Finite violation of a Bell inequality for arbitrarily large spin

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(Received 20 August 1991)

A pair of spin-j particles, prepared in a singlet state, move away from each other and are examined by two distant observers. A simple experimental procedure can produce a 24% violation of a Bell inequality, for arbitrarily large j.

PACS number(s): 03.65.Bz

Bell's inequality [1] is an upper bound on the correlations of distant events. The only assumption needed to derive that inequality is the *principle of local causes*—also called Einstein locality—which asserts that events occurring in a given space-time region are independent of external parameters that may be controlled, at the same moment, by agents located in distant space-time regions [2].

The violation of Bell's inequality by quantum theory is the most spectacular departure of quantum physics from the canons of classical realism. It is therefore interesting to examine whether quantum theory still asymptotically satisfies Bell's inequality in the limit of large quantum numbers, a limit which is commonly associated with the emergence of classical properties. It was shown long ago by Garg and Mermin [3] that a pair of spin-j particles, in a singlet state, produced correlations violating Bell's inequality, for spin measurements along nearly all pairs of directions. However, the magnitude of the violation vanished exponentially for large spin. Garg and Mermin conjectured (see footnote 15 of their Letter) that this vanishing might be due to the use of slowly varying functions of the spin components m, making their method insensitive to the rapidly varying part of the quantum correlations. The large-j limit was also investigated by other authors [4–6], who achieved various degrees of improvement.

In this Brief Report, I describe a simple *gedanken* experiment, involving two spin-j particles in a singlet state. These particles are examined by two distant observers, without mutual interaction. Each observer records the quantity $(-1)^{j-m} = \pm 1$, which is a rapidly varying function of m. For an optimal choice of macroscopic parameters controlled by the observers, and for $j \to \infty$, the Clauser-Horne-Shimony-Holt (CHSH) inequality [7] (which is an experimentally convenient variant of Bell's original inequality) is violated by a constant amount: 2.481 in lieu of 2, the classical upper bound.

The experiment is performed as follows: The two spin-j particles are prepared in an unstable singlet state [8],

$$|\Psi\rangle = (2j+1)^{-1/2} \sum_{m=-j}^{j} (-1)^{j-m} |m\rangle \otimes |-m\rangle, \quad (1)$$

and fly apart along the $\pm x$ directions (collimators eliminate those particles going toward other directions). The two distant observers, labeled 1 and 2, apply to their particles arbitrary torques around the direction of motion; for example, they make them pass through solenoids, in which each observer can control the magnetic field. The state of the pair thus becomes

$$e^{-i\theta_1 J_{1x}} e^{-i\theta_2 J_{2x}} |\Psi\rangle = e^{-i(\theta_2-\theta_1) J_{1x}} |\Psi\rangle, \quad (2)$$

because $J_{2x}|\Psi\rangle = -J_{1x}|\Psi\rangle$ for a singlet state.

Each observer then performs a Stern-Gerlach-type experiment, to measure J_{1z} and J_{2z}, respectively. There is no fundamental limitation to the resolution that can be achieved, because it is always possible, at least in principle, to position the detectors so far away from the Stern-Gerlach magnets that the $2j+1$ beams are well separated, and the corresponding m can be precisely known. (An equivalent experiment would be to apply no torque, and to rotate the Stern-Gerlach magnets, together with all the detectors, by angles θ_1 and θ_2.)

The Stern-Gerlach experiment measures (among other things) the following dichotomic variable:

$$\sum_{m=-j}^{j} |m\rangle \, (-1)^{(j-m)} \, \langle m| = e^{i\pi(j-J_z)}, \quad (3)$$

with eigenvalues ± 1. The *correlation* of the values obtained by the two observers for these dichotomic variables is the expectation value of their product:

$$C(\theta) = e^{2\pi i j} \langle\Psi| e^{i\theta J_{1x}} e^{-i\pi J_{1z}} e^{-i\pi J_{2z}} e^{-i\theta J_{1x}} |\Psi\rangle, \quad (4)$$

where $\theta = \theta_2 - \theta_1$, for brevity. Note that $e^{-i\pi J_{2z}}$ and $e^{-i\theta J_{1x}}$ commute, that $e^{-i\pi J_{2z}}|\Psi\rangle = e^{i\pi J_{1z}}|\Psi\rangle$ (because $|\Psi\rangle$ is a singlet state), and that $e^{-i\pi J_z}$ generates a rotation by an angle π around the z axis. We thus have

$$e^{-i\pi J_{1z}} e^{-i\theta J_{1x}} e^{i\pi J_{1z}} = e^{i\theta J_{1x}}, \quad (5)$$

and therefore

$$C(\theta) = e^{2\pi i j} \langle\Psi| e^{2i\theta J_{1x}} |\Psi\rangle = e^{2\pi i j} \langle\Psi| e^{2i\theta J_{1z}} |\Psi\rangle, \quad (6)$$

thanks to rotational invariance. Together with (1), this gives

$$C(\theta) = (2j+1)^{-1}\, e^{2\pi i j} \sum_{m=-j}^{j} (-1)^{2(j-m)}\, \langle m| \otimes \langle -m|\, e^{2i\theta m}\, |m\rangle \otimes |-m\rangle, \tag{7}$$

$$= (-1)^{2j}\, \sin[(2j+1)\theta]/(2j+1)\sin\theta. \tag{8}$$

We can now apply the CHSH inequality: If the first observer has a choice between parameters θ_1 and θ_3, and the second observer between θ_2 and θ_4, then it is a consequence of local realism that [7]

$$|C(\theta_1-\theta_2) + C(\theta_2-\theta_3) + C(\theta_3-\theta_4) - C(\theta_4-\theta_1)| \leq 2. \tag{9}$$

Let us take $\theta_1-\theta_2 = \theta_2-\theta_3 = \theta_3-\theta_4 = x/(2j+1)$. When $j \to \infty$, the left-hand side of (9) tends to a constant, whose maximum value is obtained for $x = 1.054$:

$$\frac{3\sin x}{x} - \frac{\sin 3x}{3x} = 2.481. \tag{10}$$

Thus, if the resolution of our instruments is good enough for discriminating between consecutive values of m, their readings may strongly violate classical local realism. This conclusion agrees with a statement by Mermin and Schwarz [9]: "... no matter how large j may be, measurements that discriminate between the $2j+1$ values of m are inherently nonclassical. There is no reason to expect classical behavior to be approached in a uniform manner."

These results can be considered as complementary to those of preceding authors [3–6]. Here, the *magnitude* of the violation (that is the *ratio* of the quantum correlation to the maximal classical one) tends to a constant, which is appreciably larger than 1. On the other hand, the range of parameters θ for which this violation is achieved becomes vanishingly small for large j.

After completion of this work, I was informed by N. Gisin that he had found other, more sophisticated operators, giving for the same $|\Psi\rangle$ the maximal violation of the CHSH inequality allowed by quantum theory [10,11], namely $2\sqrt{2}$. However, an experimental realization of Gisin's matrices is much more complicated than the simple experiment proposed here.

Note added in proof. During editorial processing of this paper, Refs. [12] and [13] appeared, which deal with the same subject.

This work was supported by the Gerard Swope Fund and by the Fund for Encouragement of Research at Technion.

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