayleigh-Bénard convection or Taylor-Couette flow, is usually discussed by means of the deterministic Navier-Stokes equations, or by suitable simplified models derived from them [1,2]. It is known, however, that these equations must in principle be supplemented by thermal noise terms to describe the fluctuation effects of molecular degrees of freedom [3]. On the other hand, certain experiments appear to require externally applied stochastic forces for a suitable physical interpretation [4]. It is thus reasonable to study the effects of external noise of arbitrary strength $F \ll 1$ on the hydrodynamic description of pattern-forming bifurcations, both to assess the influence of the known thermal noise and to model other possible sources of external noise. The present paper analyzes a simple phenomenological model of spatiotemporal pattern formation, the stochastic Swift-Hohenberg (SH) equation [5], and compares the results to presently available experiments on Rayleigh-Bénard convection [4] as well as to numerical simulations. Our primary aim is to review and clarify earlier work and to present the results in a form that will stimulate further experimental and theoretical research.

The earliest discussions of the effects of noise on Rayleigh-Bénard convection were by Zaitsev and Shliomis [6] and by Graham and co-workers [7,8], who derived the fundamental estimate of the dimensionless strength of thermal noise $F_{\text{th}} \sim k_B T/\rho d \nu^2$, where ρ is the mass density, d the plate separation, and ν the kinematic viscosity. This number represents the ratio of a microscopic to a macroscopic energy and is therefore very small (typically 10^{-9}), so that thermal noise can usually be neglected. The original calculations as well as subsequent ones by the present authors [5] were for the unrealistic free-slip boundary condition, but a later calculation by van Beijeren and Cohen [9] showed that a similar result held for the rigid case. The physical circumstances under which F_{th} can become larger have been discussed by Ahlers [10], but we shall not consider such cases here.

In 1977 the present authors introduced a simplified model [5] of the Rayleigh-Bénard instability (now known as the Swift-Hohenberg model), in order to understand the *critical behavior* associated with fluctuation effects at the bifurcation. We pointed out that although the critical region would be extremely small ($|R - R_c|/R_c \lesssim F^{2/3} \simeq 10^{-6}$), because of its symmetry the convective instability belongs to a new universality class, first discussed by Brazovskii [11]. The renormalization-group theory for this transition has never been systematically worked out, but the self-consistent solution of Brazovskii could be adapted to the relevant two-dimensional case and a fluctuation-induced first-order transition was found [5], with the size of the jump scaled by the small parameter $F^{2/3}$. Although the main current interest in the SH equation stems from its use by subsequent authors as a simplified *deterministic* model of spatiotemporal pattern formation near the convective threshold [12], the possibility of finding physical systems with a significantly larger thermal noise, and experimental evidence that non-negligible external noise (of unknown origin) is present at the Rayleigh-Bénard instability of ordinary fluids [13–15,4], warrant a reconsideration of the original stochastic SH model. The present paper will review the arguments of Ref. [5] and give a self-contained statement of the main results, correcting some minor errors and inconsistencies (and a large number of misprints), but not going beyond the basic approximations of the earlier work. One exception is that we will incorporate the estimate of E_{th} given by van Beijeren and Cohen [9] for the case of rigid horizontal boundaries, since the original work involved the unphysical approximation of free-slip boundaries. As it turns out, the expression for E_{th} is the same, when written in terms of the appropriate physical parameters.

As pointed out by Ahlers, Cross, Hohenberg, and Safran (ACHS) [13], there are physical situations under which even tiny fluctuating forces might be observable, mostly associated with the dynamic response of a system being swept through the threshold by external change of Rayleigh number $R(t)$. Experiments [13–16,4] involving linear ramps $[R(t)\sim R_0+R_1t]$ and sinusoidal modulation $[R(t)\sim R_0+R_1\cos\omega t]$ were successfully analyzed in terms of a stochastic amplitude equation [16], as well as the stochastic SH model, with the noise strength F treated as an adjustable parameter. Its value turned out to be roughly four orders of magnitude *larger* than the predicted thermal noise strength $F_{th}\sim 10^{-9}$, and there is at present no convincing interpretation of the origin of the observed noise. Although there is general agreement on the orders of magnitude of the expected thermal noise and of the observed value of F, considerable confusion remains on the precise values and on the relationship between the stochastic SH model and a single-mode stochastic amplitude equation, due to unfortunate errors and misprints in earlier work [5,7,9,13], as well as to an abundance of different definitions and conventions. The present work seeks to clarify this situation.

The analysis of experiments in terms of stochastic models involves either analytic or numerical evaluation of the predictions of the model. Since even the one-mode equation demands considerable numerical effort [16] (in order to obtain a proper average over realizations of the stochastic force), and numerical solutions of the stochastic SH model have only recently been undertaken [17], it is useful to derive approximate analytic methods. Ahlers *et al.* [13] proposed a reduction of the multimode field equation to a one-mode amplitude equation, though it was later pointed out by van Beijeren and Cohen [9] that the evaluation of the effective one-mode noise amplitude $\bar{F}$ of ACHS was not applicable to a large aspect ratio system with many linear modes initially excited. This question is considered here once again, and we show how the effective single-mode noise $\bar{F}$ depends both on the geometry and on the strength of the noise, being only simple for small noise. Besides an evaluation of $\bar{F}$, ACHS also generalized an approximation first introduced by Suzuki [18], to obtain an analytic evaluation of the response of the one-mode model to an arbitrary time-dependent Rayleigh number $R(t)$. This approximation was found to be very poor under certain circumstances, however, and in a recent paper [16], the present authors and Ahlers reconsidered this problem. These authors introduced a slightly modified approximation scheme which agreed rather well with direct simulations of the one-mode model, for ascending and descending ramps and for sinusoidal modulation of R. The present paper generalizes this approximate treatment to the stochastic SH model, and makes comparisons with available experimental and numerical data. Up to now there is little to distinguish the quantitative success of the single-mode or multimode models, but it is hoped that the present work will stimulate more detailed investigations of the latter model.

In Sec. II the derivation of the stochastic SH model is outlined, starting from the equations of hydrodynamics supplemented with fluctuating forces to represent thermal noise. The dimensionless strength of the thermal noise at the convective threshold F_{th} is obtained for free-slip boundaries and quoted from the work of van Beijeren and Cohen [9] for the rigid case. The conditions under which the system may be approximately reduced to a one-mode equation are spelled out and the corresponding value of the effective noise $\bar{F}$ derived. Section III summarizes in a self-contained way the results of SH using the Brazovskii approximation to obtain a fluctuation-induced first-order transition in an infinite system. The ensuing linear correlation function near threshold is also presented. In Sec. IV an approximate solution of the stochastic SH model generalizing the result of Ref. 16 is obtained for arbitrary time dependence of Rayleigh number $R(t)$. Comparisons of the theory with available experimental and numerical results are made in Sec. V.

II. THE STOCHASTIC SH MODEL AND THE SINGLE-MODE EQUATION

A. Stochastic SH model

The stochastic SH model was derived in Ref. [5] for a laterally infinite system with stress-free horizontal boundaries. The appropriate modification for rigid boundaries follows from the work of van Beijeren and Cohen [9]. For completeness we summarize the essential arguments here, and correct a number of errors in earlier work. The starting point is the set of Oberbeck-Boussinesq equations for the velocity $\mathbf{u}$ and temperature T, supplemented by Langevin noise terms to represent random molecular motion [5,3],

$$\partial_t\mathbf{u}+\mathbf{u}\cdot\tilde{\nabla}\mathbf{u}=-\tilde{\nabla}(P/\rho)+\nu\tilde{\nabla}^2\mathbf{u}-g\alpha T\hat{\mathbf{z}}+\tilde{\nabla}\cdot\mathbf{s}\ , \tag{2.1}$$

$$\tilde{\nabla}\cdot\mathbf{u}=0\ , \tag{2.2}$$

$$\partial_t T+\mathbf{u}\cdot\tilde{\nabla}T=\kappa\tilde{\nabla}^2T-\tilde{\nabla}\cdot\mathbf{q}_T\ , \tag{2.3}$$

where P is the pressure, ρ the density, ν the kinematic viscosity, g the acceleration of gravity, α the thermal expansion, κ the thermal diffusivity, and $\tilde{\nabla}=(\nabla,\partial_z)$ the three-dimensional gradient, with respect to the vector $(x,y,z)=(\mathbf{x},z)$. The noise terms are assumed to possess a Gaussian distribution with correlations given by

$$\langle s_{ij}(\mathbf{x},z,t)s_{lm}(\mathbf{x}',z',t')\rangle=(k_BT/\rho)2\nu\delta(\mathbf{x}-\mathbf{x}')\times\delta(z-z')\delta(t-t')\times(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})\ , \tag{2.4a}$$

$$\langle q_{Ti}(\mathbf{x},t)q_{Tj}(\mathbf{x}',t')\rangle=(k_BT^2/c_v)2\kappa\delta(\mathbf{x}-\mathbf{x}')\times\delta(z-z')\delta(t-t')\delta_{ij}\ , \quad (2.4b)$$

where c_v is the constant-volume heat capacity per unit volume. The temperature is taken to be

$$T(\mathbf{x},z,t)=T_l+(T_u-T_l)(z/d)+\theta(\mathbf{x},z,t)\ , \quad (2.5)$$

where T_u and T_l are the fixed temperatures of the upper and lower plates, respectively, which are separated by a distance d. We shall introduce dimensionless variables in which distance, time, and temperature are scaled by d, d^2/κ, and $\kappa\nu/\alpha g d^3$, respectively, and define the Rayleigh number

$$R=\frac{g\alpha d^3(T_l-T_u)}{\kappa\nu}\ . \quad (2.6)$$

A well-studied model for the deterministic problem [$\mathbf{s}=\mathbf{q}_T=\mathbf{0}$ in (2.1) and (2.3)] near the convective threshold $R=R_c$ is a two-dimensional theory involving a generalized real order parameter ψ of the form [5]

$$\tau_0\partial_t\psi(\mathbf{x},t)=[\epsilon-\tilde{\xi}_0^4(\nabla^2+q_0^2)^2-\tilde{g}_3\psi^2]\psi(x,t) =-\frac{\delta\mathcal{F}}{\delta\psi}\ , \quad (2.7a)$$

$$\mathcal{F}=-\int d^2x\{\tfrac{1}{2}\epsilon\psi^2-\tfrac{1}{4}\tilde{g}_3\psi^4-\tfrac{1}{2}\tilde{\xi}_0^4[(\nabla^2+q_0^2)\psi]^2\}\ , \quad (2.7b)$$

where the real constants τ_0, q_0, and $\tilde{\xi}_0$ given in Table I depend on the horizontal boundary conditions, and the Prandtl number $\sigma=\nu/\kappa$, and

$$\epsilon=(R-R_c)/R_c \quad (2.8)$$

is the reduced Rayleigh number. The above equation will be referred to as the (deterministic) Swift-Hohenberg model.

The real field variable ψ is related to the vertical velocity $u_z=\hat{\mathbf{z}}\cdot\mathbf{u}$ and the temperature θ in such a way that the Nusselt number is to lowest order in ϵ simply given by

$$(\mathcal{N}-1)=S^{-1}\int d\mathbf{x}\,\psi^2(\mathbf{x})\ , \quad (2.9)$$

where S is the area of the cell. Moreover, as shown in Eq. (A15) of ACHS the field ψ is related to the temperature θ, Eq. (2.5), by

$$\theta(\mathbf{x},z,t)=\tilde{c}\tilde{\theta}_0(z)\psi(\mathbf{x},t)\ , \quad (2.10)$$

where $\tilde{\theta}_0$ is a normalized eigenfunction quoted in Eq. (A24) below, and

$$\tilde{c}=\begin{cases}3q_0\sqrt{R_c}=170.89 & \text{(free slip)} \quad (2.11a)\\ 385.28 & \text{(rigid)} \quad (2.11b)\end{cases}$$

(see Appendix A). It follows that

$$\langle\theta^2\rangle=\tilde{c}^{\,2}\langle\psi^2\rangle=\tilde{c}^{\,2}(\mathcal{N}-1)\ , \quad (2.12)$$

to lowest order in ϵ.

The precise sense in which the SH model (2.7) is equivalent to the original problem is that for a solution involving small deviations from a pattern of parallel rolls perpendicular to x, both systems lead near threshold to the same amplitude equation,

$$\tau_0\partial_tA(x,y,t)=\{\epsilon+\xi_0^2[\partial_x-(i/2q_0)\partial_y^2]^2-g_0|A|^2\}A(x,y,t)\ , \quad (2.13)$$

for the complex function A defined by

$$\psi(\mathbf{x},t)=[A(\mathbf{x},t)e^{iq_0x}+\text{c.c.}]\ , \quad (2.14)$$

with [19]

$$g_0=3\tilde{g}_3\ . \quad (2.15a)$$

The correlation length coefficient ξ_0 in (2.13) is

$$\xi_0^2=4q_0^2\tilde{\xi}_0^4\ . \quad (2.15b)$$

Since $\tilde{g}_3$ is sensitive to finite size it is usually taken from a fit to static experiments [4]. The higher-order gradient terms which are present in the SH model and not in the amplitude equation (2.13) do not come from a systematic expansion of the original system (2.1)–(2.3). On the other hand, these terms lead to an equation which has an important advantage over the complex amplitude equation (2.13), in that it is rotationally invariant in the x-y plane, and it can therefore describe roll patterns with arbitrary orientation.

The stochastic equations with $\mathbf{s},\mathbf{q}_T\neq\mathbf{0}$ in (2.1)–(2.3), linearized near threshold, lead to correlation functions for velocity and temperature which were first derived by Zaitsev and Shliomis [6]. The result appears also in Eq. (13) of SH [5] or Eq. (D14) of ACHS [13] for free boundaries, and in Eq. (6.17) of Schmitz and Cohen [20] for the rigid case. From the relation between the hydrodynamic variables and the field ψ we can thus infer the correlation function for ψ in linear approximation near threshold. We then supplement Eq. (2.7) with a Gaussian white-noise term whose strength is fixed by requiring that it yield the correct $\langle\psi\psi\rangle$ correlation function (see Appendix A). The *stochastic* SH model thus becomes [21]

$$\tau_0\partial_t\psi=-\frac{\delta\mathcal{F}}{\delta\psi}+f(\mathbf{x},t)\ , \quad (2.16a)$$

$$\mathcal{F}=-\int d\mathbf{x}\{\tfrac{1}{2}\epsilon\psi^2-\tfrac{1}{4}\tilde{g}_3\psi^4-\tfrac{1}{2}\tilde{\xi}_0^4[(\nabla^2+q_0^2)\psi]^2\}\ , \quad (2.16b)$$

$$\langle f(\mathbf{x},t)f(\mathbf{x}',t')\rangle=2F\tau_0\delta(t-t')\tilde{\xi}_0^2\delta(\mathbf{x}-\mathbf{x}')\ , \quad (2.16c)$$

TABLE I. Values of basic convection parameters appearing in Eq. (2.7) in the free-slip and rigid regimes.

	Free slip	Rigid
R_c	$27\pi^4/4$	1708
τ_0	$\left[\frac{3\pi^2}{2}\right]\frac{\sigma}{\sigma+1}$	$\frac{19.65\sigma}{\sigma+0.5117}$
ξ_0^2	$8/3\pi^2$	0.148
q_0	$\pi/\sqrt{2}$	3.117
$\tilde{\xi}_0^2$	$2/\sqrt{3}\pi^2$	0.062

where the noise strength coefficient F can be shown to be (see Appendix A)

$$F = F_{\rm th} = \left[\frac{k_B T}{\rho d \nu^2} \right] \frac{2\sigma q_0}{\xi_0 \tau_0 R_c} . \tag{2.17}$$

It is remarkable that Eq. (2.17) holds for *both* free and rigid boundary conditions, when expressed in terms of the appropriate constants [22]. The thermal noise (2.17) is scaled by the small parameter $k_B T/\rho d \nu^2$, which represents the ratio of the thermal fluctuation energy $k_B T$ to a characteristic (dissipative) energy of convection $(\rho d^3)(\nu/d)^2$. It is the mismatch between the microscopic energy scale in the numerator and the macroscopic scale in the denominator of F which limits the effects of thermal fluctuations on the convective threshold. Numerical estimates of the magnitude of $F_{\rm th}$ have been given by Ahlers [10] for various materials, and under favorable circumstances its value can reach 10^{-5}. As mentioned in the Introduction, we shall use the stochastic SH model (2.16) near threshold, treating F as a single adjustable parameter.

When the ψ field satisfies the stochastic equation (2.16), the amplitude equation corresponding to (2.13) is

$$\tau_0 \partial_t A(x,y,t) = \{ \epsilon + \xi_0^2 [\partial_x - (i/2q_0)\partial_y^2]^2 - g_0 |A|^2 \} A + f_A(x,y,t) , \tag{2.18a}$$

with a complex Gaussian noise satisfying

$$\langle f_A(x,y,t) f_A^*(x',y',t') \rangle = 2F_A \xi_0^2 \tau_0 \delta(x-x')\delta(y-y')\delta(t-t') , \tag{2.18b}$$

$$\langle f_A f_A \rangle = \langle f_A \rangle = 0 , \tag{2.18c}$$

$$F_A = (\tilde{\xi}_0/\xi_0)^2 F = F/2q_0\xi_0 . \tag{2.18d}$$

This result is similar to the one in Eqs. (3.19) and (3.20) of Graham [7], except that the factor of 2 in (2.18b) above is missing from Graham's expression. [Indeed, there is an inconsistency in Graham's work [7] of a factor of 2 between Eqs. (3.19) and (3.20) which are incorrect, and Eq. (6.13), in which the factor of 2 has been restored. See the Erratum to Graham [7].]

B. Reduction to a single-mode equation

Suppose we wish to use (2.16) to describe situations near threshold where the control parameter ϵ depends on time, in particular to calculate the global quantity $\mathcal{N}$, Eq. (2.9), whose time dependence can be measured with precision by thermal means. Rather than solve the full model (2.16) and integrate the solution over space to find $\mathcal{N}(t)$, it would of course be much simpler to solve an ordinary differential equation for $\mathcal{N}(t)$ directly. The question is, under what circumstances is this reduction a reasonable approximation? The first limit, which was discussed in ACHS, is that of a finite system sufficiently close to threshold, when only one mode fits into the container

$$\psi(\mathbf{x},t) = \overline{\overline{A}}(t)\psi_0(\mathbf{x}) , \tag{2.19a}$$

where $\psi_0(\mathbf{x})$ is a normalized function satisfying the appropriate lateral boundary conditions which we do not need to specify. Then the amplitude function satisfies the equation

$$\overline{\overline{\tau}}_0 \partial_t \overline{\overline{A}} = [\overline{\overline{\epsilon}}(t) - \overline{\overline{g}}_3 \overline{\overline{A}}^2] \overline{\overline{A}}(t) + \overline{\overline{f}}(t) , \tag{2.19b}$$

$$\langle \overline{\overline{f}}(t) \overline{\overline{f}}(t') \rangle = 2\overline{\overline{F}}\,\overline{\overline{\tau}}_0 \delta(t-t') , \tag{2.19c}$$

where $\overline{\overline{\tau}}_0$, $\overline{\overline{\epsilon}}$, and $\overline{\overline{g}}_3$ are suitably renormalized coefficients which depend on system size [see, e.g., Eqs. (2.23)–(2.28) of ACHS]. Moreover, the noise strength acting on the mode $\overline{\overline{A}}$ is given, as in Eq. (D19) of ACHS, by

$$\overline{\overline{F}} \simeq F(\tilde{\xi}_0^2/S) . \tag{2.20}$$

Note, however, that the reduction of (2.16) to (2.19) holds only in the restricted range

$$\overline{\overline{\epsilon}} < \epsilon_c , \tag{2.21}$$

where ϵ_c, the threshold shift, is itself proportional to S^{-1}. Thus for systems consisting of more than a few rolls, $S = (L/d)^2 \gtrsim 1$, the direct reduction to a one-mode model employed by ACHS is not useful quantitatively.

Let us consider the opposite limit of a laterally infinite system ($S \to \infty$), and define the real amplitude function

$$\overline{A}^2(t) \equiv \mathcal{N}(t) = \langle \psi^2(\mathbf{x},t) \rangle , \tag{2.22}$$

where we assume $\overline{A} \ge 0$, and the second equality follows since the average is independent of $\mathbf{x}$ by translational invariance. In general $\overline{A}^2(t)$ satisfies a complicated equation involving derivatives of ψ and higher powers, rather than the local quantity $\overline{A}^2 = \langle \psi^2 \rangle$ itself. The crudest approximation is an equation of the form (2.19), where the coefficients τ_0 and ϵ are those of (2.7) because of the infinite geometry, and we choose the nonlinear coefficient to satisfy Eq. (2.9),

$$\tau_0 \partial_t \overline{A} = [\epsilon(t) - g_3 \overline{A}^2] \overline{A}(t) + \overline{f}(t) , \tag{2.23a}$$

$$\langle \overline{f}(t) \overline{f}(t') \rangle = 2\overline{F}\tau_0 \delta(t-t') , \tag{2.23b}$$

with

$$g_3 = g_0/2 = 3\tilde{g}_3/2 , \tag{2.23c}$$

and the strength $\overline{F}$ to be determined. Although in our quantitative analysis we shall use an approximation which is somewhat more accurate than (2.23) for general $\epsilon(t)$, this equation is useful for estimating the order of magnitude of $\overline{F}$. Let us first assume a simple jump in ϵ,

$$\epsilon(t) = \begin{cases} -\epsilon_0 < 0, & t < 0 \quad (2.24a) \\ \epsilon_1 > 0, & t > 0 . \quad (2.24b) \end{cases}$$

Then for large times both (2.16) and (2.23) yield (to lowest order in $\epsilon_1 \ll 1$)

$$\langle \psi^2 \rangle = \langle \overline{A}^2 \rangle = 2\epsilon_1/3\tilde{g}_3 = \epsilon_1/g_3 , \tag{2.25}$$

so the form of the nonlinear term in (2.23a) seems to be reasonable. To determine the unknown parameter $\overline{F}$ we look at early times, specifically at the onset time $t_{\rm on}$, where $\langle \overline{A}^2(t) \rangle$ first becomes appreciable. We first calculate the initial rise from the linearized equation ($\tilde{g}_3 = 0$), and then check the self-consistency of this assumption. The calculation to be presented in Sec. IV yields in the linear approximation for the step (2.24),

$$\langle \psi_L^2(t)\rangle \sim [F/4\sqrt{2\pi t/\tau_0}](\epsilon_0^{-1}+\epsilon_1^{-1})\exp(2\epsilon_1 t/\tau_0)\ , \quad t\epsilon_1/\tau_0 \gg 1\ . \tag{2.26}$$

Alternatively the one-mode equation (2.23) gives in the linear case,

$$\langle \bar{A}_L^2(t)\rangle \sim \bar{F}(\epsilon_0^{-1}+\epsilon_1^{-1})\exp(2\epsilon_1 t/\tau_0)\ , \quad t\epsilon_1/\tau_0 \gg 1 \tag{2.27}$$

whence comparison with (2.26) yields

$$\bar{F}_{\text{eff}} = F/[4\sqrt{2\pi t/\tau_0}]\ . \tag{2.28}$$

The reason for the absence of the factor S^{-1}, which appeared in the small geometry, Eq. (2.20), was already commented upon by van Beijeren and Cohen [9]. For a large system there is a whole continuum of modes around $q=q_0$ which experience growth above threshold, and their contribution adds up to eliminate the factor S^{-1} present in Eq. (2.20), replacing it by a factor proportional to $t^{-1/2}$ at long times.

It should be stressed, however, that the argument leading to (2.28) is entirely tied to the linear approximation, which must fail at long times. In particular, when $\langle \psi^2\rangle$ has grown sufficiently large so that nonlinearity is dominant, it has the effect of suppressing most of the excited modes leading to (2.28). For the ideal situation in which a perfect roll pattern is created, the phase space must shrink to a point ($\mathbf{q}=\mathbf{q}_1=q_0\hat{\mathbf{x}}$, say) at long times. The noise in the original equation (2.16) acting on that particular mode has magnitude

$$\langle f_{\mathbf{q}_1}(t) f_{-\mathbf{q}_1}(t')\rangle = \frac{2F\tilde{\xi}_0^2}{S}\tau_0\delta(t-t')\ , \tag{2.29}$$

as in (2.20). It is thus legitimate to ask whether the time scale for the growth of this mode will be set by the noise (2.28) acting on all the linearly unstable modes, or by the noise (2.29) acting on the mode $\mathbf{q}_1$ which eventually wins out. Since the $\mathbf{q}\neq\mathbf{q}_1$ modes act to suppress the growth of the $\mathbf{q}_1$ mode during the late stages, it is not obvious that they contribute to the effective noise which stimulates the $\mathbf{q}_1$ mode during the initial stages. This, however, seems to be the case, *if* the noise is sufficiently weak, as we shall see.

Although we have not solved the difficult theoretical problem of actually calculating $\langle \psi^2(t)\rangle$ from (2.13) for an $\epsilon(t)$ given by (2.24), we may appeal to the work of Kawasaki, Yalabik, and Gunton [23], who considered a quench for the slightly simpler case where $(\nabla^2+q_0^2)^2$ in Eq. (2.7) is replaced by ∇^2. Building on work of Suzuki [18], these authors argued that for sufficiently weak noise the growth of the ordered pattern (in their case a simple ferromagnetic domain) can be divided into two stages: an initial stochastic one where the dynamical equation can be linearized, and a later nonlinear stage which is deterministic, i.e., in which the stochastic force f can be neglected. In Sec. IV we shall derive a general formula for $\langle \psi^2(t)\rangle$ which is an adaption of these ideas to our problem. For our present purposes we may summarize the result by saying that it corresponds to the one-mode equation (2.23) with an effective noise given by the linear estimate (2.28). A similar argument was recently shown to hold for front propagation in the SH model by Elder and Grant [24].

A necessary condition for the validity of the above scheme is that the noise should be sufficiently weak that the linear approximation should still hold at a time when $\langle \psi^2(t)\rangle$ has grown sufficiently so that the stochastic force may be neglected. From the equation

$$\frac{\tau_0}{2}\partial_t\langle \psi^2(t)\rangle = \epsilon(t)\langle \psi^2(t)\rangle - \tilde{\xi}_0^4\langle \psi(\nabla^2+q_0^2)^2\psi\rangle - \tilde{g}_3\langle \psi^4\rangle + F\xi_0 q_0/4\pi\ , \tag{2.30}$$

which follows from (2.16), we take this condition to be

$$\epsilon(t)\langle \psi^2(t)\rangle \gg \max[F\xi_0 q_0/4\pi, \tilde{g}_3\langle \psi^4\rangle]\ , \tag{2.31}$$

for some range of time t. Replacing $\langle \psi^2\rangle$ and $\langle \psi^4\rangle$ by their linear (Gaussian) approximations $\langle \psi_L^2\rangle$ and $3\langle \psi_L^2\rangle^2$, respectively, and estimating these by use of Eq. (4.1) given in Sec. IV, we find the condition

$$\epsilon_1^2 \gg 3\tilde{g}_3 F\xi_0 q_0/4\pi\ , \tag{2.32}$$

for a jump (2.24), and

$$\beta\tau_0 \gg 3\tilde{g}_3 F\xi_0 q_0/4\pi\ , \tag{2.33}$$

for a ramp $\epsilon=\beta t$. Thus for sufficiently weak noise the stochastic effects occur entirely in the linear domain and the replacement of the multimode field equation (2.16) by the single-mode equation (2.23) is a reasonable first approximation.

The one-mode stochastic equation can be solved numerically for arbitrary time-dependent $\epsilon(t)$. An approximate analytic solution scheme for the weak-noise case $F \ll 1$ was proposed by ACHS, and recently modified [16] and shown to reproduce numerical results on the same model rather well, though not perfectly. Before attempting to extend the analytic work to the stochastic field equation (2.16), we wish to summarize our present knowledge concerning the "thermodynamics" of the stochastic SH model, i.e., the behavior of steady-state averages such as $\langle \psi^2\rangle$ for time independent ϵ in the thermodynamic limit of infinite volume ($S\to\infty$) [5].

III. THERMODYNAMICS OF THE STOCHASTIC SH MODEL

A. Phase transition

This section summarizes the results of our paper [5] based on Brazovskii's work [11]. Since we have not succeeded in carrying out a systematic renormalization-group analysis in any dimension we use diagrammatic perturbative theory based on weak coupling, and find a fluctuation-induced first-order transition. As explained earlier [5], the result may be shown to be self-consistent for weak noise, but it is by no means a full solution of the problem.

For constant Rayleigh number the stationary probability distribution for the model (2.16) is of the form

$$\mathcal{P}(\psi)=Z^{-1}\exp\{-\mathcal{F}(\psi)/\tilde{\xi}_0^2\}\ , \tag{3.1}$$

or in the scaled units of Appendix A,

$$\mathcal{P}(\bar{\psi})=\bar{Z}^{-1}\exp\{-\bar{\mathcal{F}}(\bar{\psi})\}\ , \tag{3.2}$$

where Z and $\bar{Z}$ are normalization constants. The rescaled form of the free-energy function is given in Eq. (A27),

$$\bar{\mathcal{F}}=\int d\bar{\mathbf{x}}\left\{-\tfrac{1}{2}\epsilon\bar{\psi}^2+\tfrac{1}{2}[(\bar{\nabla}^2+\bar{q}_0^2)\psi]^2+\frac{\lambda}{4!}\bar{\psi}^4\right\}\ , \tag{3.3}$$

$$\lambda=6\tilde{g}_3F,\quad \bar{q}_0=(q_0\xi_0/2)^{1/2}\ , \tag{3.4}$$

which is essentially Eq. (B1) of SH except that the derivative $(\bar{\nabla}^2+\bar{q}_0^2)^2$ was replaced by $(-i\bar{\nabla}-q_0)^2=(\bar{q}-q_0)^2$. We will retain the more accurate form (3.3). We wish to calculate the propagator

$$\bar{g}(\bar{x})=\langle\bar{\psi}(\bar{\mathbf{x}})\bar{\psi}(0)\rangle\ , \tag{3.5}$$

in the ordered and disordered phases in order to compare the corresponding values of the free energy

$$\bar{\Phi}=-\ln\langle e^{-\bar{\mathcal{F}}}\rangle\ . \tag{3.6}$$

In the disordered phase [25] we have the Hartree expression [Eq. (B2) of SH]

$$r_-=\bar{g}^{-1}(\bar{q}=\bar{q}_0)=-\epsilon+\tfrac{1}{8}\lambda r_-^{-1/2}\ , \tag{3.7}$$

where the factor $\frac{1}{8}$, coming from the integral in the Hartree diagram [5] replaces the constant α [26]. In the ordered phase $\epsilon>0$, where

$$\langle\bar{\psi}(\bar{x})\rangle=2a\cos(\bar{\mathbf{q}}_1\cdot\bar{\mathbf{x}}),\quad \bar{\mathbf{q}}_1\approx\bar{\mathbf{q}}_0\ , \tag{3.8}$$

we have [Eq. (B8) of SH]

$$r_+=-\epsilon+\tfrac{1}{8}\lambda r_+^{-1/2}+\lambda a^2\ , \tag{3.9}$$

and the order parameter a is given by the zero-field condition of Brazovskii [Eq. (11) of Ref. [11]],

$$h=a(r_+-\tfrac{1}{2}\lambda a^2)=0\ . \tag{3.10}$$

The free-energy difference between the ordered and disordered phases can be shown to be

$$\Delta\bar{\Phi}=-\frac{r_-^2}{2\lambda}-\frac{1}{8}r_-^{1/2}-\frac{r_+^2}{2\lambda}+\frac{1}{8}r_+^{1/2}\ , \tag{3.11}$$

replacing the corresponding equation of Brazovskii [Eq. (14)], which has incorrect factors. Equations (3.7) and (3.9) with (3.10) may be solved to find $r_\pm(\epsilon)$, from which $\Delta\bar{\Phi}(\epsilon)$ can be calculated. This quantity changes sign at $\epsilon=\epsilon_1=1.69(\tilde{g}_3F)^{2/3}$, where the system makes a (first-order) transition to the ordered state with $a\neq0$. To make contact with measurable quantities we evaluate the average of the Nusselt number which according to Eqs. (2.9) and (A20) is

$$\mathcal{N}-1=\langle\psi^2\rangle=F\langle\bar{\psi}^2\rangle=(F/4)r_-^{-1/2},\quad \epsilon<0 \tag{3.12a}$$

$$\mathcal{N}-1=(F/4)[r_+^{-1/2}+16r_+/\lambda],\quad \epsilon>0\ . \tag{3.12b}$$

Equations (3.7)–(3.12c) allow us to evaluate the Nusselt number as a function of ϵ, in terms of the quantity shown in Fig. 1,

$$3.63\tilde{g}_3^{1/3}F^{-2/3}[\mathcal{N}(\epsilon)-1]=\tilde{\mathcal{N}}[\epsilon/(3\tilde{g}_3F/4)^{2/3}]\equiv\tilde{\mathcal{N}}(\tilde{\epsilon})\ . \tag{3.12c}$$

The scale of the fluctuation contribution is set by $F^{3/2}$, which measures both the size of the critical region and the magnitude of the jump in Nusselt number.

B. Correlation function

The correlation function of the linearized stochastic equation [(2.16) and (2.7b) with $\tilde{g}_3=0$] may be calculated exactly for arbitrary $\epsilon(t)$. Let us for concreteness assume

$$\epsilon(t)=-\epsilon_0<0,\quad t<t_0 \tag{3.13}$$

$$\epsilon(t)\ \text{arbitrary},\quad t>t_0\ . \tag{3.14}$$

Then for $t>t'$ the linear correlation function takes the form

$$\langle\psi_L(\mathbf{x},t)\psi_L(\mathbf{x}',t')\rangle =F\tilde{\xi}_0^2\int\frac{d^2q}{(2\pi)^2}e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')+Q_q(t)+Q_q(t')}\times\left[\frac{1}{\epsilon_{0q}}+\frac{2}{\tau_0}\int_{t_0}^{t'}ds\,e^{-2Q_q(s)}\right]\ , \tag{3.15}$$

where

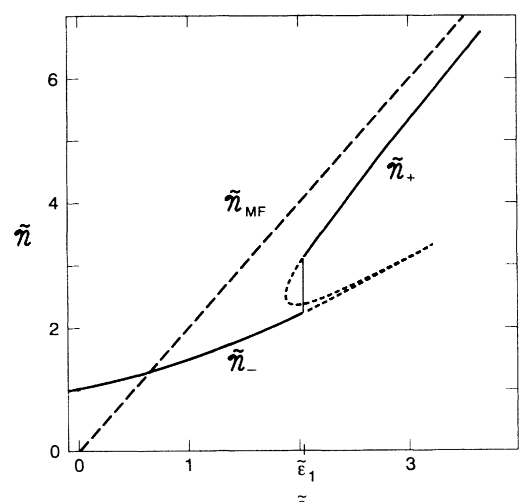

FIG. 1. The reduced Nusselt number $\tilde{\mathcal{N}}=3.63\tilde{g}_3^{1/3}F^{-2/3}(\mathcal{N}-1)$ as a function of the reduced Rayleigh number $\tilde{\epsilon}=\epsilon/(3\tilde{g}_3F/4)^{2/3}$, as obtained from the self-consistent Brazovskii approximation given in Eqs. (3.7)–(3.12). The effective free energies of the disordered ($\tilde{\mathcal{N}}_-$) and ordered ($\tilde{\mathcal{N}}_+$) branches cross at $\tilde{\epsilon}=\tilde{\epsilon}_1=2.03$, where the Nusselt number has a jump $\Delta\tilde{\mathcal{N}}\approx1$. Unstable and metastable portions of the $\tilde{\mathcal{N}}_\pm$ branches are represented by dashed lines. The scale of Nusselt number and Rayleigh number is set by the small parameter $F^{2/3}$. In these units the mean-field dependence $\mathcal{N}_{MF}-1=2\epsilon/3\tilde{g}_3$ is given by $\tilde{\mathcal{N}}_{MF}=2\tilde{\epsilon}$ (long dashed line).

$$\epsilon_{0q}=\epsilon_0+\xi_0^4(q^2-q_0^2)^2 , \tag{3.16a}$$

$$Q_q(t)=\tau_0^{-1}\int_{t_0}^{t}ds\,\epsilon_q(s) , \tag{3.16b}$$

$$\epsilon_q(t)=\epsilon(t)-\xi_0^4(q^2-q_0^2)^2 . \tag{3.16c}$$

Let us first calculate the "equilibrium" correlation function below threshold, i.e., for the case $\epsilon(t)=-\epsilon_0<0$. We find, for $t>t'$,

$$\langle\psi_L(\mathbf{x},t)\psi_L(\mathbf{x}',t')\rangle=\frac{F\xi_0^2}{2\pi}\int_{\Lambda_l}^{\Lambda_u}q\,dq\,J_0(q|\mathbf{x}-\mathbf{x}'|)\times\frac{\exp[-\epsilon_{0q}(t-t')/\tau_0]}{\epsilon_{0q}}, \tag{3.17}$$

where $J_0(\alpha)$ is a Bessel function, and we have introduced upper and lower cutoffs Λ_l and Λ_u for the q integral. It turns out that the quantities we calculate are insensitive to the precise values of the cutoffs. If for convenience we pick $\Lambda_u=\sqrt{2}q_0$ and $\Lambda_l=0$ we may easily transform (3.17) to

$$\langle\psi_L(\mathbf{x},t)\psi_L(\mathbf{x}',t')\rangle=\frac{Fe^{\epsilon_0(t-t')/\tau_0}}{\sqrt{2}\pi c_1}L\left[c_1\left[\frac{t-t'}{2\tau_0}\right]^{1/2},\frac{2\epsilon_0}{c_1^2},q_0|\mathbf{x}-\mathbf{x}'|\right], \tag{3.18}$$

with

$$L(\alpha,\beta,\gamma)=\frac{1}{2}\int_{-1}^{1}du\,J_0(\gamma\sqrt{1+u})\frac{e^{-\alpha^2u^2}}{\beta+u^2} , \tag{3.19a}$$

$$J_0(\eta)=\int_0^1 dv\,\cos[\eta\cos(2\pi v)] , \tag{3.19b}$$

$$c_1=\sqrt{2}(\xi_0q_0)^2=\xi_0q_0/\sqrt{2}=0.85 \quad \text{(rigid)} . \tag{3.19c}$$

The temperature correlation function $\langle\theta\theta\rangle$ may be obtained from (3.18) by use of (2.10).

IV. RESPONSE TO TIME-DEPENDENT CONTROL PARAMETER

Let us now seek an approximate formula for the Nusselt number in cases where the control parameter depends on time $\epsilon(t)$. This was done in Ref. [16] for the one-mode equation [2.23]. The idea was to solve the *linear* stochastic problem exactly for the given $\epsilon(t)$, to solve the nonlinear *deterministic* problem exactly, and to match in between. For the multimode model (2.16) the second step is not possible. As mentioned above, Kawasaki, Yalabik, and Gunton [23] have given an approximate treatment for the case of relaxation to a steady state in a model where $q_0=0$, based on an extension of the Suzuki approximation [18]. We propose a similar scheme which essentially uses the one-mode deterministic solution of the nonlinear problem and the exact linear stochastic solution (3.15), with the same matching as in Ref. [16] (referred to as I in what follows).

We begin by evaluating the local average $\langle\psi_L^2(\mathbf{x},t)\rangle$ from the linear expression (3.15) using the same cutoffs as in Sec. III. For this case the integrals in (3.15)–(3.16c) may be rewritten in the form

$$\langle\psi_L^2(t)\rangle=R_1^2(t)[\tilde\psi_0^2(t)+R_3^\psi(t)]=R_1^2(t)\tilde\psi^2(t) , \tag{4.1}$$

$$\tilde\psi_0^2(t)=\left[\frac{F}{c_1\pi\sqrt{2}}\right]K\left[c_1\left[\frac{t-t_0}{\tau_0}\right]^{1/2},\frac{2\epsilon_0}{c_1^2}\right] , \tag{4.2}$$

$$R_3^\psi(t)=\left[\frac{Fc_1}{\tau_0\pi\sqrt{2}}\right]\int_{t_0}^{t}ds\,R_1^{-2}(s)H\left[c_1\left[\frac{t-t_0}{\tau_0}\right]^{1/2}\right] , \tag{4.3}$$

with

$$K(\alpha,\beta)\equiv\int_0^1 du\,e^{-\alpha^2u^2}(\beta+u^2)^{-1} , \tag{4.4a}$$

$$H(\alpha)\equiv\int_0^1 du\,e^{-\alpha^2u^2} , \tag{4.4b}$$

$$R_1(t)=\exp\left[\tau_0^{-1}\int_{t_0}^{t}\epsilon(s)ds\right] . \tag{4.4c}$$

The expression in (4.4a) depends rather sensitively on the assumption made above that $\epsilon(t)=-\epsilon_0<0$ for $t<t_0$. More generally, we may approximate the function K in (4.4a) as $K(\alpha,\beta)\simeq K(0,\beta)H(\alpha)$ and replace (4.2) by

$$\tilde\psi_0^2(t)=\tilde\psi_{00}^2H(c_1(t-t_0)^{1/2}\tau_0^{-1/2}) , \tag{4.5}$$

where $\tilde\psi_{00}$ is a fitting parameter, to be determined by a matching condition as in I. Comparing Eqs. (4.1)–(4.5) to Eqs. (A4)–(A10) of I we see that the multiplicity of modes is thus summarized entirely in the function $H(\alpha)$. This leads to an effective noise which is nonlocal in time, as can be seen by comparing Eq. (4.3) above with Eq. (A10) of I. At long times we have $H(\alpha)\sim\sqrt{\pi}/2\alpha$, $\alpha\gg1$, and Eqs. (4.1)–(4.5) lead to an effective one-mode noise given by Eq. (2.28) above.

Our proposed solution of the nonlinear stochastic SH equation (2.16) is now the same as in the one-mode case [27] discussed in I, except that the exact linear result of the multimode equation given in (4.1) is used in place of $\langle A_L^2(t)\rangle$, Eq. (A6) of I. In Appendix B we give a self-contained summary of the approximation scheme of I as it applies to the stochastic SH model.

V. COMPARISON WITH EXPERIMENT AND WITH NUMERICAL SIMULATIONS

A. Experiments

The most convenient experimental tests of the stochastic Swift-Hohenberg model for our purposes are those of Meyer, Ahlers, and Cannell [4] on Rayleigh-Bénard convection with time-dependent Rayleigh number. The results of the analysis for temporal ramps ($\epsilon=\beta t$) going through threshold were already discussed by Meyer, Ahlers, and Cannell [4], who compared the measured time-dependent convective current with a calculation of $\langle\psi^2(t)\rangle$ based on the approximation discussed in Sec. IV, treating F as an adjustable parameter. The first result found by these authors was that the shape of $\langle\psi^2(t)\rangle$ fitted the experiment well (better than a simple deterministic model) for dimensionless ramp rates in the range

$0.01 \lesssim \beta \lesssim 0.3$. The fitted value of F was independent of ramp rate in the interval $0.05 \lesssim \beta < 0.3$ and was given by

$$F_{\rm expt} = 3.2 \times 10^{-5} = 1.9 \times 10^4 F_{\rm th} , \quad (5.1)$$

i.e., four orders of magnitude larger than the predicted thermal noise. The origin of the stochastic force acting on the system and causing the onset of convection remains unknown. For $\beta \lesssim 0.05$ there was some indication that the fitted F becomes larger, but the authors noted that the experiments are more difficult in this range. Finally we should mention that the one-mode stochastic equation (2.23) also gives a good fit to $\langle \psi^2(t) \rangle$, but the value of $\bar{F}$ obtained in the fit has a systematic dependence on ramp rate, which is consistent with an estimate based on Eq. (2.28) above. Thus the onset time experiments give some support for the model, though so far no theoretical analysis of the emerging patterns has been carried out, and there is no understanding of the physical origin of the stochastic force.

Turning to periodic modulation experiments, with

$$\epsilon(t) = \epsilon_0 + \delta \cos\omega t , \quad (5.2)$$

these may be analyzed with either the multimode (2.16) or the one-mode (2.23) stochastic equations, using the approximations of Ref. [16] and Sec. IV. Detailed comparisons with experiment [29] have only been made for the average convective current in the periodic state reached at long times ($t \to \infty$),

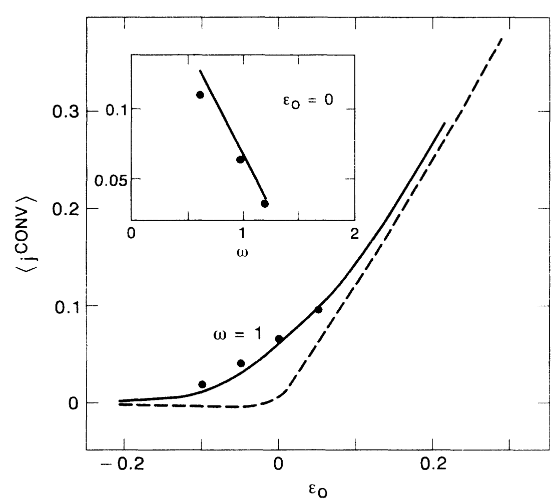

FIG. 2. Convective contribution to the heat current averaged over a period for convection with external modulation of the Rayleigh number in the form $\epsilon(t) = \epsilon_0 + \delta \cos\omega t$, plotted as a function of ϵ_0, for fixed dimensionless frequency $\omega = 1$. The solid and dashed lines represent the experimental results of Meyer, Ahlers, and Cannell [29] in the present and absence of modulation, respectively. The solid points are the calculated values using the stochastic SH model (2.16) and the solution method of Appendix B, with $F = 3.2 \times 10^{-5}$, as obtained by Meyer, Ahlers, and Cannell [4] from a separate experiment in the same cell. The modulation strength $\delta = 0.3$, which varied slightly with ϵ_0, was also taken from experiment. The inset shows the experimental (solid line) and theoretical (points) $\langle j^{\rm conv} \rangle$ at fixed $\epsilon_0 = 0$, for different frequencies.

$$\langle j^{\rm conv} \rangle = (\omega/2\pi) \int_t^{t+2\pi/\omega} ds \langle \psi^2(s) \rangle , \quad (5.3)$$

which is independent of t. In Fig. 2 we show $\langle j^{\rm conv} \rangle = \langle \mathcal{N} \rangle - 1$ as a function of ϵ_0 at fixed $\omega = 1$ and as a function of frequency at $\epsilon_0 = 0$. It is seen that the transition is rounded in the presence of modulation by an amount which increases as the frequency decreases. As explained by Ahlers, Hohenberg, and Lücke [28], this rounding is entirely due to the forcing, since for $f = 0$ the bifurcation is unshifted by modulation for an equation that is first order in time. (The argument of Ahlers, Hohenberg, and Lücke was given for the one-mode equation, but it holds for the field equation as well.) The agreement between experiment and theory shown in Fig. 2 can be considered quite satisfactory since it involves no adjustable parameters, the noise strength F being taken from an independent ramp experiment in the same cell. We have found no difference in the quality of the fit between the field equation (2.16) and the one-mode approximation (2.23) if the appropriate field strength is used.

B. Numerical simulations

In Ref. [16] we have compared the analytic approximation scheme of Sec. IV to numerical simulations of the one-mode stochastic equation and found good, though not perfect, agreement. In this section we briefly compare the theory to recent simulations [17] of the stochastic field equation (2.16), for the case of a ramp $\epsilon = \beta t$, starting with $\psi \equiv 0$ at $t = t_0 < 0$, i.e., in the disordered state, and crossing the threshold at $t = 0$. The parameters used were close to those of Meyer, Ahlers, and Cannell [4], namely $\tilde{\xi}_0 = 0.249$, $\tau_0 = 0.0552$, $q_0 = 3.117$, $F = 3.2 \times 10^{-5}$, and $\tilde{g}_3 = 0.5$, and two values of ramp rate, $\beta = 0.27$ (with $t_0 = -2$) and $\beta = 5$ (with $t_0 = -0.5$). In the theoretical expression given in Appendix B the nonlinear effects are treated in an approximation based on the one-mode model, Eq. (B10), according to which $\langle \psi^2(t) \rangle$ reaches the one-mode adiabatic answer,

$$\langle \psi^2(t) \rangle_{\rm ad} = \epsilon(t)/g_3 = 2\epsilon(t)/3\tilde{g}_3 , \quad (5.4)$$

at long times. Thus, in addition to neglecting the deterministic rearrangements of the many modes present in the actual solution, the approximation in Appendix B neglects the higher powers of ϵ in the static dependence of the convective current on ϵ, $\langle \psi^2 \rangle_{\rm st}(\epsilon)$. For the SH model (2.7) this dependence may be calculated analytically in perturbation theory in ϵ, or numerically using a time-independent ϵ. The adiabatic dependence is then in general given by

$$\langle \psi^2(t) \rangle_{\rm ad} = \langle \psi^2 \rangle_{\rm st}[\epsilon(t)] , \quad (5.5)$$

whose lowest-order expansion in ϵ is exhibited in Eq. (5.4). As an approximation to the exact $\langle \psi^2 \rangle_{\rm st}$ we will write

$$\langle \psi^2 \rangle_{\rm st} = (\epsilon/g_3)(1 + \bar{c}\epsilon) , \quad (5.6)$$

and note that the perturbative result for Eq. (2.7) is $\bar{c} = 0.015$. In order to improve on the approximation (5.4) in a simple way we then use (5.6) with $\bar{c}$ adjusted to fit the exact $\langle \psi^2 \rangle_{\rm st}$ at a convenient value of ϵ. This allows us to correct the analytic theory perturbatively to take into account the "mean-field" effects of mode interactions at larger ϵ. Any remaining discrepancies between the analytic theory and the numerical results can then be ascribed to intrinsic dynamical effects arising from the presence of many modes.

Figure 3 shows the average convective order parameter $\langle \psi^2 \rangle$ (corresponding to the experimentally measured convective heat current $\langle j^{\rm conv} \rangle$) as a function of time. The solid line is the theoretical calculation using the approximation of Appendix B with a nonlinear correction given in Eq. (5.6), the points are the numerical simulation by Xi, Viñals, and Gunton [17], and the dashed line is the exact adiabatic result calculated numerically [17]. In contrast to the fit to experiment made in Ref. [4], the noise strength is not an adjustable parameter here, so the test is more stringent. For the case $\beta=0.27$ shown in Fig. 3(a) there is a small discrepancy between theory and simulation in the stochastic region at early times ($t \lesssim 1.5$), which amounts to an effective noise strength F which is roughly 20% larger in the simulation than in the theory. Although part of the discrepancy might be explainable by finite size or finite grid effects in the simulation, a comparison of theory and simulation for $\epsilon<0$ (where the linear theory is valid) suggests that this only explains about $\frac{1}{3}$ of the difference. We are at present unable to explain the remaining 15%. The data for $\beta=5$ in Fig. 3(b), on the other hand, show better agreement at early times.

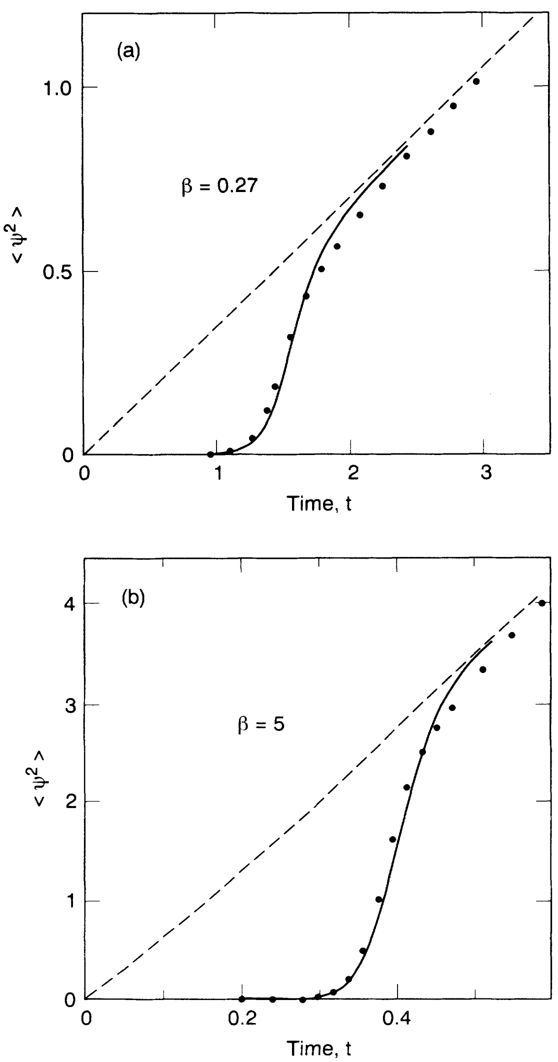

FIG. 3. Average convective order parameter $\langle \psi^2 \rangle$ as a function of time for the stochastic SH model (2.16), with $\epsilon(t)=\beta t$ and (a) $\beta=0.27$ and (b) $\beta=5$. The parameters are given in the text and correspond to those in the experiments of Meyer, Ahlers, and Cannell [4]. The solid line is the approximate analytic calculation of Appendix B, the points are numerical simulation results of Xi, Viñals, and Gunton [17], and the dashed line is the adiabatic result $\langle \psi^2 \rangle_{\rm ad}$ calculated by Xi, Viñals, and Gunton.

In the deterministic region at later times the theoretical curve depends on the constant $\bar{c}$ of Eq. (5.6). For $\beta=0.27$ it was determined by fitting $\langle \psi^2 \rangle_{\rm st}$ at $\epsilon=0.6$ ($t=2.22$), and was thus changed from its perturbative value $\bar{c}=0.015$ to -0.06, still a very small correction. It is seen that the one-mode theory is a reasonable approximation to the data, but it fails to account quantitatively for the slow relaxation to the adiabatic value at long times. This effect is magnified for the steeper ramp $\beta=5$ shown in Fig. 3(b). Here $\langle \psi^2 \rangle_{\rm st}$ was fitted at $\epsilon=2.5$ ($t=0.5$), leading to $\bar{c}=0.02$, surprisingly close to the perturbative value. The figure clearly illustrates the agreement at early times and the failure of the theory to account precisely for the long-time behavior. It is clear that in this purely deterministic region [23] a more sophisticated theoretical treatment is necessary [24], and our work suggests that a careful experimental study of the long-time relaxation (perhaps with steps in ϵ rather than a ramp) might be of considerable interest. Since the disagreement at long times between the analytic theory and the simulation evident in Fig. 3(a) did not show up in the experimental fits in Ref. [4], it may well be that the SH model does not describe the slow relaxation to steady state satisfactorily away from threshold.

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APPENDIX A: THE STOCHASTIC SH MODEL

In this appendix we derive the noise strength for the stochastic Swift-Hohenberg model starting from the hydrodynamic equations, which we write in the dimensionless variables of Sec. II as

$$(\partial_t+\mathbf{u}\cdot\tilde{\nabla})\mathbf{u}=-\tilde{\nabla}P+\sigma\tilde{\nabla}^2\mathbf{u}+\sigma\theta\hat{\mathbf{z}}+\tilde{\nabla}\cdot\mathbf{s}\ , \tag{A1a}$$

$$(\partial_t+\mathbf{u}\cdot\tilde{\nabla})\theta=\tilde{\nabla}^2\theta+Ru_z-\tilde{\nabla}\cdot\mathbf{q}_T\ , \tag{A1b}$$

$$\tilde{\nabla}\cdot\mathbf{u}=0\ , \tag{A1c}$$

where $\tilde{\nabla}=(\nabla,\partial_z)$, R is given in (2.6), σ is the Prandtl number, and the dimensionless noise correlations are

$$\langle s_{ij}(\mathbf{x},z,t)s_{lm}(\mathbf{x}',z',t')\rangle=\left[\frac{k_BT}{\rho d\nu^2}\right]2\sigma^3\delta(\mathbf{x}-\mathbf{x}')\delta(z-z')\times\delta(t-t')(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})\ , \tag{A2}$$

$$\langle q_{Ti}(\mathbf{x},z,t)q_{Tj}(\mathbf{x}',z',t')\rangle=\frac{2(Td^3\alpha g/\kappa\nu)^2}{(c_vd^3/k_B)}\delta(\mathbf{x}-\mathbf{x}')\times\delta(z-z')\delta(t-t')\delta_{ij}\ . \tag{A3}$$

We eliminate the pressure by taking the curl of the curl of (A1a) and use (A1c) to obtain

$$\partial_t\tilde{\nabla}^2u_z=\sigma\tilde{\nabla}^4u_z+\sigma\nabla^2\theta+f_z\ , \tag{A4}$$

$$\partial_t\theta=\tilde{\nabla}^2\theta+Ru_z+f_T\ , \tag{A5}$$

with

$$f_z=-\{\tilde{\nabla}\times[\tilde{\nabla}\times(\tilde{\nabla}\cdot\mathbf{s})]\}_z\ , \tag{A6}$$

$$f_T=-\tilde{\nabla}\cdot\mathbf{q}_T\ . \tag{A7}$$

The expression for the Nusselt number in terms of the dimensionless variables is

$$\mathcal{N}-1=R^{-1}\langle\theta u_z-\partial_z\theta\rangle_h\ , \tag{A8}$$

where $\langle\ \rangle_h$ means an average over a horizontal plane. We define two-dimensional Fourier transforms by the relations

$$\psi(\mathbf{x},t)=\int\frac{d^2q}{(2\pi)^2}e^{i\mathbf{q}\cdot\mathbf{x}}\psi(\mathbf{q},t)\ , \tag{A9}$$

$$u_z(\mathbf{q},t)=2\int_0^1dz\int d\mathbf{x}\,e^{-i\mathbf{q}\cdot\mathbf{x}}\sin(\pi z)u_z(\mathbf{x},z,t)\ , \tag{A10}$$

with a similar equation for $\theta(\mathbf{q},t)$. In Eq. (A9) ψ is the linear combination of u_z and θ corresponding to the slow eigenmode of the linear instability with normalization (2.9) (it corresponds to the variable w of SH).

Specifically,

$$\psi(\mathbf{q},t)=a(q)[u_z(\mathbf{q},t)+b(q)\theta(\mathbf{q},t)]\ , \tag{A11}$$

where for free-slip horizontal boundaries,

$$|a(q)|^2=[2(1+\sigma)^2(q^2+\pi^2)]^{-1}\ , \tag{A12}$$

$$b(q)=\sigma(q^2+\pi^2)/R_c\ . \tag{A13}$$

We now take a linear combination of Eqs. (A1a) and (A1b) to obtain an equation for ψ of the form [(2.16a), (2.16b)] with a force correlation defined by Eq. (2.16c). This yields

$$2F\tau_0\xi_0^2\delta(t-t')(2\pi)^2\delta(\mathbf{q}+\mathbf{q}')=4\tau_0^2\int_0^1dz\int_0^1dz'\int d\mathbf{x}\int d\mathbf{x}'e^{-i(\mathbf{q}\cdot\mathbf{x}+\mathbf{q}'\cdot\mathbf{x}')}\sin(\pi z)\sin(\pi z')a(\mathbf{q})a(\mathbf{q}')\times\left\langle\left[\frac{-f_z(\mathbf{x}',z',t')}{(q'^2+\pi^2)}+b(\mathbf{q}')f_T(\mathbf{x}',z',t')\right]\left[\frac{-f_z(\mathbf{x},z,t)}{(q^2+\pi^2)}+b(\mathbf{q})f_T(\mathbf{x},z,t)\right]\right\rangle\ . \tag{A14}$$

We may now evaluate Eq. (A14), making use of the expressions for q_0, ξ_0, τ_0, and R_c in Table I, to find for free-slip boundary conditions,

$$F\xi_0^2=F_{\rm ACHS}=\left[\frac{k_BT}{\rho d\nu^2}\right]\left[\frac{\sigma^2}{\sigma+1}\right]\frac{1}{(3R_c)^{1/2}}\left[1+\frac{g\alpha\rho Td}{c_v(T_l-T_u)}\right]\ . \tag{A15}$$

The second term in the square bracket of (A15) is estimated to be $O(10^{-4})$ and may be neglected. Equation (A15) is thus smaller than the result in Eq. (D18) of ACHS by a factor of 3. Use of the identities in Table I then shows that (A15) is identical to Eq. (2.17) of the text, or equivalently Eq. (A22) below.

For rigid boundaries we take the result of van Beijeren and Cohen (BC), which we must translate to our notation. From their Eq. (16), we have

$$\mathcal{N}-1=\frac{d^2}{\kappa\nu R_c\tau_0}\sum_{\mathbf{q}}\langle|\alpha_{\mathbf{q}}|^2\rangle=\frac{q_0S}{\kappa\nu R_c\tau_0}\int\frac{dq}{(2\pi)}\langle|\alpha_{\mathbf{q}}|^2\rangle\ , \tag{A16}$$

where the integral on the right-hand side is only over $q=|\mathbf{q}|$ and S is the cell cross section (q is dimensionless). According to Eqs. (9)–(12) of BC we have, for a constant $\epsilon=-\epsilon_0$, $(\Lambda\to q_0\xi_0)$,

$$\langle|\alpha_q|^2\rangle=F_{\rm BC}/|\lambda(q)|\ , \tag{A17}$$

$$\lambda(q)=-(\tau_0t_D)^{-1}[\epsilon_0+\xi_0^2(q-q_0)^2]\ , \tag{A18}$$

where $t_D=d^2/\kappa$. We may carry out the integral in (A16) to find

$$\mathcal{N}-1=\frac{q_0SF_{\rm BC}d^2}{2\kappa^2\nu R_c\xi_0\sqrt{\epsilon_0}}\ . \tag{A19}$$

On the other hand, from Eqs. (A25) and (A27) below we have in linear approximation ($\lambda=0$),

$$\mathcal{N}-1=F/4\sqrt{\epsilon_0}\ , \tag{A20}$$

which upon insertion of the result in Eq. (10b) of BC,

$$F_{\rm BC}=k_BT\kappa/\rho d^2\tau_0V\ , \tag{A21}$$

where $V=Sd$, yields the result

$$F=\left[\frac{k_B T}{\rho d \nu^2}\right]\frac{2\sigma q_0}{\xi_0 \tau_0 R_c}\ , \tag{A22}$$

quoted in Eq. (2.17). The same result may be found from Eq. (19) of BC, valid for a ramp $\epsilon(t)=\beta t$, rewritten in our notation as $(t/t_D \to t, a_c \to q_0)$

$$\langle \psi_L^2(t)\rangle \sim \left[\frac{k_B T}{\rho d \nu^2}\right]\frac{q_0\sigma}{\xi_0\tau_0 R_c}\frac{e^{\beta t^2/\tau_0}}{(2\beta t)^{1/2}}\ . \tag{A23a}$$

On the other hand, Eqs. (4.3) and (4.5) yield

$$\langle \psi_L^2(t)\rangle \sim F e^{\beta t^2/\tau_0}/2(2\beta t)^{1/2}\ , \tag{A23b}$$

from which Eq. (A22) also follows.

The constant of proportionality $\tilde{c}$ between $\psi(\mathbf{x})$ and the dimensionless temperature $\theta(\mathbf{x})$, quoted in Eq. (2.11) can be extracted from Eqs. (A11)–(A15) of ACHS for stress-free boundaries and from the appendix of Cross [30] for the rigid case. If we define $\tilde{\theta}_0$ so that its square integrates to unity we have

$$\tilde{\theta}_0=\begin{cases}\sqrt{2}\sin\pi z \quad (\text{free slip}) & \text{(A24a)}\\ (1.99\times 10^{-3})(650.68\cosh(i3.9784\tilde{z})+\{(39.277+i0.433)\cosh[(5.195-i2.126)\tilde{z}]+\text{c.c.}\})\quad (\text{rigid})\ , & \text{(A24b)}\end{cases}$$

$$\tilde{z}=z+\tfrac{1}{2}\ . \tag{A24c}$$

The constant $\tilde{c}$ then turns out to be

$$\tilde{c}=[R_0(|\theta_0|^2)_m/(w_0^*\theta_0)_m]^{1/2}\ , \tag{A24d}$$

$$\tilde{c}=\begin{cases}3q_0\sqrt{R_c}=170.89 \quad (\text{free slip}) & \text{(A24e)}\\ 385.3 \quad (\text{rigid})\ , & \text{(A24f)}\end{cases}$$

[where $(\ )_m$ denotes a vertical average.

Let us now compare our result with Eqs. (19)–(22) of SH. We define

$$\bar{\psi}=\psi/F^{1/2},\quad \bar{t}=t/\tau_0,\quad \bar{\mathbf{x}}=\mathbf{x}/\bar{\xi}_0,\quad \bar{f}=f/F^{1/2}\ , \tag{A25}$$

and transform Eqs. (2.16) to

$$\partial_{\bar{t}}\bar{\psi}=-\frac{\delta\bar{\mathcal{F}}}{\delta\bar{\psi}}+\bar{f}\ , \tag{A26}$$

$$\begin{aligned}\bar{\mathcal{F}}&=(F\bar{\xi}_0^2)^{-1}\mathcal{F}\\ &=\int d\bar{\mathbf{x}}\left\{-\frac{1}{2}\epsilon\bar{\psi}^2+\frac{1}{2}[(\bar{\nabla}^2+\bar{q}_0^2)\psi]^2+\frac{\lambda}{4!}\bar{\psi}^4\right\}\ ,\end{aligned} \tag{A27}$$

$$\langle \bar{f}(\bar{\mathbf{x}},\bar{t})\bar{f}(\bar{\mathbf{x}}',\bar{t}')\rangle=2\delta(\bar{\mathbf{x}}-\bar{\mathbf{x}}')\delta(\bar{t}-\bar{t}')\ , \tag{A28}$$

$$\lambda=6\bar{g}_3F,\quad \bar{q}_0^2=q_0^2\bar{\xi}_0^2=q_0\xi_0/2\ . \tag{A29}$$

Comparison with Eqs. (19)–(22) of SH shows that a factor of 2 is missing from (19) of SH, Eq. (21) of SH should be replaced by (A27), and Eq. (22b) of SH by $[p_z=\pi$, $P\to\sigma$, $l\to d$, and $g_0\to\bar{g}_3]$

$$\begin{aligned}\lambda=&\left[\frac{k_BT}{\rho\nu^2 d}\right]\left[\frac{\bar{g}_3\sigma^2}{1+\sigma}\right]\left[\frac{6q_0^3}{(q_0^2+\pi^2)^{3/2}}\right]\\ &\times\left[1+\frac{g\alpha\rho T d}{c_v(T_l-T_u)}\right]\ .\end{aligned} \tag{A30}$$

The free energy (A18) is equivalent to that used in Eq. (B1) of SH, with $\tau\to-\epsilon$, except that the quartic derivative $(\bar{\nabla}^2+\bar{q}_0^2)^2$ was replaced by $(\bar{q}-\bar{q}_0)^2$ rather than $4\bar{q}_0^2(\bar{q}-\bar{q}_0)^2$, i.e., the "nonessential constant" $4\bar{q}_0^2=2q_0\xi_0$ was dropped. We retain the form (A18) in our analysis.

APPENDIX B: ALGORITHM FOR APPROXIMATE SOLUTION

We consider the stochastic SH model (2.16) with an arbitrary $\epsilon(t)$. The approximate calculation of $\langle\psi^2(t)\rangle$ is the same as the calculation of $\langle A^2(t)\rangle$ described in the appendix of Ref. [16], except that the evaluation of the linear function A_L^2 in Eqs. (A4)–(A6) of I is replaced by Eqs. (4.1)–(4.5) in Sec. IV. The calculation involves the following steps.

(i) For a general function $\epsilon(t)$, $t>t_0$ we denote by t_i^+ and t_i^- the ith zero crossing of $\epsilon(t)$ with $d\epsilon/dt>0$ at t_i^+ and $d\epsilon/dt<0$ at t_i^-.

(ii) The distribution function is obtained from

$$P(\psi,t)=\tfrac{1}{2}[\tilde{P}(\psi)+\tilde{P}(-\psi)]\ , \tag{B1}$$

where

$$\begin{aligned}\tilde{P}(\psi)&=\tilde{P}_+(\psi)\\ &=(2\pi\tilde{\psi}^2)^{-1/2}\left[\frac{\partial\psi_0(\psi)}{\partial\psi}\right]\\ &\quad\times\exp\{-[\psi_0(\psi)-\psi_1]^2/2\tilde{\psi}^2\}\ ,\end{aligned} \tag{B2}$$

for $\epsilon(t)>0$, and

$$\begin{aligned}\tilde{P}(\psi)&=\tilde{P}_-(\psi)\\ &=(2\pi\psi_L^2)^{-1/2}\exp\{-[\psi-\psi_D(\psi_1)]^2/2\psi_L^2\}\ ,\end{aligned} \tag{B3}$$

for $\epsilon(t)<0$. In the above formulas the quantities are defined as

$$\psi_0(\psi)=\psi(R_1^2-R_2\psi^2)^{-1/2}\ , \tag{B4}$$

$$\tilde{\psi}^2(t)=\tilde{\psi}_0^2(t)+R_3^{\psi}(t)\ , \tag{B5}$$

$$\psi_L(t)=\tilde{\psi}(t)R_1(t)\ , \tag{B6}$$

$$\psi_D(\psi_1)=R_1\psi_1(1+R_2\psi_1^2)^{-1/2}\ , \tag{B7}$$

$$\tilde{\psi}_0^2=\tilde{\psi}_{00}^2H[c_1(t-t_i)^{1/2}\tau_0^{-1/2}]\ , \tag{B8}$$

$$R_1(t)=\exp[\tau_0^{-1}\int_{t_1}^{t}ds\ \epsilon(s)]\ , \tag{B9}$$

$$R_2(t)=2g_3\tau_0^{-1}\int_{t_i}^{t}ds\, R_1^2(s)=3\tilde{g}_3\tau_0^{-1}\int_{t_i}^{t}ds\, R_1^2(s)\ , \quad \text{(B10)}$$

$$R_3^{\psi}(t)=(Fc_1/\pi\sqrt{2}\tau_0)\times\int_{t_i}^{t}R_1^{-2}(s)H[c_1(t-s)^{1/2}\tau_0^{-1/2}]\ , \quad \text{(B11)}$$

$$H(\alpha)=\int_0^1 du\,\exp(-\alpha^2u^2)\ , \quad \text{(B12)}$$

$$c_1=\sqrt{2}(q_0\tilde{\xi}_0)^2=(q_0\xi_0)/\sqrt{2}=0.85\ , \quad \text{(B13)}$$

where the values for the rigid case (see Table I) have been used. The quantity t_i in (B8)–(B11) denotes the preceding crossing point (i.e., t_i^- for $t_i^- \le t \le t_{i+1}^+$, $\epsilon<0$, and t_i^+ for $t_i^+ \le t \le t_i^-$, $\epsilon>0$). An examination of the above formulas shows that they determine $P(\psi)$ in each interval in terms of only two constants, $\tilde{\psi}_{00}$ [Eq. (B8)] and ψ_1 [Eq. (B7)], which characterize the initial distribution for that interval.

(iii) To determine the parameters $\tilde{\psi}_{00}(t_i)$ and $\psi_1(t_i)$ at the ith crossing point the distribution function is represented in the form

$$\tilde{P}(\psi,t_i)=(2\pi\tilde{\psi}_0^2)^{-1/2}\exp[-(\psi-\psi_1)^2/2\tilde{\psi}_0^2]\ , \quad \text{(B14)}$$

and is fitted to the distribution in the preceding interval,

$$\tilde{P}_-(\psi,t_i^+)=\lim_{\eta\to 0}\tilde{P}_-(\psi,t_i^- -\eta)\ , \quad \text{(B15)}$$

$$\tilde{P}_-(\psi,t_i^-)=\lim_{\eta\to 0}\tilde{P}_+(\psi,t_i^- -\eta)\ . \quad \text{(B16)}$$

In this way $P(\psi)$ can be evaluated in succeeding intervals once it is assumed to be known at the earliest time $t=t_0$. A set of simplified matching conditions that are often sufficient is [16]

$$\tilde{\psi}_{00}=\psi_L(t^+),\quad \psi_1=0\quad \text{at } t=t^+\ , \quad \text{(B17)}$$

and

$$\tilde{\psi}_{00}=R_1(t^-)[3\sqrt{6}R_2^{3/2}(t^-)\tilde{\psi}^2(t^-)]^{-1}\ ,$$
$$\psi_1=R_1(t^-)R_2^{-1/2}(t^-)\quad \text{at } t=t^-\ . \quad \text{(B18)}$$

(iv) The expressions for the distribution functions in Eqs. (B1)–(B13) imply the following formula for the moments:

$$\langle\psi^{2n}\rangle=(2\pi)^{-1/2}\int_{-\infty}^{\infty}d\xi\, e^{-(\xi-\xi_1)^2/2}\left[\frac{\psi_L^2\xi^2}{1+\tilde{\tau}\xi^2}\right]^n\ , \quad \text{(B19)}$$

where

$$\tilde{\tau}=\begin{cases}R_2\tilde{\psi}^2, & \epsilon>0\\ 0, & \epsilon<0\end{cases} \quad \text{(B20)}$$

and

$$\xi_1=\begin{cases}R_1\psi_1/\psi_L, & \epsilon>0\\ \psi_D/\psi_L, & \epsilon<0\ .\end{cases} \quad \text{(B21)}$$

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[19] In Eq. (2.5) of ACHS the coefficient $g=g_3$ was erroneously written in place of $\tilde{g}_3=2g_3/3$. We remind the reader that Meyer, Ahlers, and Cannell (Ref. [4]) used the notation g_3 for what we call $\tilde{g}_3$.

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[21] Note that the definition of the noise strength F in (2.14) differs from that in ACHS: $F=F_{\text{ACHS}}/\tilde{\xi}_0^2=F_{\text{ACHS}}(2q_0/\xi_0)$.

[22] Equation (2.17) is the same as Eq. (10b) of Ref. [9], valid for rigid boundaries, when allowance is made for differences in notation, and when the second term in the brackets of (10b) [of relative order 10^{-5}] is neglected in comparison with unity. For the free case, Eq. (2.17)

corrects the corresponding relation (D18) of ACHS by dividing it by a factor of 3, as in Eq. (A15) below.

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[25] In order to conform to the notation appropriate for convection we are making a crucial change with respect to SH. We are referring the phases to the value of ϵ, rather than to $\tau=-\epsilon$. Thus, in contrast to SH, the *disordered* phase now has the subscript $-$, whereas the ordered phase is denoted by $+$.

[26] There is also a threshold shift $\epsilon_c \propto \lambda$, but this is small (of relative order $\lambda^{1/3} \lesssim 10^{-2}$) compared to the fluctuation effects we keep, so we will neglect it.

[27] In particular the one-mode nonlinear coefficient $g_3 = 3\bar{g}_3/2$ is used in Eq. (B10) below in order to agree with the static result (2.25) for time independent ϵ.

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