Alternative to the Pomraning-Eddington approach to radiative transfer

K. Razi Naqvi and A. El-Shahat
Department of Physics, University of Trondheim, N-7055 Dragvoll, Norway

S. A. El-Wakil
Physics Department, Faculty of Science Mansoura University, Mansoura, Egypt
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The modified Eddington approximation, introduced in 1969 by Pomraning [J. Quant. Spectrosc. Radiat. Transfer **9**, 407 (1969)], is used for solving the radiative-transfer equation for plane-parallel media. The specific intensity $I(x,\mu)$ is expressed as the sum of an even and an odd function of the angular variable μ, i.e., $I(x,\mu)=E(x)\epsilon(x,\mu)+F(x)o(x,\mu)$; the integro-differential transport equation is thereby transformed into two coupled first-order differential equations involving the energy density $E(x)$, the radiative flux $F(x)$, and $D(x)\equiv\int_{-1}^{1}\mu^2\epsilon(x,\mu)d\mu$. In Pomraning's treatment, one of these two exact equations is replaced by an approximate version derived by neglecting the spatial dependence of $D(x)$; despite this simplification, the coupled equations usually defy, if one is treating inhomogeneous media, attempts at analytic solutions; the boundary conditions for the two differential equations are fabricated by multiplying the condition satisfied by $I(x,\mu)$ at each boundary by a prescribed weight function and integrating with respect to μ. We also outline an alternative strategy wherein the integral equations satisfied by $D(x)$, $E(x)$, and $F(x)$ are deduced and solved by means of the variational method. The advantages of the latter approach are twofold. First, since the pertinent boundary conditions are automatically incorporated in the integral equations, the problem of inferring suitable weight functions is obviated; second, the spatial dependence of $D(x)$ is taken into account explicitly. Numerical results are presented to illustrate the performance of the adapted Pomraning-Eddington approach.

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I. INTRODUCTION

Radiative transfer in a scattering and absorbing medium is a problem of much academic and practical interest [1–3]. Since the underlying transport equation is not easy to solve, a variety of analytic approximations have bee sought; of these the simplest and perhaps the most familiar is that known variously as the Eddington approximation, the P_1 approximation, or elementary diffusion theory. It has been established [4–6] that this approximation, which amounts to replacing the integro-differential transport equation by two coupled first-order ordinary differential equations, works well only for weakly absorbing homogeneous media, i.e., when the single-scattering albedo $c\equiv\sigma/(\sigma+\alpha)=\sigma/\Sigma$ is close to unity, where σ, α, and Σ denote the scattering cross section, the absorption cross section, and the total cross section, respectively.

The Eddington approximation [7] was the forerunner of what have come to be known as two-stream models [8]; a great variety of such methods, the most popular of which appears to be that named after Kubelka and Munk [9–11], have been in use for some time. Higher-order approximations can be implemented by considering four, six, or still larger number of streams, and setting up as many coupled differential equations [12]; alternatively, recourse may be made to the so-called spherical harmonics or P_n approximation ($n=1,3,5,\ldots$), which entails solving a system of $(n+1)$ differential equations [4–6].

Originally, the spherical harmonics method suffered from a serious blemish: the boundary conditions to be imposed on the solution of the resulting differential equation(s) could not be inferred in a straightforward manner. A significant advance was made when Pomraning [13] and Federighi [14] demonstrated that this shortcoming could be overcome by appealing to variational arguments. Subsequently, Pomraning [15] succeeded, by means of a simple but ingenious approach, in extending the range of validity of the standard Eddington approximation without sacrificing its basic simplicity. To supply the boundary conditions for the two coupled ordinary differential equations, Pomraning [15] invoked the same variational argument as he had employed earlier in connection with the P_1 approximation. He called this approach the extended Eddington approximation [15], but we would like to honor his contribution by alluding to it as the Pomraning-Eddington (PE) approximation. Notwithstanding its conceptual superiority to other two-stream rivals, the PE approximation has not gained much vogue, presumably because of the esoteric nature of the variational arguments without which one cannot deduce appropriate boundary conditions for problems other than those considered by Pomraning.

Recently [16], variational calculus was used to develop an alternative form of the P_n approximation which dispenses with the very problem of the boundary conditions, and yields better results than those obtained by Pomraning [13] and Federighi [14]. It seems natural to carry the argument a step further and apply the same reasoning to the PE approximation; the results obtained by applying this modified PE approach to an anisotropically scattering homogeneous medium and a homogeneous medium with isotropic scattering are presented in this article.

II. GENERAL FORMULATIONS

We consider the transfer problem for finite media

$$\mu \frac{\partial I(x,\mu)}{\partial x} + I(x,\mu) = \omega(x) \int_{-1}^{1} P(\mu,\mu') I(x,\mu') d\mu' , \quad (1)$$

with the boundary conditions

$$I(0,\mu>0) = f_1(\mu) , \quad (2)$$

$$I(a,\mu<0) = f_2(\mu) . \quad (3)$$

The scattering kernel is taken as

$$P(\mu,\mu') = L\delta(\mu-\mu') + \frac{n}{2} , \quad (4)$$

in which L represents the importance of forward scattering and $n = 1-L$. Inserting from Eq. (4) into (1), the transfer equation becomes

$$\mu \frac{\partial I(x,\mu)}{\partial x} + [1 - L\omega(x)] I(x,\mu) = \frac{n\omega(x)}{2} \int_{-1}^{1} I(x,\mu) d\mu . \quad (5)$$

We commence by following Pomraning [15] and setting

$$I(x,\mu) = E(x)\epsilon(x,\mu) + F(x)o(x,\mu) , \quad (6)$$

where $\epsilon(x,\mu)$ and $o(x,\mu)$ are even and odd functions in μ, respectively, with the following normalizations:

$$\int_{-1}^{1} \epsilon(x,\mu) d\mu = 1 , \quad (7)$$

$$\int_{-1}^{1} \mu o(x,\mu) d\mu = 1 . \quad (8)$$

Integration of Eq. (5) over all μ gives

$$\frac{dF(x)}{dx} + [1-\omega(x)]E(x) = 0 . \quad (9)$$

We now define a new function $D(x)$,

$$D(x) = \int_{-1}^{1} \mu^2 \epsilon(x,\mu) d\mu , \quad (10)$$

and observe that multiplication of Eq. (5) by μ and integration over μ yields an equation involving $F(x)$ and $D(x)$:

$$\frac{d}{dx}[D(x)E(x)] + [1 - L\omega(x)]F(x) = 0 . \quad (11)$$

The pair of coupled equations (9) and (11) involving the quantities of utmost interest in radiative transfer, viz., the energy density $E(x)$ and the radiative flux $F(x)$, are exact; however, their solution requires knowledge of the spatial dependence of $D(x)$ and appropriate boundary conditions. To make further progress, one may employ a maneuver designed by Pomraning [15], or follow an alternative devised by the present authors; these options are spelled out in the next two sections.

III. POMRANING'S METHOD

Pomraning [15] substitutes Eq. (6) into Eq. (5), neglects the spatial dependence of the angular functions $\epsilon(x,\mu)$ and $o(x,\mu)$, and separates the even and odd parts of the resulting equation to arrive at the two equations given below:

$$\frac{\mu\epsilon(x,\mu)}{D(x)} \frac{d}{dx}[D(x)E(x)] + [1 - L\omega(x)]F(x)o(x,\mu) = 0 , \quad (12)$$

$$\mu o(x,\mu) \frac{dF(x)}{dx} + [1 - L\omega(x)]E(x)\epsilon(x,\mu) = \tfrac{1}{2} n\omega(x)E(x) . \quad (13)$$

Manipulation of these equations now leads to the following expressions for $\epsilon(x,\mu)$ and $o(x,\mu)$:

$$o(x,\mu) = \mu \frac{\epsilon(x,\mu)}{D(x)} , \quad (14)$$

$$\epsilon(x,\mu) = \frac{1}{\gamma(x)(1-\nu^2\mu^2)} , \quad (15)$$

where

$$\gamma(x) = \frac{2[1-L\omega(x)]}{n\omega(x)} , \quad (16)$$

$$\nu^2(x) = \frac{1-\omega(x)}{D(x)[1-L\omega(x)]} . \quad (17)$$

Multiplication of Eq. (15) by μ^2 and integration over all μ yields

$$\frac{[1-\omega(x)]\gamma(x)}{[1-L\omega(x)]\nu^2(x)} = \int_{-1}^{1} \frac{\mu^2 d\mu}{(1-\nu^2\mu^2)} . \quad (18)$$

The requisite boundary conditions are to be procured by utilizing the following corrollaries of the exact boundary conditions stated in Eqs. (2) and (3):

$$\int_0^1 d\mu\, W_0(\mu)[I(0,\mu) - f_1(\mu)] = 0 , \quad (19)$$

$$\int_{-1}^{0} d\mu\, W_a(\mu)[I(a,\mu) - f_2(\mu)] = 0 . \quad (20)$$

The appropriate weight functions remain to be specified; Pomraning appeals to arguments from the variational calculus to obtain these functions [13,15,17,18].

IV. THE NEW APPROACH

Our scheme, which we shall designate as the adapted Pomraning-Eddington (APE) approach, consists in solving Eq. (5) for $I(x,\mu>0)$ and $I(x,\mu<0)$ and deriving integral equations for $E(x)$ and $F(x)$. That is to say, we begin by writing

$$I(x,\mu>0) = f_1(\mu) e^{-y(x)/\mu} + \frac{n}{2\mu} \int_0^x E(x')\omega(x') e^{-[y(x)-y(x')]/\mu} dx' . \quad (21)$$

$$I(x,\mu<0)=f_2(\mu)e^{-[y(a)-y(x)]/\mu}+\frac{n}{2\mu}\int_x^a E(x')\omega(x')e^{-[y(x')-y(x)]/\mu}dx' , \quad (22)$$

where we have set

$$y(x)=\int_0^x [1-L\omega(x')]dx' . \quad (23)$$

The last two equations and Eq. (6) imply that $E(x)$ satisfies the following integral equation:

$$E(x)=G(x)+\frac{n}{2}\int_0^a \omega(x')E(x')E_1(|y(x)-y(x')|)dx' , \quad (24)$$

where $E_1(x)$ is the exponential integral function of order unity (see below) and

$$G(x)=\int_0^1 f_1(\mu)e^{-y(x)/\mu}d\mu+\int_0^1 f_2(\mu)e^{-[y(a)-y(x)]/\mu}d\mu . \quad (25)$$

Equation (24) can be given a more symmetric and compact rendering, namely,

$$\mathcal{E}(x)=\mathcal{G}(x)+\int_0^a K(x,x')\mathcal{E}(x')dx' , \quad (26)$$

if one introduces the following symbols:

$$\mathcal{E}(x)=\sqrt{\omega(x)}E(x) , \quad (27)$$

$$\mathcal{G}(x)=\sqrt{\omega(x)}G(x) , \quad (28)$$

$$K(x,x')=\frac{n}{2}\sqrt{\omega(x)\omega(x')}E_1(|y(x)-y(x')|) . \quad (29)$$

Combining the definition of $D(x)$ given in Eq. (10) with Eqs. (21) and (22), one gets

$$D(x)E(x)=H_1(x)+\frac{n}{2}\int_0^a \omega(x')E(x')\times E_3(|y(x)-y(x')|)dx' , \quad (30)$$

where

$$H_1(x)=\int_0^1 \mu^1 f_1(\mu)e^{-y(x)/\mu}+\int_0^1 \mu^2 f_2(\mu)e^{-[y(a)-y(x)]/\mu} . \quad (31)$$

From Eq. (9) it follows that

$$F(x)=F(0)-\int_0^x [1-\omega(x')]E(x')dx' , \quad (32)$$

where $F(0)$ is obtained from Eqs. (6), (21), (22), and the boundary conditions:

$$F(0)=H_2(x)-\frac{n}{2}\int_0^a \sqrt{\omega(x')}\mathcal{E}(x')E_2(y(x'))dx' , \quad (33)$$

$$H_2(x)=\int_0^1 \mu f_1(\mu)d\mu+\int_0^1 \mu f_2(\mu)e^{-y(a)/\mu}d\mu . \quad (34)$$

Finally, we provide the definition of the exponential integral function of order r:

$$E_r(x)=\int_0^\infty \frac{e^{-xu}du}{u^r} .$$

The integrals on the right-hand sides of Eqs. (26), (30), and (33) are to be determined with the aid of variational calculus, a concise and perspicuous account of which has been given by Rowlands [19]. A more recent review of variational methods has been given by Duderstadt and Martin [20], who have also listed in tabular form the common variational principles used in transport theory, and discussed all the functionals that are used in this work; nonetheless, for the sake of clarity and completeness, we will sketch out, in the following section, the crucial steps involved in variational calculations of quantities tabulated in this articles.

V. NUMERICAL EXAMPLES

In this section we shall compare the PE and APE approaches by considering three examples: the classic Milne problem in a semi-infinite capturing medium [15,20], and the calculation of transmission and reflection coefficients of a slab composed of an absorbing material with a single-scattering albedo $\omega(x)$ given by the expression

$$\omega(x)=\omega_0 e^{-\beta x} , \quad 0\le\omega_0\le 1 , \quad 0\le\beta . \quad (35)$$

Milne problem. Here one simply sets $\omega=\text{const}=c$, $L=0$, (i.e., $n=1$), $f_1(\mu)=0$, and lets $a\to\infty$. As shown by Pomraning [15], the PE approximation now gives

$$E(x)=Ae^{\nu x}+Be^{-\nu x} , \quad (36)$$

where A and B are constants and ν is the root of the transcendental equation

$$\frac{2\nu}{c}=\ln\left|\frac{1+\nu}{1-\nu}\right| . \quad (37)$$

To obtain the linear extrapolation distance λ, defined by the relation

$$\lambda=\frac{E(0)}{[dE/dx]_{x=0}} , \quad (38)$$

Pomraning [15] chooses the weight function

$$W_0(\mu)=\mu[E(0)\epsilon(0,\mu)-F(0)o(0,\mu)] , \quad (39)$$

and arrives at the following expression:

$$\lambda=\frac{1}{\nu^2}[\nu^2+(1-\nu^2)\ln(1-\nu^2)]^{1/2} . \quad (40)$$

In the APE approximation, one starts by transforming the transport equation into the integral equation shown below [cf. Eq. (24)]:

$$E(x)=\frac{c}{2}\int_0^\infty E(x')E_1(|x-x'|)dx' , \quad (41)$$

and seeks to solve it by using the variational principle and a suitable trial function. In analogy with the practice adopted in Ref. [16] (where the spherical harmonics method served to provide a trial function), we can now use the solution supplied by the PE approximation, namely, that displayed in Eq. (36), as the trial function. To this end, we rewrite Eq. (36) as

$$E(x)=A[e^{\nu x}+Ce^{-\nu x}]\ , \tag{42}$$

and take over LeCaine's work [21]. She has manipulated the homogeneous integral equation for $E(x)$ and shown that the function $q_2(x)$ defined by the relation

$$E(x)=A[e^{\nu x}+q_2(x)] \tag{43}$$

satisfies the inhomogeneous integral equation

$$q_2(x)=\frac{c}{2}\int_0^\infty q_2(x')E_1(|x-x'|)dx' + \frac{G_2(\nu,x)+G_3(\nu,x)}{2}\ , \tag{44}$$

in which $G_2(\nu,x)$ and $G_3(\nu,x)$ are given functions. One can now use $q=Ce^{-\nu x}$ as a trail function for $q_2(x)$ and use the variational principle to calculate the extrapolated end point x_0, which is related to λ as follows:

$$\lambda=\frac{1}{\nu}\tanh(\nu x_0)\ . \tag{45}$$

As it happens, x_0 has already been computed by LeCaine [21], who availed herself of Marshak's functional [22]; the corresponding values of λ are compared with Pomraning's results [15] in Table I.

A slab of finite thickness. Since precise values of the reflection coefficient R and transmission coefficient T of a slab are available [23–25] for the case $f_1(\mu)=1$, $f_2(\mu)=0$, we now apply both the PE and the APE method to this problem; for this comparison, we will follow Pomraning and set (with $b=a$ or 0)

$$W_b(\mu)=\mu I^*(b,\mu)=\mu I(b,-\mu)\ . \tag{46}$$

For the problem at hand, the boundary conditions imply that $R\equiv 2\int_0^1\mu I(0,-\mu)d\mu$ and $T\equiv 2\int_0^1 \mu I(a,\mu)d\mu$. On using the expression for the specific intensity, one sees that the reflection and transmission coefficients can now be expressed in a single formula (with $S_+\equiv R$ and $S_-\equiv T$),

TABLE I. Dependence of λ, the linear extrapolation length, on c, the single-scattering albedo. Pomraning's values [15] (PE) are compared with those obtained in this work by means of the amended Pomraning-Eddington (APE) approximation and the exact results (reproduced from Pomraning's paper).

c	PE	APE	Exact
0.0	1.0000	1.0000	1.0000
0.1	1.0000	1.0000	1.0000
0.2	0.9993	0.9993	0.9993
0.3	0.9888	0.9889	0.9889
0.4	0.9601	0.9604	0.9605
0.5	0.9190	0.9199	0.9200
0.6	0.8733	0.8747	0.8749
0.7	0.8277	0.8295	0.8299
0.8	0.7843	0.7864	0.7871
0.9	0.7440	0.7461	0.7472
1.0	0.7071	0.7083	0.7104

$$S_\pm=\frac{-E(b)}{\gamma(b)\nu^2(b)}\ln[1-\nu^2(b)] \pm\frac{2F(b)}{\gamma(b)D(b)\nu^2(b)}\left[1-\frac{1}{2\nu(b)}\ln\left[\frac{1+\nu(b)}{1-\nu(b)}\right]\right]\ , \tag{47}$$

in which b equals 0 or a according to whether S_+ or S_- is under consideration. The quantities $D(b)$, $E(b)$, and $F(b)$ are obtainable from Eqs. (26), (30), and (33); this will be illustrated by outlining the calculation of $F(0)$ and $D(0)E(0)$, which now satisfy the following equations:

$$F(0)=\tfrac{1}{2}-\frac{n}{2}\int_0^a \mathcal{E}(x')[\omega(x')]^{1/2}E_2(y(x'))dx'\ , \tag{48}$$

$$D(0)E(0)=\tfrac{1}{3}+\frac{n}{2}\int_0^a \mathcal{E}(x')[\omega(x')]^{1/2}E_3(y(x'))dx'\ . \tag{49}$$

The integral equation governing $E(x)$ will first be abbreviated as [cf. Eqs. (24)–(29)]

$$\mathcal{T}\mathcal{E}(x)=f(x)\ , \tag{50}$$

where $f(x)$ is to be identified with $\sqrt{\omega(x)}E_2(y(x))$, and

$$\mathcal{T}\mathcal{E}(x)\equiv\int_0^a[\delta(x-x')-K(x,x')]\mathcal{E}(x')dx'\ . \tag{51}$$

To find $F(0)$ by the variational principle, one evaluates the integral $(\mathcal{E},f)$ appearing on the right-hand side of Eq. (48) by choosing a trial function $\overline{\mathcal{E}}$ and optimizing the functional

$$Q[\overline{\mathcal{E}}]=2(\overline{\mathcal{E}},f)-(\overline{\mathcal{E}},\mathcal{T}\overline{\mathcal{E}})\ , \tag{52}$$

which equals $(\mathcal{E},f)$ if $\overline{\mathcal{E}}$ happens to be identical with $\mathcal{E}$.

Since $D(0)E(0)$ involves the integral $(\mathcal{E},\sqrt{\omega(x)}E_3(y(x)))$, we set up the so-called adjoint equation (with $\mathcal{T}^\dagger=\mathcal{T}$)

$$\mathcal{T}^\dagger\mathcal{E}^\dagger(x)=g^\dagger(x),\ \ [g^\dagger(x)=\sqrt{\omega(x)}E_3(y(x))]\ , \tag{53}$$

and call into play the bivariational method, which is based on finding an extreme of the functional

$$Q[\overline{\mathcal{E}^\dagger},\overline{\mathcal{E}}]=(g^\dagger,\overline{\mathcal{E}})+(f,\overline{\mathcal{E}^\dagger})-(\overline{\mathcal{E}^\dagger},\mathcal{T}\overline{\mathcal{E}}) \tag{54}$$

with respect to arbitrary variations in the two trial functions $\overline{\mathcal{E}^\dagger}$ and $\overline{\mathcal{E}}$. It will be observed that $Q[\overline{\mathcal{E}^\dagger},\overline{\mathcal{E}}]=(f,\mathcal{E})$ if $\overline{\mathcal{E}}=\mathcal{E}$ and $\overline{\mathcal{E}^\dagger}=\mathcal{E}^\dagger$.

We take our cue from Attia and coauthors [23], and choose the trial solutions to be

$$\overline{\mathcal{E}}=[A_0+A_1y(x)+A_2y^2(x)]\frac{y'(x)}{\sqrt{\lambda(x)}}\ , \tag{55}$$

$$\overline{\mathcal{E}^\dagger}][B_0+B_1y(x)+B_2y^2(x)]\frac{y'(x)}{\sqrt{\lambda(x)}}\ , \tag{56}$$

where prime denotes differentiation with respect to x. One gets, by differentiating the functional with respect to all the coefficients and setting the partial derivatives equal to zero, a system of linear algebraic equations whose solutions yield the coefficients and thereby the required variational estimates for $D(b)$, $E(b)$, and $F(b)$.

TABLE II. The dependence of the reflection coefficient (for a homogeneous slab of thickness a) on L, n, and ω_0. The results under the columns APE and PE pertain to the present work; those under the columns AEE and DSY, to Refs. [23] and [24].

(L,n)	ω_0	a	APE	PE	AEE	DSY
$(\frac{1}{3},\frac{2}{3})$	0.900	1	0.2833	0.2851	0.2737	0.2737
		3	0.3969	0.3875	0.3876	0.3880
		5	0.3966	0.4049	0.4042	0.4060
	0.950	1	0.3191	0.3200	0.3113	0.3113
		3	0.4858	0.4776	0.4776	0.4779
		5	0.5160	0.5177	0.5174	0.5183
	0.999	1	0.3597	0.3596	0.3541	0.3541
		3	0.6117	0.6065	0.6066	0.6067
		5	0.7163	0.7135	0.7138	0.7138
$(\frac{2}{3},\frac{1}{3})$	0.900	1	0.1478	0.2028	0.1649	0.1650
		3	0.2830	0.2663	0.2624	0.2626
		5	0.2983	0.2855	0.2857	0.2864
	0.950	1	0.1877	0.2206	0.1913	0.1913
		3	0.3501	0.3396	0.3358	0.3359
		5	0.3976	0.3862	0.3859	0.3863
	0.999	1	0.2237	0.2394	0.2224	0.2224
		3	0.4506	0.4457	0.4438	0.4438
		5	0.5684	0.5625	0.5624	0.5625

Tables II and III show the numerical values for R and T, respectively, for anisotropically scattering homogeneous slabs of different thickness, while Tables IV and V display the same quantities for an isotropically scattering inhomogeneous slab of unit thickness ($a=1$); for the sake of comparison, we have included what appear to be the most reliable results available in the literature [23–25]. The discrepancy between the calculations of Attia and co-workers [23] (AEE) and those of Garcia and Siewert [25] appears to have been due to some programming er-

TABLE III. The dependence of the transmission coefficient (for homogeneous slab of thickness a) on L, n, and ω_0. The headings have the same meanings as in Table II.

(L,n)	ω_0	a	APE	PE	AEE	DSY
$(\frac{1}{3},\frac{2}{3})$	0.900	1	0.5202	0.6443	0.5536	0.5536
		3	0.2348	0.2448	0.2179	0.2176
		5	0.1599	0.1018	0.0921	0.0905
	0.950	1	0.5443	0.6692	0.5961	0.5961
		3	0.2768	0.3085	0.2845	0.2843
		5	0.1711	0.1600	0.1489	0.1481
	0.999	1	0.5774	0.6995	0.6439	0.6439
		3	0.3558	0.4079	0.3874	0.3874
		5	0.2594	0.2887	0.2764	0.2763
$(\frac{2}{3},\frac{1}{3})$	0.900	1	0.6981	0.8903	0.6630	0.6630
		3	0.3531	0.4017	0.3324	0.3322
		5	0.2408	0.2097	0.1744	0.1739
	0.950	1	0.6830	0.8858	0.7163	0.7163
		3	0.3998	0.4789	0.4222	0.4221
		5	0.2679	0.2947	0.2629	0.2626
	0.999	1	0.7017	0.8816	0.7756	0.7756
		3	0.4952	0.5893	0.5502	0.5502
		5	0.3903	0.4521	0.4276	0.4276

TABLE IV. Dependence of the reflection coefficient (R) for an isotropically scattering ($L=0, n=1$) inhomogeneous slab of unit thickness ($a=1$) on β. The results under columns AEE and GS originate from Refs. [23] and [25].

ω_0	β	APE	AEE	GE
0.7	0.100	0.2228	0.2115	0.2116
	0.010	0.2414	0.2208	0.2210
	0.001	0.2434	0.2218	0.2220
0.9	0.100	0.3396	0.3301	0.3302
	0.010	0.3620	0.3502	0.3503
	0.001	0.3644	0.3524	0.3525
1.0	0.100	0.4175	0.4124	0.4125
	0.010	0.4484	0.4428	0.4429
	0.001	0.4518	0.4462	0.4462

TABLE V. Dependence of the transmission coefficient (T) for an isotropically scattering ($L=0, n=1$) inhomogeneous slab of unit thickness ($a=1$) on β. The column headings have been explained in Table IV.

ω_0	β	APE	AEE	GS
0.70	0.100	0.3748	0.3584	0.3583
	0.010	0.3798	0.3699	0.3698
	0.001	0.3804	0.3712	0.3711
0.90	0.100	0.4450	0.4475	0.4474
	0.010	0.4626	0.4718	0.4718
	0.001	0.4646	0.4745	0.4744
0.95	0.100	0.4701	0.4777	
	0.010	0.4929	0.5074	
	0.001	0.4955	0.5110	

ror in the former work, for it has not been confirmed by our calculations; it should be noted in this context that though we have labeled the second last columns in Tables II–V as AEE, all results reported under this heading have been computed afresh by following the recipe published in Ref. [23].

The results assembled in Tables I–III indicate that, so far as homogeneous media are concerned, there is little to choose between the PE and the APE approximation; the latter does slightly better in delaying with the Milne problem (Table I) and in predicting transmission coefficients (Table III), but the former meets with greater success in forecasting reflection coefficients (Table II); both compare favorably with other, exact or near-exact calculations. For an inhomogeneous medium, we have found it easier to implement our integral-equation approach; the data given in Table IV and V show that the utility of the APE is not confined to homogeneous media. For the Milne problem, the PE expression for the energy density $E(x)$, namely Eq. (36), has been used as the trial function in the APE treatment; but for finite homogeneous slabs (Tables II and III), the trial function has a different form. Further work [26] has shown that if Eq. (36) is used throughout, the APE approach yields values for R and T which are superior to those ensuing from the PE approximation and, in many cases, practically indistinguishable from their exact counterparts; the results of these calculations will be presented elsewhere.

VI. CONCLUDING REMARKS

An impediment to the widespread use of the Pomraning-Eddington approximation has been the problem of inferring suitable boundary conditions. The removal, thanks to the integral-equation formulations adopted here, of this long-standing difficulty has paved the way for developing a straightforward alternative to the prevalent two-stream approximations. In this work we have obtained very encouraging results for homogeneous media that are isotropically scattering and for inhomogeneous media that scatter isotropically; application of the adapted Pomraning-Eddington method to problems involving sources and/or an inhomogeneous medium that scatters anisotropically is planned to be the subject of a later publication.

An approximate description of radiative transfer, couched in terms of a family of flux-limited diffusion theories, has recently been published by Sanchez and Pomraning [27]. It would be interesting to apply their promising approach to the problems considered here; this exercise is consigned to future or other workers.

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[1] K. Stamnes, Rev. Geophys. **24**, 299 (1986).
[2] A. Ishimaru, App. Opt. **28**, 2210 (1989).
[3] W. I. Cheong, S. A. Prahl, and A. J. Welch, IEE J. Quantum Electron. **26**, 2166 (1990).
[4] B. Davison, *Neutron Transport Theory* (Oxford University Press, Oxford, 1957), pp. 139–142.
[5] E. Amaldi, in *Encyclopedia of Physics, Neutrons and Related Gamma Ray Problems,* edited by S. Flügge (Springer, Berlin, 1959), Vol. 38/2, pp. 562–568.
[6] K. M. Case and P. F. Zweifel, *Linear Tansport Theory* (Addison-Wesley, Reading, MA, 1967), p. 200.
[7] A. Eddington, *The Internal Constitution of the Stars* (Dover, New York, 1926), p. 322.
[8] W. E. Meador and W. R. Weaver, J. Atmos, Sci. **37**, 630 (1980).
[9] P. Kubelka and F. Munk, Z. Tech. Phys. **12**, 593 (1931).
[10] P. Kubelka, J. Opt. Soc. Am. **38**, 448 (1948).
[11] P. Kubelka, J. Opt. Soc. Am. **44**, 330 (1954).
[12] W. E. Meador and W. R. Weaver, Appl. Opt. **15**, 3155 (1976).
[13] G. C. Pomraning, Nucl. Sci. Eng. **18**, 528 (1964).
[14] F. D. Federighi, Nucleonik **6**, 21 (1964).
[15] G. C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **9**, 407 (1969).

[16] K. Razi Naqvi, A. El-Shahat, and K. J. Mork, Phys. Rev. A **44**, 994 (1991).
[17] G. C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **36**, 69 (1986).
[18] G. C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **46**, 221 (1991).
[19] G. Rowlands, J. Nucl. Energy, Part A **13**, 69 (1961).
[20] J. J. Duderstadt and W. R. Martin, *Transport Theory* (Wiley, New York, 1979), Chap. 7.
[21] J. LeCaine, Can. J. Res., Sect. A **28**, 742 (1949).
[22] R. E. Marshak, Phys. Rev. **71**, 688 (1947).
[23] M. T. Attia, A. El-Sheikh, and A. El-Shahat, Astrophys. Space Sci. **141**, 133 (1988).
[24] C. Devaux, C. E. Siewert, and Y. Yener, J. Quant. Spectrosc. Radiat. Transfer **21**, 505 (1979).
[25] R. D. M. Garcia and C. E. Siewert, J. Quant. Spectrosc. Radiat. Transfer **27**, 141 (1978).
[26] K. Razi Naqvi, S. A. El-Wakil, and E. M. Abulwafa (unpublished).
[27] R. Sanchez and G. C. Pomraning, J. Quant. Spectrsoc. Radiat. Transfer **45**, 313 (1991).