Zeros of the Husimi functions of the spin-boson model

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We study the distribution of zeros of the Husimi functions for the spin-boson model, following an approach introduced by Leboeuf and Voros. The interest lies in the model's double feature of possessing both a classical integrable to chaotic transition and an unbounded four-dimensional phase space. The latter gives rise to several new questions regarding the Husimi zeros which are discussed and partially answered. Some significant results occur in spite of the fact that we treat the case of spin $\frac{1}{2}$.

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I. INTRODUCTION AND SUMMARY

The spin-boson model is a very popular model in physics, with a large variety of applications (see, for example, [1–3] and references therein). Recently, there arose some renewed interest connected with the model's chaotic classical dynamics [1,3]. In particular, it has a very rich structure from the point of view of level-statistics transitions [4] and Husimi distributions [5,6]. In fact, from the correspondence principle we expect that in the semiclassical limit the quantum behavior reflects somehow the nature of the classical dynamics.

In this paper we have also studied Husimi distributions of eigenstates of the spin-boson model but the emphasis is on different aspects than the ones treated before, akin to the theory developed in [7]. There it was found that in the case of quantum systems having a two-dimensional compact phase space an arbitrary state of Hilbert space is completely determined by the zeros of its Husimi function. Moreover, for *eigenstates,* the phase-space distribution of zeros reflects, in the semiclassical limit, the underlying classical dynamics in a precise way: the distribution is one dimensional (i.e., concentration of zeros along curves) for integrable systems whereas it becomes highly spread out (diffusive behavior) for chaotic systems.

The two main features distinguishing the spin-boson model from those treated in [7] (shared by other interesting models with chaotic dynamics [8,9]) are (i) the unbounded phase space and (ii) the dimension of that space, which is now four instead of two. It is therefore of conceptual interest to know whether the aforementioned picture suggested in [7] remains valid for the spin-boson model. One simple way to overcome the second difficulty is through the study of quantum Poincaré sections, which are introduced in Sec. II. In what concerns the first point, two additional questions arise naturally: (a) is the number of zeros finite or infinite, and, more importantly, (b) do these zeros still determine the wave function? Problems (a) and (b) are difficult and one cannot hope for a general solution, but in this paper we provide partial answers to these questions in the special case of the spin-boson model. In its most general form, we describe the spin-boson model by the Hamiltonian

$$H=\hbar\omega a^{\dagger}a+\frac{\omega_0}{s}S_z+\sqrt{\hbar/2}\,\frac{\lambda}{2s}[(S_+a+S_-a^{\dagger})+\epsilon(S_+a^{\dagger}+S_-a)] \tag{1.1}$$

on the tensor product $\mathbf{C}^{2s+1}\otimes\mathcal{F}$, where $a^{\dagger}$, a are standard creation and annihilation operators corresponding to one boson with $[a,a^{\dagger}]=1$ and acting on Fock space $\mathcal{F}$. S_x, S_y, and S_z are dimensionless spin operators corresponding to a spin quantum number s and satisfying SU(2) commutation relations $[S_x,S_y]=iS_z$ (with cyclic permutations). Moreover, $S_{\pm}$ are defined as usual by $S_{\pm}=S_x\pm iS_y$. The frequencies ω and ω_0 are real positive constants and λ is the coupling constant between the spin and the boson. We have introduced in the coupling term an additional parameter ϵ which gives a generalized version of the model. In the nonrelativistic theory of interaction between atoms and the cutoff radiation field the well-known spin-boson Hamiltonian [10]

$$H_{\mathrm{SB}}=\hbar\omega a^{\dagger}a+\frac{\omega_0}{s}S_z+\sqrt{\hbar/2}\,\frac{\lambda}{s}S_x(a+a^{\dagger}) \tag{1.2}$$

is obtained by setting $\epsilon=1$ in (1.1). On the other hand, when the boson is interpreted as one mode of the electromagnetic field interacting with a $(2s+1)$-level atom, the term $S_+a+S_-a^\dagger$ is the resonant one, and hence $\epsilon=0$ in (1.1) corresponds to the rotating-wave approximation of (1.2). Furthermore, if $\epsilon=0$ an additional quantum constant of the motion (apart from energy conservation) appears and the classical system corresponding to (1.1) is integrable, while, if $\epsilon=1$, chaotic solutions to the dynamics exist [10]. We therefore make ϵ into a variable parameter ranging from 0 to 1 and we keep ϵ as a measure of the deviation of the system from classical integrability. Besides this regular-to-chaotic transition in ϵ, other possible transitions of the model are of interest. Firstly, variation of $\hbar$ in (1.1) permits interpolation between classical and quantum boson regimes. Secondly, we may also consider a transition between quantum and classical spin regimes by changing the spin quantum number s. However, the full classical limit of the model is only obtained when $\hbar\to 0$ and $s\to\infty$ simultaneously, with $\hbar s=1$. In fact, (1.1) is written in a way which is adapted to this latter limit and hence $\hbar$ is replaced by $1/s$ when multiplying a (dimensionless) spin operator.

The main purpose of this paper is to analyze the distribution of the zeros of the Husimi function for the eigenstates of (1.1) and to study how they evolve when we vary the parameter ϵ in the above-mentioned regular-to-chaotic transition. The changes on the distribution of zeros can be studied in the different semiclassical regimes already mentioned. Unfortunately, for technical reasons explained later on, we are forced to restrict ourselves to the special case of $s=\frac{1}{2}$ in our treatment of the zeros in Secs. III and IV, which represents the extreme quantum limit for the spin. Nevertheless, we keep for the moment (1.1) with an arbitrary s in order to motivate the discussion in Sec. II, which is of wider scope and introduces concepts of general applicability.

The organization of the paper is as follows. In Sec. II we discuss the appropriate coherent-state representation of this model in order to study the eigenfunctions of Hamiltonian (1.1) in phase space. This leads us to introduce the Glauber and spin coherent states [11,12] and to find closed-form expressions for the Husimi functions associated with the eigenstates. The quantum Poincaré sections are then defined.

The spin components $|\psi_m^E\rangle$ ($m=-s,-s+1,\dots,s$) of an eigenstate of H corresponding to energy E may be expanded in the harmonic-oscillator basis $|n\rangle$ ($n=0,1,2,\dots$). The large-n behavior of the coefficients of $|\psi_m^E\rangle$ in this expansion determines the order [13] of the (entire analytic) wave function $\psi_m^E(z)$ in the (Glauber) coherent-state representation. This large-n behavior is a crucial point when trying to elucidate items (a) and (b) already mentioned; these topics are discussed in Sec. III. There we show that, at least for $s=\frac{1}{2}$ and for special values of the parameter ϵ and the energy E, the zeros of the Husimi function associated to $|\psi_{\pm 1/2}^E\rangle$ completely determine the eigenstate. The limitations and shortcomings of this result are further discussed there.

Finally, Sec. IV presents numerical computations of the distribution of zeros of the Husimi functions for $\hbar$ small and different values of ϵ, with a careful analysis of the spurious zeros associated to the process of truncation of the entire analytic function $\psi_{1/2}^E(z)$.

II. REDUCED FRAMEWORK

For clarity we first consider the general case (1.1) of a spin-s particle interacting with a boson field. We denote by $|n\rangle$ and $|m\rangle$ the eigenstates of $a^\dagger a$ and S_z, respectively, labeled by the number of bosons $n=0,\dots,\infty$ and the projection of the spin over the z axis $m=-s,-s+1,\dots,s$. Then the boson and spin coherent states [11, 12] are defined as

$$|z_b\rangle=e^{\bar{z}_b a^\dagger}|0\rangle\ , \tag{2.1}$$

$$|z_s\rangle=e^{\bar{z}_s S_+}|-s\rangle\ , \tag{2.2}$$

where $|0\rangle\equiv|n=0\rangle$ and $|-s\rangle\equiv|m=-s\rangle$ are, respectively, the ground states of the bosonic and spin degrees of freedom. z_b and z_s are two complex variables labeling the coherent states, while the bar indicates complex conjugation. These states are not normalized: $\langle z_b|z_b\rangle=\exp(|z_b|^2)$, $\langle z_s|z_s\rangle=(1+|z_s|^2)^{2s}$. Using Eqs. (2.1) and (2.2) the coherent-state representation $\psi(z_b,z_s)=\langle z_bz_s|\psi\rangle$, $|z_bz_s\rangle\equiv|z_b\rangle\otimes|z_s\rangle$, of an arbitrary quantum state $|\psi\rangle$ is an analytic function of the two complex variables z_b and z_s:

$$\psi(z_b,z_s)=\sum_{n=0}^{\infty}\sum_{m=-s}^{s}\frac{\psi_{nm}}{\sqrt{n!}}\begin{bmatrix}2s\\ s+m\end{bmatrix}^{1/2} z_b^n z_s^{s+m}\ , \tag{2.3}$$

where $\binom{a}{b}$ denotes binomial coefficients. If $|\psi\rangle$ is an eigenstate of (1.1), the coefficients $\psi_{nm}=\langle nm|\psi\rangle$, $|nm\rangle\equiv|n\rangle\otimes|m\rangle$ are obtained by diagonalizing that Hamiltonian. The phase space for the boson is a two-dimensional plane spanned by the variables (q,p), while for the spin it is a two-dimensional sphere spanned by the spherical angles (θ,φ). These coordinates are related to z_b and z_s via

$$z_b=\frac{1}{\sqrt{2\hbar}}(q-ip)\ , \tag{2.4}$$

$$z_s=\left[\cot\frac{\theta}{2}\right]e^{i\varphi}\ . \tag{2.5}$$

Equation (2.5) corresponds to a stereographic projection of the sphere onto the complex z_s plane through the north pole. The variables $(\cos\theta,\varphi)$ are canonically conjugated, $\cos\theta$ being the classical continuous version of the normalized discrete quantum projection over the z axis, $\cos\theta\sim m/s$.

The normalized squared modulus of $\psi(z_b,z_s)$ associates with each eigenstate of the system a quasiprobability distribution function in phase space, usually called the Husimi function [14–16]. In our case, it is given by

$$\begin{aligned}W_\psi(z_b,z_s)&=\frac{|\psi(z_b,z_s)|^2}{\langle z_bz_s|z_bz_s\rangle}\\ &=(1+|z_s|^2)^{-2s}e^{-|z_b|^2}\\ &\quad\times\left|\sum_{n=0}^{\infty}\sum_{m=-s}^{s}\frac{\psi_{nm}}{\sqrt{n!}}\begin{bmatrix}2s\\ s+m\end{bmatrix}^{1/2} z_b^n z_s^{s+m}\right|^2\end{aligned} \tag{2.6}$$

which, using (2.5), can be written as

$$W_\psi(z_b,z_s)=e^{-|z_b|^2}\left|\sum_{n=0}^{\infty}\sum_{m=-s}^{s}\begin{bmatrix}2s\\ s+m\end{bmatrix}^{1/2}\left[\cos\frac{\theta}{2}\right]^{s+m}\left[\sin\frac{\theta}{2}\right]^{s-m}e^{im\varphi}\frac{\psi_{nm}}{\sqrt{n!}}z_b^n\right|^2 . \tag{2.7}$$

This is a positive function defined in the four-dimensional phase space. Just as in the classical counterpart, we can define Poincaré sections of it. Let us consider, for example, a Poincaré surface of section defined by $\cos\theta=\text{const}=\cos\theta_0$, i.e., we fix the spin projection onto the z axis. Classically, this corresponds to plotting a point in the bosonic phase space (q,p) each time the particle crosses the parallel defined by $\cos\theta_0$ on the sphere, the angle φ being fixed by energy conservation $\varphi=\varphi(E,p,q,\cos\theta_0)$. Since in the semiclassical limit W_ψ is, for an eigenstate, sharply peaked on the energy shell [16], an equivalent definition is [17]

$$\begin{aligned}R_{\cos\theta_0}(p,q)&=\int_0^{2\pi}d\varphi\, W_\psi(p,q,\cos\theta_0,\varphi)\\ &=2\pi\sum_{m=-s}^{s}\begin{bmatrix}2s\\ s+m\end{bmatrix}\left[\cos\frac{\theta_0}{2}\right]^{2(s+m)}\left[\sin\frac{\theta_0}{2}\right]^{2(s-m)}e^{-|z_b|^2}\left|\sum_{n=0}^{\infty}\frac{\psi_{nm}}{\sqrt{n!}}z_b^n\right|^2 .\end{aligned} \tag{2.8}$$

This Poincaré section was already used in order to show the emergence of classical structures in the eigenfunctions of two-dimensional chaotic systems [16]. The quantity

$$\psi_m(z_b)=\sum_{n=0}^{\infty}\frac{\psi_{nm}}{\sqrt{n!}}z_b^n \tag{2.9}$$

appearing in (2.8) is just the coherent-state (or Bargmann) representation for the bosonic degree of freedom computed for the mth component of the spin, and

$$R_m(p,q)=e^{-|z_b|^2}|\psi_m(z_b)|^2\ ,\quad m=-s,\ldots,s \tag{2.10}$$

its phase-space quasiprobability distribution function. We call $R_m(p,q)$ the *reduced quantum Poincaré section* of $W_\psi(z_b,z_s)$ over the plane fixed by the discrete index m. There are obviously $2s+1$ of them.

The full section $R_{\cos\theta_0}(p,q)$ is a sum over all the reduced sections (2.10) with a binomial weight

$$\begin{bmatrix}2s\\ s+m\end{bmatrix}I_0^{s+m}(1-I_0)^{s-m}\ ,$$

$$I_0=\cos^2\frac{\theta_0}{2}=\frac{\cos\theta_0+1}{2}\ .$$

In the classical limit $s\to\infty$, the binomial distribution becomes highly peaked around $m_0/s=2I_0-1=\cos\theta_0$, so that only the spin component m_0 significantly contributes to the sum (2.8), and $R_{\cos\theta_0}(p,q)$ and $R_{m_0}(p,q)$ essentially coincide.

In any case, it is clear that the set of $2s+1$ functions (2.9) for $m=-s,\ldots,s$ contains full information about the quantum state. For systems with a compact phase space, the analytic function (2.9) is polynomial-like [7] and, by factorization, completely determined by a *finite* number of zeros. In our case, due to the unbounded nature of the bosonic phase space, (2.9) is an entire function and the question is whether or not it can be completely characterized just by its zeros, and if that number is finite or infinite. These complications arise from the fact that an entire function is determined by its zeros only up to multiplication by a nonvanishing entire function. Obviously, truncation of the basis (which is always needed in practical computations) faces us with the problem of approximating an entire function by a finite polynomial. We will come back to these problems in Sec. III. Let us finally mention that, by construction, the function $R_m(p,q)$ will have the same zeros as $\psi_m(z_b)$.

We now consider the special case $s=\frac{1}{2}$, which is used in our treatment of the zeros in Secs. III and IV. Hamiltonian (1.1) becomes in this case

$$\begin{aligned}H=&\hbar\omega a^\dagger a+2\omega_0 S_z\\ &+\sqrt{\hbar/2}\lambda[(S_+a+S_-a^\dagger)+\epsilon(S_+a^\dagger+S_-a)]\ .\end{aligned} \tag{2.11}$$

The conserved parity operator [1] is given by

$$P=2S_zT\ , \tag{2.12}$$

with

$$T=(-1)^{a^\dagger a}\ .$$

Then, the eigenstate $|\psi^E\rangle$ of H corresponding to eigenvalue E of (for definiteness) positive parity (i.e., eigenvalue $+1$ of the operator P) may be written as

$$\begin{aligned}|\psi^E\rangle&=\sum_{n=0}^{\infty}\sum_{m=-1/2}^{1/2}\psi_{nm}|n\rangle\otimes|m\rangle\\ &\equiv\sum_{n\text{ even}}\psi_{n,1/2}|n\rangle\otimes|\tfrac{1}{2}\rangle\\ &\quad+\sum_{n\text{ odd}}\psi_{n,-1/2}|n\rangle\otimes|-\tfrac{1}{2}\rangle\ ,\end{aligned} \tag{2.13}$$

where $S_z|\pm\frac{1}{2}\rangle=\pm\frac{1}{2}|\pm\frac{1}{2}\rangle$. We now restrict ourselves to a reduced section (2.10), which we take to be $m=+\frac{1}{2}$ ("spin up"). Hence, denoting by $|\psi^E_{1/2}\rangle$ the "spin-up" component of the eigenstate (2.13), one has

$$|\psi^E_{1/2}\rangle\equiv\sum_{n\text{ even}}c_n|n\rangle\ , \tag{2.14}$$

where we wrote c_n for $\psi_{n,1/2}$. This wave function may be written in the Bargmann representation as

$$\psi_{1/2}^E(z) \equiv \langle z | \psi_{1/2}^E \rangle = \sum_{n \text{ even}} \frac{c_n}{\sqrt{n!}} z^n , \tag{2.15}$$

where we set $z_b = z$ in (2.9). From (2.10) the *reduced quantum Poincaré section* is thus

$$R_{1/2}(z) = e^{-|z|^2} |\psi_{1/2}^E(z)|^2 \equiv W_{\psi_{1/2}^E}(z) . \tag{2.16}$$

As mentioned before, this is precisely the Husimi function associated to the "spin-up" component (2.14). We have numerically verified that the distribution of zeros of $W_{\psi_{1/2}^E}(z)$ does not depend qualitatively, either on the section ($m = \pm\frac{1}{2}$) or on the parity ($P = \pm 1$). We shall therefore henceforth consider only the "spin-up" case with positive parity. In order to search for eigenvalues and eigenvectors of Hamiltonian (2.11) it is convenient to introduce the following unitary operator on $\mathbf{C}^2 \otimes \mathcal{F}$:

$$U = (S_z + iS_y) + (S_z - iS_y) T . \tag{2.17}$$

It may be proved [3,18] that H, given by (2.11), is unitarily equivalent through U to a much simpler Hamiltonian $\tilde{H}$ involving only the operators S_z, a, and $a^\dagger$:

$$\tilde{H} = UHU^{-1} , \tag{2.18}$$

with

$$\tilde{H} = \hbar\omega a^\dagger a + 2\omega_0 S_z T + \sqrt{\hbar/2}\lambda \left\{ \frac{1+\epsilon}{2}(a + a^\dagger) + (1-\epsilon) S_z (a^\dagger - a) T \right\} . \tag{2.19}$$

This Hamiltonian is of tridiagonal type and therefore easily numerically diagonalizable. Clearly, from (2.18)

$$|\tilde{\psi}^E\rangle = U |\psi^E\rangle \tag{2.20}$$

is an eigenstate of $\tilde{H}$ corresponding to the same eigenvalue E:

$$\tilde{H} |\tilde{\psi}^E\rangle = E |\tilde{\psi}^E\rangle , \tag{2.21}$$

and the parity operator (2.12) becomes

$$\tilde{P} = UPU^{-1} = 2S_z .$$

Our choice of section ($m = \frac{1}{2}$) corresponds to restricting ourselves to the eigenspace associated to the eigenvalue $+1$ of $\tilde{P}$. Hence $|\tilde{\psi}^E\rangle$ may be written as

$$|\tilde{\psi}^E\rangle = \sum_{n=0}^{\infty} \tilde{c}_n |n\rangle \otimes |\tfrac{1}{2}\rangle , \tag{2.22}$$

where the coefficients are $\tilde{c}_n = \psi_{n,1/2}$ for n even and $\tilde{c}_n = \psi_{n,-1/2}$ for n odd.

III. THE COEFFICIENTS: RECURSION RELATIONS AND ORDER OF $\psi_{1/2}^E(z)$

The coefficients c_n may be found from (2.13), (2.20), and (2.22) if we determine the $\tilde{c}_n$'s. It follows readily from (2.19), (2.21), and (2.22) that the $\tilde{c}_n$'s satisfy the following recursion relations:

$$\lambda\sqrt{n+1}\tilde{c}_{n+1} = -\lambda\epsilon\sqrt{n}\tilde{c}_{n-1} - \sqrt{2/\hbar}(\hbar\omega n + \omega_0 - E)\tilde{c}_n \quad \text{for } n \text{ even} , \tag{3.1a}$$

$$\lambda\epsilon\sqrt{n+1}\tilde{c}_{n+1} = -\lambda\sqrt{n}\tilde{c}_{n-1} - \sqrt{2/\hbar}(\hbar\omega n - \omega_0 - E)\tilde{c}_n \quad \text{for } n \text{ odd} , \tag{3.1b}$$

with

$$\tilde{c}_1 = -\sqrt{2/\hbar}\frac{1}{\lambda}(\omega_0 - E)\tilde{c}_0 . \tag{3.1c}$$

The $\tilde{c}_n$'s depend rather sensitively on the eigenvalue E, and it was seen that their determination from the above recursion relations (with E given by numerical diagonalization of $\tilde{H}$) was plagued by numerical instabilities, which led to a divergence of the $\tilde{c}_n$'s for large n. On the other hand, because of the $\sqrt{n!}$ in the denominator in (2.15), it is the first coefficients $\tilde{c}_n$ (i.e., for small n) which are important, but the determination of these by numerical diagonalization of (2.19) was seen to be unreliable. We therefore found the $\tilde{c}_n$'s for small n by the recursion (3.1), and for large n by numerical diagonalization of (2.19). Equations (3.1a)–(3.1c) show that all coefficients are uniquely determined when $\tilde{c}_0$ is given. The first coefficient $\tilde{c}_0$ was chosen in such a way that the two branches (for large and small n) join at a certain point. The resulting curve turned out to be smooth (see Appendix), giving us confidence in the procedure. Unfortunately, this method does not work for higher spin values, because no simple recursion relation is available in this case and we have not found a substitute.

In the rest of this section, we will address problems (a) and (b) mentioned in the Introduction. Since $\psi_{1/2}^E(z)$ is not simply a polynomial of finite degree, these questions require a detailed discussion.

The function $\psi_{1/2}^E(z)$, defined by (2.15), is an entire analytic function of z. Let $M(r)$ denote the maximum modulus of $\psi_{1/2}^E(z)$ for $|z| = r$. The order of $\psi_{1/2}^E(z)$ is a number ρ such that, for every positive δ but for no negative δ

$$M(r) = O(e^{r^{\rho+\delta}})$$

when $r \to \infty$ (see [13]). When the order is finite, it depends on the asymptotic behavior of the coefficients [see (2.15)]

$$d_n \equiv \frac{c_n}{\sqrt{n!}} \tag{3.2}$$

in a simple way:

$$\rho = \lim_{n\to\infty} \sup \frac{n \ln n}{\ln(1/|d_n|)} . \tag{3.3}$$

How does the order relate to problems (a) and (b) of the Introduction? The basic ingredient is Hadamard's theorem on the factorization of entire functions: if $f(z)$ is an entire function of order ρ with a k-fold zero at the origin, we have

$$f(z) = z^k e^{Q(z)} P(z) ,$$

where $Q(z)$ is a polynomial of degree $q \le \rho$ and $P(z)$ is a canonical product [13] built from the zeros (other than $z=0$) of $f(z)$. A corollary is that, if $\rho<1$, $Q(z)$ is a constant and therefore $f(z)$ is determined by its zeros. This concerns problem (b). Problem (a) has a clear-cut answer at least in one special case: an entire function of nonintegral order has an infinite number of zeros. A well-known counterexample for integral order is provided by the exponential function $f(z)=\exp(z)$, which has order $\rho=1$, and no zero at all. However, the function $\cosh z=[\exp(z)+\exp(-z)]/2$, which has the same integral order $\rho=1$, has an infinite number of zeros located along the imaginary axis, illustrating the fact that the situation for integral order is much more delicate.

It follows from (3.3) that if the c_n's are such that

$$c_n = O((n!)^{-\beta}) \tag{3.4}$$

when $n>N_0$ for some fixed but large N_0, then the order of $\psi^E_{1/2}(z)$ is

$$\rho=\frac{1}{\beta+\frac{1}{2}} \ . \tag{3.5}$$

Figure 1 shows the logarithmic plot of $|c_n/c_{N_0}|$ as a function of $n \ge N_0$, for the special case of an eigenstate of Hamiltonian (2.11) for one set of parameters in the regular region ($\epsilon=0.01$). The curves representing $\ln[(N_0!/n!)^{1/2}]$ and $\ln[(N_0!/n!)^{3/2}]$ are shown for comparison. Clearly in this case (3.4) holds with $\frac{1}{2}<\beta<\infty$, which implies, by (3.5),

$$0<\rho<1 \ . \tag{3.6}$$

This particular Husimi function is studied in our treatment of zeros in Sec. IV. From (3.6) and the previously cited theorems, we may conclude that, in this special case, the zeros of $W_{\psi^E_{1/2}}(z)$ determine the function $\psi^E_{1/2}(z)$, and that there is an infinity of zeros. There are, however, several limitations to this result. The asymptotic behavior (3.4) depends sensitively on E, so that it is not clear whether in each region of parameters $(\lambda,\epsilon,\hbar,E)$—integrable or chaotic—a representative $|\psi^E_{1/2}\rangle$ may be found, such that the zeros of $W_{\psi^E_{1/2}}(z)$ determine $\psi^E_{1/2}(z)$. In fact, for one set of parameters in the chaotic region ($\epsilon=1$, the one where the distribution of zeros was also found in Sec. IV), a similar plot for the coefficients of the corresponding eigenstate led to inconclusive results, i.e., it was not possible to assert that $\rho<1$ conclusively, because there the range of n values was too small.

What conclusions may be drawn from these remarks? In the case of unbounded phase space, the zeros of $W_{\psi^E_{1/2}}(z)$ may determine $\psi^E_{1/2}(z)$, and the set of zeros of $W_{\psi^E_{1/2}}(z)$ may be infinite. It is not clear "how often" this occurs in physical models with unbounded phase space (i.e., whether this is an exception, or the rule, or neither), but there is, of course, not a priori reason for any of these properties to hold [19]. However, even if the zeros do not completely determine $\psi^E_{1/2}(z)$ in some cases (e.g., for the case shown in Sec. IV which corresponds to the classically chaotic region), it is interesting to remark that a transition from a one-dimensional alignment to a partially diffusive behavior occurs, although then the picture conveyed by the zeros is rougher with regard to the form of the Husimi function.

We end this section with the following remark. It is often wrongly asserted that the order of $\psi^E_{1/2}(z)$ is 2, because

$$|\psi^E_{1/2}(z)| \le e^{|z|^2/2} \ . \tag{3.7}$$

Equation (3.7) is indeed true, and follows from (2.15) and the Schwartz inequality, together with the normalization condition

$$\sum_{n=0}^{\infty} |c_n|^2 = 1 \ .$$

However, Eq. (3.7) gives just an upper bound, which implies that $\rho \le 2$. In fact, the case

$$f(z)=\sum_{n=0}^{\infty}\frac{z^n}{(n!)^{\alpha}}$$

is standard, and corresponds to order $\rho=1/\alpha$ [the same happens if $f(z)=\sum_{n=0}^{\infty}c_n z^n$, with $c_n=O[(n!)^{-\alpha}]$ see [20], example 1, p. 255]. If $\frac{1}{2}\le\alpha\le\infty$, (3.7) remains true but $0\le\rho\le 2$.

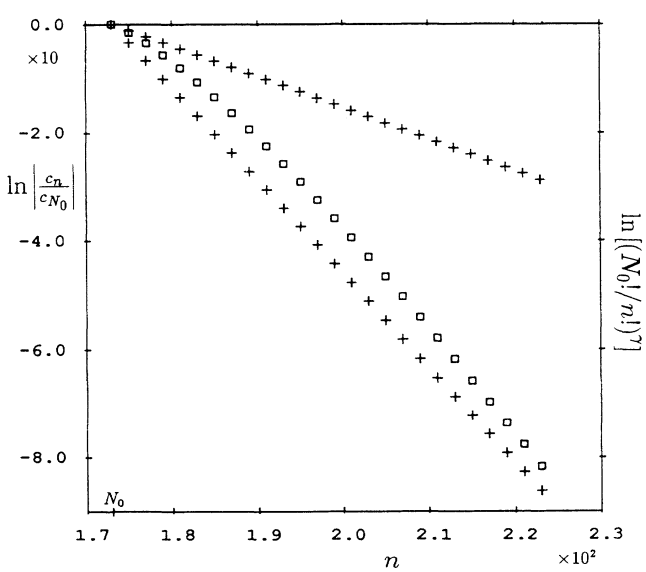

FIG. 1. Large-n behavior of the coefficients c_n of $|\psi^E_{1/2}\rangle$ with $E=-2.0485\times10^{-3}$ and $\epsilon=0.01$, $\lambda=1.5$, $\hbar=0.01$, $\omega=\omega_0=1$ in (2.11). The □ plot shows $\ln|c_n/c_{N_0}|$ vs n for $n\ge N_0$ ($=172$). The + plots show $\ln[(N_0!/n!)^\gamma]$, $n\ge N_0$, for $\gamma=\frac{1}{2}$ (upper curve) and $\frac{3}{2}$ (lower curve).

IV. SOME NUMERICAL RESULTS

We have studied the distribution of zeros of the Husimi function associated with the "spin-up" component of several eigenstates of Hamiltonian (2.11). We get the eigenvalues of H by diagonalizing a 1200×1200 matrix and the coefficients c_n in (2.15) have been obtained as explained in Sec. III. Here we present two significant results. In (2.11) we fix $\hbar$ "small" ($\hbar=0.01$ corresponding to a classical boson regime), $\lambda=1.5$, $\omega=\omega_0=1$, and we vary ϵ from 0 to 1. Figure 2 shows the distribution of zeros for the "spin-up" component of an eigenstate corresponding to a classical integrable case ($\epsilon=0.01$) whereas Fig. 3 corresponds to a classical chaotic case ($\epsilon=1$). These two eigenstates have comparable energy $E\simeq 0$ (see figure captions). Similar results for the Husimi zeros have also been found for some other eigenstates in the same energy range. We have verified the aforementioned classical behavior—integrable or chaotic—in the corresponding Poincaré sections, which are not shown here. In Figs. 2 and 3 a clear transition from regular (one-dimensional) to partially diffusive behavior takes place for the zeros. One remarks in Fig. 3 that some zeros still concentrated along curves; this coexistence of some diffusive zeros with others concentrated on lines is similar to what has been observed, in the semiclassical limit, for the distribution of roots of eigenstates of classically mixed systems [7]. In our case, this behavior is probably due to the fact that the spin quantum number is indeed $s=\frac{1}{2}$ (extreme quantum limit for the spin) and we are not therefore in the full semiclassical regime. But since the role of the spin is to bring about a nonlinear effective self-interaction in the boson field, it may not be too surprising that some classical features are still visible in (1.1) for $\hbar$ small even if $s=\frac{1}{2}$. One might think that for a large s these curves will disappear and that the distribution of zeros will become fully spread out. We have not verified this conjecture which clearly requires further study.

Moreover, we point out that the triangle located at the origin of Fig. 2 represents 98 zeros which are all concentrated in a little neighborhood near the origin. This is related to the fact that we are not far from the $\epsilon=0$ case for which (2.11) is analytically diagonalizable and where it is easy to see that each $\psi_{1/2}^{E}(z)$ has only a multiple zero at the origin [only one eigenstate $|n\rangle$ contributes to the sum (2.14) and so the Husimi function has a monomial form]. For increasing values of ϵ this structure breaks down and the zeros progressively leave the origin. This seems to occur in Figs. 2 and 3: for $\epsilon=0.01\gtrsim 0$ the zeros are close to the origin (some of them remain localized very near the center) whereas for higher values of ϵ ($\simeq 1$), the state has its roots spread out farther on the phase space.

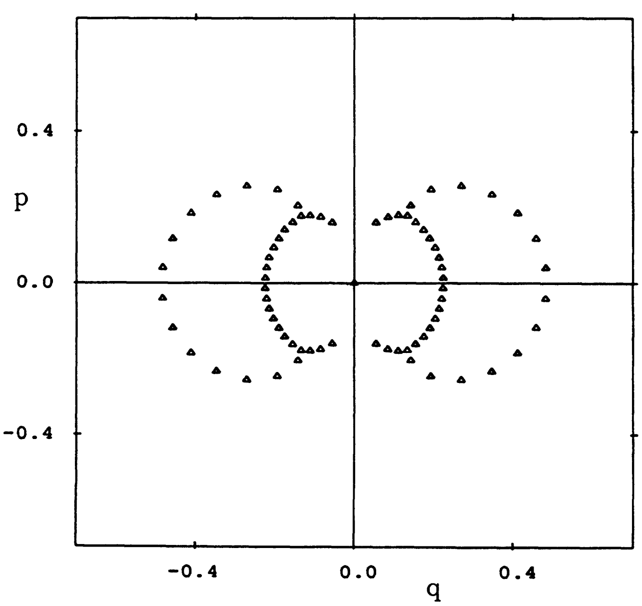

FIG. 2. Phase-space distribution of zeros in the boson plane of $\psi_{1/2}^{E}(z)$ ($E=-2.0485\times 10^{-3}$), classical regular regime: $\epsilon=0.01$, $\lambda=1.5$, $\hbar=0.01$, $\omega=\omega_0=1$ in (2.11).

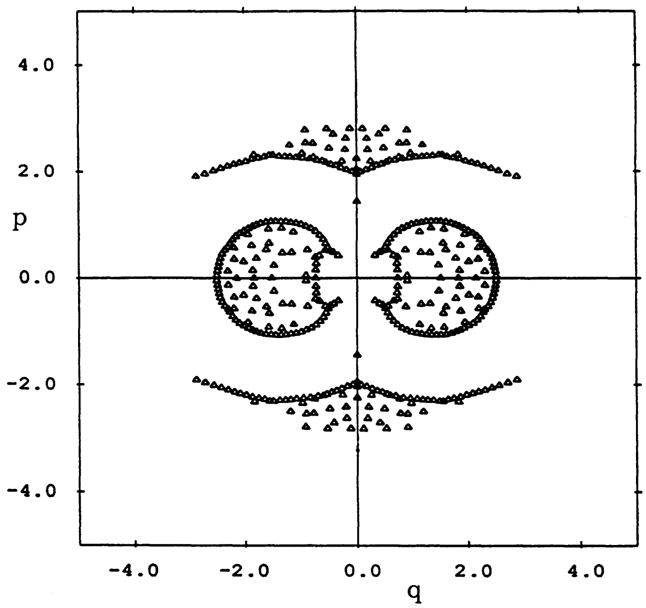

FIG. 3. Phase-space distribution of zeros in the boson plane of $\psi_{1/2}^{E}(z)$ ($E=5.2288\times 10^{-3}$), classical chaotic regime: $\epsilon=1$, $\lambda=1.5$, $\hbar=0.01$, $\omega=\omega_0=1$ in (2.11).

The numerical method used to find these distributions of zeros is based on the well-known result of complex analysis relating the number of zeros of an analytic function on a certain domain with the variation of its phase on the boundary. By dividing the phase space into smaller cells, one can determine the position of the zeros very precisely. We have checked their stability with respect to the numerical precision of the coefficients and with respect to the size of the diagonalized matrix. In an Appendix we discuss also how the "spurious zeros" introduced by truncation of the power series in (2.15) were handled.

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APPENDIX

In this appendix we consider the effect of spurious zeros introduced by the process of truncation of the power series representing the function $\psi^E_{1/2}(z)$ [given by (2.15)] at some value $n=N$, thereby replacing $\psi^E_{1/2}(z)$ by

$$\psi^{E,N}_{1/2}(z) \equiv \sum_{n=0}^{N} \frac{c_n}{\sqrt{n!}} z^n . \tag{A1}$$

With

$$\sum_{n=0}^{\infty} |c_n|^2 < \infty ,$$

it is clear that $\psi^E_{1/2}(z)$ (no truncation) is an entire analytic function, and that

$$\psi^{E,N}_{1/2}(z) \to \psi^E_{1/2}(z) \tag{A2}$$

as $N\to\infty$ uniformly in compact subsets of **C**. Hence, by Hurwitz theorem (a corollary of Rouché's theorem [20]), z_0 is a zero of $\psi^E_{1/2}(z)$ if, and only if, it is a limit point of the set of zeros of the functions $\psi^{E,N}_{1/2}(z)$ (points which are zeros for an infinity of values of N being counted as limit points). Because of (A2), when $N\to\infty$ one has

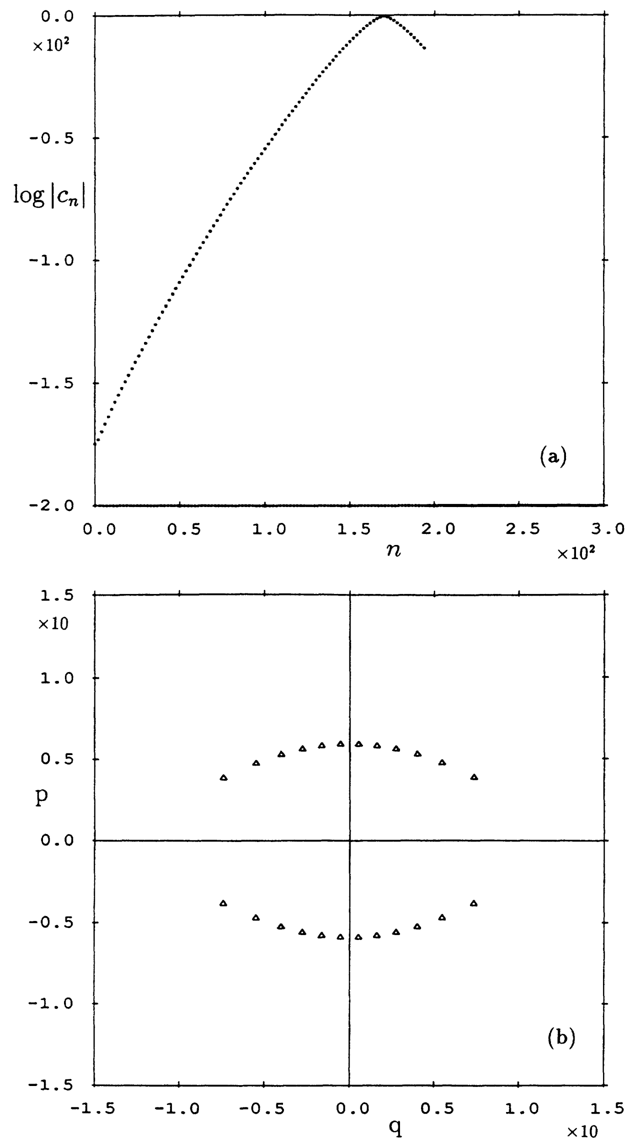

FIG. 4. (a) Logarithmic plot of the coefficients $|c_n|$ as a function of n (truncated at $N_1=195$) for the state $|\psi^E_{1/2}\rangle$ of Fig. 2 (regular case). The logarithm is base 10. (b) Corresponding set of spurious zeros of $\psi^{E,N_1}_{1/2}(z)$, see (A1). The stable zeros shown in Fig. 2 are not drawn here because of scale reasons.

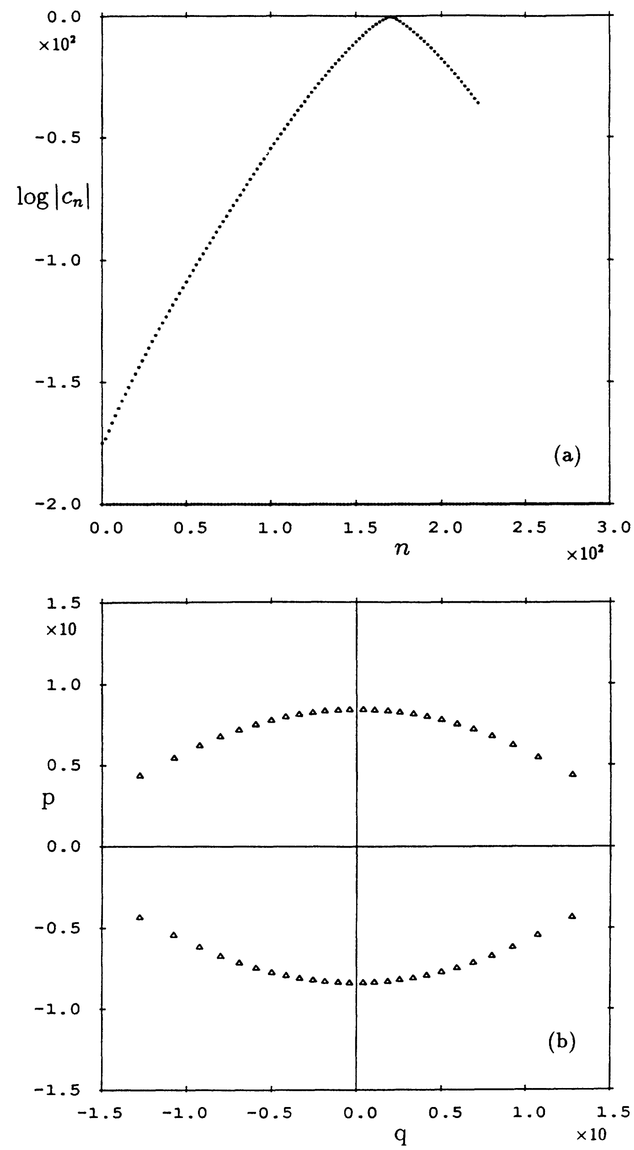

FIG. 5. The same as Figs. 4(a) and 4(b) but for a truncation value $N_2=223$.

$$\frac{1}{2\pi i}\oint_C \frac{[\psi^{E,N}_{1/2}(z)]'}{\psi^{E,N}_{1/2}(z)}dz \to \frac{1}{2\pi i}\oint_C \frac{[\psi^{E}_{1/2}(z)]'}{\psi^{E}_{1/2}(z)}dz ,$$

where C is the boundary of a fixed subset of **C**. Hence the spurious zeros [i.e., zeros of $\psi^{E,N}_{1/2}(z)$ which do not tend to zeros of $\psi^{E}_{1/2}(z)$] must leave any finite region, as $N \to \infty$.

To test for these spurious zeros, we compare two truncations [two different N's in (A1)] for the states of Figs. 2 and 3. Notice that for these different truncations we determine always the coefficients c_n in the same way, keeping, in particular, unchanged the size of the diagonalized matrix. We first consider the case of Fig. 2 ($\epsilon=0.01$). Figures 4(a) and 5(a) are logarithmic plots showing the coefficients $|c_n|$ as a function of n, for $n \le N_1$ and $n \le N_2$ ($N_1 < N_2$), respectively. Figures 4(b) and 5(b) show the corresponding sets of spurious zeros (not shown in Fig. 2): notice that their location has varied (the variation in number is of course normal). In contrast, the position of those zeros shown in Fig. 2 remained rigorously stable under this process of truncation. Similarly, Figs. 6(a) and 7(a) show the logarithmic plots of the coefficients $|c_n|$ for two truncations N_1' and N_2' in the case of Fig. 3 ($\epsilon=1$). In Figs. 6(b) and 7(b) we show the corresponding distributions of zeros. Again, we see that the location of some zeros has changed and Fig. 3 displays only those which remained unchanged under this process.

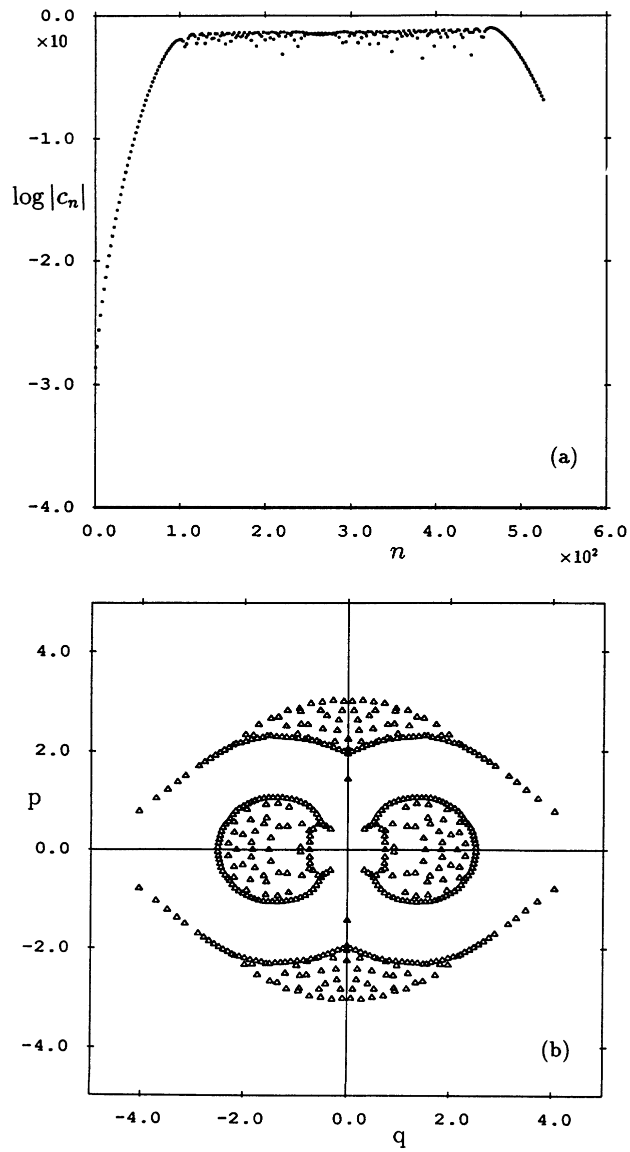

FIG. 6. (a) Logarithmic plot of the coefficients $|c_n|$ as a function of n (truncated at $N_1'=527$) for the state $|\psi^E_{1/2}\rangle$ of Fig. 3 (chaotic case). The logarithmic is base 10. (b) Corresponding distribution of zeros of $\psi^{E,N_1'}_{1/2}(z)$, see (A1).

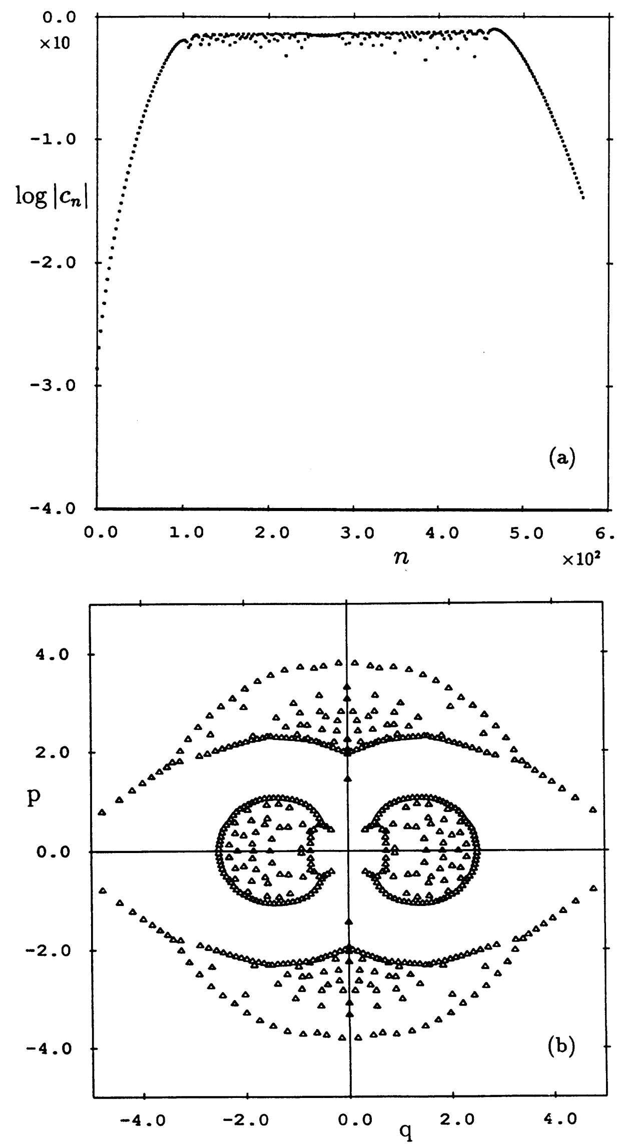

FIG. 7. The same as Figs. 6(a) and 6(b) but for a truncation value $N_2'=571$.

As indicated in Sec. III, the coefficients shown in Figs. 4(a) and 5(a) and 6(a) and 7(a) have been found by the recursion relation (3.1) for small n (i.e., in the increasing part of the plot) and by usual diagonalization routines for large n (i.e., in the other part of the plot). The two branches join at a certain point, and note that the plots of the coefficients c_n as a function of n have the smooth behavior referred to in Sec. III.

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