Wave-function quantum stochastic differential equations and quantum-jump simulation methods

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The quantum-stochastic-differential-equation formulation of driven quantum-optical systems is carried out in the interaction picture, and quantum stochastic differential equations for wave functions are derived on the basis of physical principles. The Ito form is shown to be the most practical, since it already contains all the radiation reaction terms. The connection between this formulation and the master equation is shown to be very straightforward. In particular, a direct connection is made to the theory of continuous measurements, which leads directly to the method of quantum-jump simulations of solutions of the master equation. It is also shown that all conceivable spectral and correlation-function information in output fields is accessible by means of an augmentation of the simulation process. Finally, the question of the reality of the jumps used in the simulations is posed.

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I. INTRODUCTION

The introduction of Markovian quantum stochastic methods into quantum optics was a major innovation that enabled computations to be carried out rather simply, but quite accurately. Haken [1] and Lax [2], following from the pioneering work of Senitzky [3], developed methods based on quantum "Langevin equations," which were based on a Heisenberg picture formulation. For practical computations with reliable controllable approximations, quantum Langevin equations are not very useful, largely because of the technical difficulties associated with the fact that they are normally nonlinear operator equations. Louisell [4] introduced the idea of the Markovian master equation into quantum optics, and the use of such master equations has proved one of the most powerful methods in quantum optics.

The rigorous basis of these two methods has been available for some time. The foundations of the master equation method are summarized in the book by Davies [5] and the review article by Spohn [6] gives some idea of the validity of these methods in physics. The rigorous formulation of quantum stochastic differential equations has been established in the work of Hudson and Parthasarthy [7], and applied by Barchielli [8], while the work of Collett and Gardiner [9, 10] showed the relationship between the physical "quantum Langevin equations" and the more mathematically precise "quantum stochastic differential equations." This yielded what was called an "input-output formalism," which was indispensible for the understanding of the production of beams of squeezed light. Applications from a mathematical point of view can be found in Ref. [11].

Note, however, the work of Collett and Gardiner was based entirely on the equations of motion for operators, expressed as quantum stochastic differential equations—that is, it was based on a Heisenberg picture formulation, which, as mentioned above, can be very difficult to calculate with. In fact, a formulation of quantum stochastic differential equations based on the Schrödinger picture has been available for some time, but has not found much use in practice. Of course, such a formulation could be very useful, because of the linearity of the resulting equations of motion.

The aim of this paper is to develop the physical basis for such a formulation, which we will do in the interaction picture, rather than in the Schrödinger picture. Choosing the interaction picture leaves all the fast but uncomplicated optical time development in the operators, while the slower, but more difficult fluctuation and damping phenomena take place in the state vector, and are thus represented by a linear equation.

In order to make the paper reasonably self-contained and conceptually accessible, some effort is devoted in Secs. II and III to developing the quantum-stochastic-differential-equation formalism, and establishing the physical approximations that are used in this process. Quantum stochastic integration is defined simply, and the technical methodology made clear, but no rigorous basis is given, since this is available elsewhere to those who require it [7, 8]. The concepts of the Stratonovich and the Ito equations are described, and the physical basis for each kind of equation is explained. Briefly, in the Stratonovich equation in quantum optics the system is driven by both an incoming field and a self-field. In the Ito equation, the self-field is explicitly solved for, and eliminated, leaving an equation that is driven only by an input field, but that has an added damping term which is the radiation reaction generated by the self-field.

From a physical point of view, it is clear that the equation in which the effect of radiation reaction has been explicitly included is much to be preferred. There is a further advantage, which is always stressed from a mathematical point of view, that the driving field in the Ito

formulation is stochastically independent of the variables in the equation it drives. This is because it is effectively the field in the immediate future, which has not yet been affected by the system it drives.

In Sec. IV, the relationship between our quantum-stochastic-differential-equation formulation and the continuous-measurement formalism of Srinivas and Davies [12] is explained. It is shown that the equations of motion that arise from the continuous-measurement formalism are derivable directly from the state-vector quantum stochastic differential equations. On the other hand, the continuous-measurement formalism has the interpretation as a probabilistic description of the process of quantum jumps in a system in interaction with a light field. Thus the continuous-measurement theory can be simulated probabilistically, and this simulation yields a sequence of wave functions, which jump at times $t_1, t_2, t_3, \ldots$, by a rule that is directly related to the structure of the appropriate quantum stochastic differential equation.

In Sec. V we show how the continuous-measurement theory can be extended with the aid of quantum-stochastic-differential-equation theory to yield methods of computing a range of different kinds of spectra of output fields. This is done by separating the concept of a driving field from that of a measurable output field. From this we can define extra state vectors, obtained by operating on the vector $|\varphi, t\rangle$, which represents the state of the system with various combinations of output field operators. These equations are coupled to each other and to the vector $|\varphi, t\rangle$, but $|\varphi, t\rangle$ itself is not coupled to these extra vectors. From this we develop a generalization of continuous-measurement theory which enables us to determine arbitrary output spectra, without really a great deal of extra computation.

The method of computing the intensity spectrum is interesting, in that it uses the gauge or photon-counting process $d\Lambda(t)$, introduced in Ref. [7] and applied by Barchielli [8] in a somewhat formal context.

Simulations based on quantum jumps have been recently proposed by Dum, Zoller, and Ritsch [13] as an economical method of computing the time evolution of quantum optical systems, and Dalibard, Castin, and Mølmer [14] and Carmichael [15] have also proposed similar methods. Because the work presented here is based on a wave-function picture which includes the field, we are able not only to give a firm foundation to this simulation method, but also to show directly how to compute spectra by a very similar simulation method.

The paper ends with a comparison with the work of Mollow [16], who as long ago as 1975 proposed a treatment of the interaction of light and atoms which is physically almost the same as our use of quantum stochastic differential equations for wave functions; however, it is not quite as general or as easy to apply, because all the work that is done by the Ito formalism in our method has to be done by hand in Mollow's treatment.

This paper is the first of a pair; in the second paper [17] we will demonstrate applications of the quantum jump method to a variety of practical problems.

II. DEVELOPMENT OF THE FORMALISM

We want to consider a one-sided one-dimensional setup, consisting of a "system" (perhaps an atom) interacting with the one-dimensional electromagnetic field. We can write down a formal Hamiltonian for this as

$$H = H_{\rm sys} + \int_0^\infty dx \kappa(x)\{c^\dagger \dot{A}^{(+)}(x,t) + c\dot{A}^{(-)}(x,t)\} + \frac{1}{2}\int_0^\infty dx \left\{\dot{A}(x,t)^2 + \frac{1}{c^2}[\partial_x A(x,t)]^2\right\} . \quad (1)$$

Here $A(x,t)$ is a one-dimensional vector potential, $H_{\rm sys}$ is the Hamiltonian for the small "system" (i.e., atom), and $\kappa(x)$ is a function that in practice is almost a delta function. The vector potential has the expansion

$$A(x,t) = A^{(+)}(x,t) + A^{(-)}(x,t), \quad (2)$$

where

$$A^{(+)}(x,t) = \int_0^\infty d\omega \sqrt{\frac{\hbar}{2\pi\omega c}} \cos\left\{\frac{\omega x}{c}\right\} b(\omega) e^{-i\omega t} \quad (3)$$

and the canonical commutation relations are

$$[b(\omega), b^\dagger(\omega')] = \delta(\omega - \omega'). \quad (4)$$

This is an approximate description. A rotating-wave approximation has been made to eliminate the terms $c^\dagger A^{(-)}$ and $cA^{(+)}$, recoil of the atom has been assumed to be negligible, and of course the description is only one dimensional.

We can of course write a more simplified description by carrying out the x integrals; it is

$$H = H_{\rm sys} + i\hbar \int_0^\infty d\omega \tilde{\kappa}(\omega)\{c^\dagger b(\omega) - c b^\dagger(\omega)\} + \int_0^\infty d\omega \hbar\omega b^\dagger(\omega) b(\omega) \quad (5)$$

with

$$\tilde{\kappa}(\omega) = -\sqrt{\frac{\omega}{2\pi\hbar c}} \int_0^\infty dx\, \kappa(x) \cos\frac{\omega x}{c} . \quad (6)$$

The general structure of a realistic description of an atom in interaction with a light field, involving three dimensions, and two polarizations is similar. The definition for $\tilde{\kappa}(\omega)$ may change, and there will be several $b(\omega), b^\dagger(\omega)$ corresponding to the various angular momenta and polarizations. This is easily incorporated into the fundamental formalism.

A. The equations of motion for the wave function

Let us consider an atom in which there are operators, corresponding to possible transitions between energy levels, which we will call X_m^-, X_m^+, such that

$$[H_{\rm sys}, X_m^\pm] = \pm\hbar\omega_m X_m^\pm, \quad (7)$$

and let us suppose that the operator c is one of these, for which we use $\omega_m \to \Omega$ for brevity. We can then

consider these to be Schrödinger picture operators, and the Schrödinger picture equation of motion is

$$\frac{d|\varphi,t\rangle_s}{dt} = \left\{ \frac{\mathrm{i}}{\hbar}[H_{\rm sys}+H_B] + \int_0^\infty d\omega\tilde{\kappa}(\omega)[cb^\dagger(\omega)-c^\dagger b(\omega)]\right\}|\varphi,t\rangle_s. \tag{8}$$

We now move to an interaction picture, defined by

$$|\varphi,t\rangle = \exp\left[\frac{\mathrm{i}}{\hbar}[H_{\rm sys}+H_B](t-t_0)\right]|\varphi,t\rangle_s \tag{9}$$

and the operators of course have a time development given by

$$X_m^\pm(t)_I = \exp[\pm\mathrm{i}\omega_m(t-t_0)]X_m^\pm, \tag{10}$$

$$c(t) = \exp[-\mathrm{i}\Omega(t-t_0)]c, \tag{11}$$

$$b(\omega,t)_I = \exp[-\mathrm{i}\omega(t-t_0)]b(\omega), \tag{12}$$

where t_0 is the initial time at which the two pictures coincide. In the interaction picture the Schrödinger equation is

$$\frac{d}{dt}|\varphi,t\rangle = \int_0^\infty d\omega\tilde{\kappa}(\omega)[c(t)b^\dagger(\omega,t)_I - c^\dagger(t)b(\omega,t)_I]\,|\varphi,t\rangle\,. \tag{13}$$

Here

$$\tilde{\kappa}(\Omega) = \sqrt{\frac{\gamma}{2\pi}}\,. \tag{14}$$

We can now substitute Eqs. (10) and (12) into (13) to obtain the equation

$$\frac{d}{dt}|\varphi,t\rangle = \sqrt{\gamma}[cb^\dagger(t)-c^\dagger b(t)]|\varphi,t\rangle\,, \tag{15}$$

where

$$b(t) = \frac{1}{\sqrt{\gamma}}\int_0^\infty \tilde{\kappa}(\omega)b(\omega)e^{-\mathrm{i}(\omega-\Omega)(t-t_0)}d\omega\,. \tag{16}$$

It is important to realize that this definition of $b(t)$ shows that $b(t)$ is in fact a linear combination of *Schrödinger picture* operators, the parameter t arising from the coefficients in the linear combination. This will be very significant in the description of the initial states. As well, because $c, c^\dagger, b(t), b^\dagger(t)$ are essentially Schrödinger picture operators representing different degrees of freedom, $c, c^\dagger$ commute with $b(t), b^\dagger(t)$.

Up to this stage no approximations have been made in the derivation from the Hamiltonian (5) to obtain (15). We are now in a position to make an approximation which in the end yields a Markovian equation.

B. Markov approximation

Let us consider the commutator of $b^\dagger(t'), b(t)$,

$$[b(t),b^\dagger(t')] = \int_0^\infty d\omega\, e^{\mathrm{i}\omega(t-t')}e^{-\mathrm{i}\Omega(t-t')}|\tilde{\kappa}(\omega)|^2/\gamma\,. \tag{17}$$

The Markov approximation involves the following points.

(1) We assume that the solution of (15), the interaction picture equation of motion, is a rather slowly varying function of time, having a time scale which we shall call τ_D, the damping time. Thus it is assumed that

$$\tau_D \gg 1/\omega_m. \tag{18}$$

(2) It is also assumed that $\tau_D \gg 1/\Omega$, in particular.

(3) We also assume that $|\tilde{\kappa}(\omega)|^2$ is a very slowly varying function of ω over the range $|\omega-\Omega| \sim 1/\tau_D$. This means, from (6), that $\kappa(x)$ is very sharply peaked in a range $x/c \sim \tau_D$. In practice, what this means is that the atom is very much smaller in diameter than the distance that light could travel during one damping time. This is an approximation that is universally valid in quantum optics.

(4) In this case, most of the contribution of the integral (5) comes from $\omega \simeq \Omega$, and we can approximate the integrand by its value at $\omega = \Omega$:

$$|\tilde{\kappa}(\omega)|^2 = |\tilde{\kappa}(\Omega)|^2 = \gamma/2\pi, \tag{19}$$

and take the lower limit of the integral as $-\infty$ since Ω is so large compared with the inverse time scales of interest. In this case, we obtain

$$[b(t),b^\dagger(t')] = \delta(t-t')\,. \tag{20}$$

Although the commutator is very simple, it must not be forgotten that the $\delta(t-t')$ is nonetheless a very singular function, and one must be very careful when integrating.

(5) The formula (16) shows that $b(t)$ is a function of Ω, which is the frequency of oscillation of the interaction picture operator $c(t)$. This means that if more than one coupling $c_r(t)$ to the electromagnetic field occurs, each will have its own $b_r(t)$. Normally the frequency difference between the different Ω_r will be so large compared to the time scales of the motion in the interaction picture that we can still safely make a white-noise approximation simply by choosing the range of integration over ω in definition of the $b_r(t)$ to be only in a rather narrow bandwidth around Ω, sufficiently narrow that this does not overlap the corresponding range for any other $b_r(t)$.

After all such approximations are taken care of, it is possible to move back to the Schrödinger picture, although the white-noise approximation is not strictly valid, because of the rapid time variation of the wave function arising from optical time scales. Essentially what happens is that there are only a few well-defined frequencies Ω_r and the $b_r(t)$ become noises at these frequencies, with a bandwidth much larger than the inverse time scale of the change of the wave function on the very slow damping that modulates the high-frequency optical motion.

(6) Notice that if we make the assumption that $\kappa(\omega) = \sqrt{\gamma/2\pi}$ in consonance with (19) [which simply involves fixing the phase of $b(\omega)$], then $b(t)$ and the initial input field are closely related. We make the narrow bandwidth approximation in (3) by setting $\omega \to \Omega$ in the square root so that

$$A^{(+)}(x,t_0)=\frac{1}{2}\sqrt{\frac{\hbar}{\Omega c}}\left\{b(t_0+x/c)+b(t_0-x/c)\right\}\ . \quad (21)$$

This means that $b(t_0+x/c)$ is the operator that describes the incoming part of the initial field $A^{(+)}(x,t_0)$, while $b(t_0-x/c)$ describes the outgoing part.

Clearly, $b(t_0-x/c)$ does not affect the future development of the system, since the outgoing part of the field propagates away. However, time evolution to time t means that the incoming part $b(x/c+t_0)$ at the point $x=c(t-t_0)$, i.e., $b(t)$, is brought into contact with the system at that time. Thus $b(t)$ is the driving field for the equation of motion at time t.

Thus the parameter t should be interpreted to mean *the time at which the initial incoming field at the point* $x=c(t-t_0)$ *will interact with the system*, rather than as specifying that $b(t)$ is a time-dependent operator at time t.

III. QUANTUM STOCHASTIC INTEGRATION

The commutator (20) acquires the δ function form because of the Markovian approximations of Sec. II. The Markovian equations that result have a greatly simplified form, but this simplification does not arise without some cost. This cost is the requirement to define stochastic calculus with as much care as in the classical case, leading to the concepts of Ito and Stratonovich stochastic integration in much the same way as in the classical case.

We will give here a heuristic explanation of the differences between these two types of integration, which have been defined rigorously by the mathematicians Hudson and Parthasarthy [7], and have been used in quantum optics from a Heisenberg picture point of view by Collett and Gardiner [10, 9] and Barchielli [8].

A. Integration with respect to $b(t), b^\dagger(t)$: Ito and Stratonovich

Let us consider first a situation in which the field is a vacuum, so that $b(t')|0\rangle=0$, and thus $\langle b(t')b^\dagger(t)\rangle\equiv\langle 0|b(t')b^\dagger(t)|0\rangle=\delta(t-t')$, and $\langle b^\dagger(t)b(t')\rangle=0$. The singular nature of this average means that it is not in fact possible to integrate a function of $b(t)$—one can see that it is rather like classical white noise $\xi(t)$, which is δ correlated:

$$\langle\xi(t)\xi(t')\rangle=\delta(t-t'), \quad (22)$$

and which has infinite variance. On the other hand, as in the case of classical white noise, we can make more sense out of the integral of $b(t)$. Namely we define $B(t)$ by

$$B(t)-B(t_0)=\int_{t_0}^{t}dt'b(t')\ . \quad (23)$$

Taking averages in the vacuum state, we find

$$\langle B(t)-B(t_0)\rangle=0, \quad (24)$$

$$\langle[B^\dagger(t)-B^\dagger(t_0)]^2\rangle=\langle[B(t)-B(t_0)]^2\rangle=0, \quad (25)$$

$$\langle[B(t_1)-B(t_0)][B^\dagger(t)-B^\dagger(t_0)]\rangle=\min(|t_1-t_0|,|t-t_0|)\ . \quad (26)$$

Let us now consider two definitions of quantum stochastic integration: *Ito*,

$$\left\{\int_0^t f(t')dB(t')\right\}_{\mathcal{I}}=\lim_{n\to\infty}\sum_{i=0}^{n}f(t_i)[B(t_{i+i})-B(t_i)]; \quad (27)$$

Stratonovich,

$$\left\{\int_0^t f(t')dB(t')\right\}_{\mathcal{S}}=\lim_{n\to\infty}\sum_{i=0}^{n}\frac{f(t_i)+f(t_{i+1})}{2}[B(t_{i+1})-B(t_i)]. \quad (28)$$

In both of these $f(t)$ is a *nonanticipating* function, i.e., a function that does not depend on $B(s)$ for $s>t$. Such a function arises naturally as the solution of any physical problem, in which the behavior of the system does not depend on the driving field evaluated in the future, but may of course depend on its past values. (Barchielli [8] and Hudson and Parthasarthy [7] use the term *adapted function* instead of nonanticipating function.)

There are analogous definitions for the Ito and Stratonovich versions of

$$\int_0^t dB(t')f(t'),\quad \int_0^t dB^\dagger(t')f(t'),\quad \int_0^t f(t')dB^\dagger(t'). \quad (29)$$

The Ito and Stratonovich versions of these integrals are not the same. The basic difference arises from the fact that in the Ito form, the terms $f(t_i)$ and $[B(t_{i+1})-B(t_i)]$ are independent of each other, whereas in the Stratonovich form the term $f(t_i)+f(t_{i+1})$ is not independent of $[B(t_{i+1})-B(t_i)]$.

As an example, if we use the properties (26), we find, for the Ito and Stratonovich versions, respectively,

$$\left\langle\left\{\int_0^t dB(t')B^\dagger(t')\right\}_{\mathcal{I}}\right\rangle=0, \quad (30)$$

$$\left\langle\left\{\int_0^t dB(t')B^\dagger(t')\right\}_{\mathcal{S}}\right\rangle=\frac{1}{2}|t|\ . \quad (31)$$

In fact this example shows the three main principles involved.

(1) *Stratonovich integration follows the rules of conventional calculus.* For example, the conventional differential of $B(t)B^\dagger(t)$ is the Stratonovich differential

$$\{d[B(t)B^\dagger(t)]\}_{\mathcal{S}}=dB(t)B^\dagger(t)+B(t)dB^\dagger(t) \quad (32)$$

so that

$$B(t)B^\dagger(t)|_0^t = \left\{\int_0^t dB(t')B^\dagger(t')\right\}_S + \left\{\int_0^t B(t')dB^\dagger(t')\right\}_S \tag{33}$$

and using (26) and (28) and taking averages, we find

$$|t| = \frac{1}{2}|t| + \frac{1}{2}|t|, \tag{34}$$

as should be the case.

(2) *Ito integration increments are independent of and commute with the integrand.* From this independence we see that we can factorize the average so that

$$\left\langle\left\{\int_0^t dB(t')B^\dagger(t')\right\}_{\mathcal{I}}\right\rangle = \int_0^t \langle dB(t')\rangle\langle B^\dagger(t')\rangle = \int_0^t 0 = 0\,. \tag{35}$$

For Ito differentials the conventional rule (32) is replaced by the following rules: (1) Expand all differentials to *second* order. (2) Use the multiplication rules [in the vacuum state—otherwise use the rules (59)–(63)]

$$\begin{aligned} dt^2 = dB^2 = dB^{\dagger 2}(t) = dB^\dagger(t)dB(t) = 0,\\ dB(t)dt = dB^\dagger(t)dt = dtdB(t) = dtdB^\dagger(t) = 0,\\ dB(t)dB^\dagger(t) = dt\,. \end{aligned} \tag{36}$$

Using these rules, we find

$$\begin{aligned} \{d\left[B(t)B^\dagger(t)\right]\}_{\mathcal{I}} &= dB(t)B^\dagger(t) + B(t)dB^\dagger(t) + dB(t)dB^\dagger(t)\\ &= dB(t)B^\dagger(t) + B(t)dB^\dagger(t) + dt\,, \end{aligned} \tag{37}$$

so that using the Ito integral, we get

$$\langle B(t)B^\dagger(t)\rangle|_0^t = |t| = 0 + 0 + |t|\,, \tag{38}$$

in agreement with (34). These rules are sufficient for ordinary manipulation, but are defined only when the state is the vacuum. If the state is not the vacuum, there are a number of possibilities, which will be dealt with in Sec. III E, in which we also show how to derive the rules (36).

(3) *The mean value of an Ito integral is always zero.* This follows because $dB(t) = B(t+dt) - B(t)$, which is independent of $f(t)$, which has been assumed to be nonanticipating.

B. Equations of motion as stochastic differential equations

As in the case of classical stochastic differential equations [18, 19], the equation of motion (15) cannot be rigorously considered as a *differential* equation, since the terms involving $b^\dagger(t), b(t)$, are in some sense infinite. However, this equation of motion can be regarded as an *integral* equation, using the *Stratonovich* definition. The Stratonovich definition is necessary because the rules of calculus are the same as in ordinary calculus, and these have been implicitly assumed in all manipulations leading to (15). Thus a more correct way of writing the equation of motion would be the *integral* equation

$$|\varphi,t\rangle - |\varphi,t_0\rangle = \sqrt{\gamma}\left\{\int_{t_0}^t \left\{dB^\dagger(t')c - dB(t')c^\dagger\right\}|\varphi,t'\rangle\right\}_S . \tag{39}$$

The more usual *symbolic* abbreviation is

$$\{d|\varphi,t\rangle\}_S = \sqrt{\gamma}\{dB^\dagger(t)c - dB(t)c^\dagger\}|\varphi,t\rangle, \tag{40}$$

which is known as a *stochastic differential equation*, although it should be understood as merely a simplified notation for (39). While the Stratonovich form has the merit of satisfying ordinary calculus, the equation as it stands is rather like an implicit algorithm for the solution of a differential equation, and this leads to the following problem. Using the definition (28), we can see that in a discretized form, this equation would take the form

$$|\varphi,t+h\rangle - |\varphi,t\rangle = \sqrt{\gamma}\{\Delta_h B^\dagger(t)c - \Delta_h B(t)c^\dagger\} \times \frac{|\varphi,t\rangle + |\varphi,t+h\rangle}{2}\,, \tag{41}$$

where

$$\Delta_h B(t) = B(t+h) - B(t)\,. \tag{42}$$

In this definition, $|\varphi,t+h\rangle$ is not independent of $\Delta_h B(t)$, and this makes manipulation rather difficult. By converting to the Ito form, we can get a modified equation, in which $\Delta_h B(t)$ and $|\varphi,t+h\rangle$ are independent, and manipulation is consequently easier.

C. Conversion to Ito equations

We can also express the equation of motion in an Ito form, that is, as an integral equation defined in terms of Ito integrals. The procedure is analogous to that employed in the conversion between classical Ito and Stratonovich integrals [18]. The only modifications occur in the use of noncommuting $\Delta B, \Delta B^\dagger$ operators. A derivation of the general formulas for converting between Ito and Stratonovich equations is given in Appendix A, since the full generality is not needed here.

The exact solution of the equation of motion (39) is well known—it is given in terms of the time-ordered product

$$|\varphi,t\rangle = T\left\{\exp\left[\sqrt{\gamma}c\int_{t_0}^t dB^\dagger(t') - \sqrt{\gamma}c^\dagger\int_{t_0}^t dB(t')\right]\right\}|\varphi,t_0\rangle\,. \tag{43}$$

This leads to a rather straightforward conversion to the Ito form, by noting that the stochastic integrals in (43) have no t' dependence in the integrand [apart from $dB(t'), dB^\dagger(t')$] and can therefore be interpreted as either Ito or Stratonovich integrals. We can therefore write

$$|\varphi,t+dt\rangle = \exp\left\{\sqrt{\gamma}c\int_t^{t+dt} dB^\dagger(t') - \sqrt{\gamma}c^\dagger\int_t^{t+dt} dB(t')\right\}|\varphi,t\rangle. \tag{44}$$

We note that we can then make the substitution

$$\int_t^{t+dt} dB(t') \to \{dB(t)\}_{\mathcal{I}} \tag{45}$$

and expand the exponential to *second* order to get

$$\{d|\varphi,t\rangle\}_{\mathcal{I}} = \Big\{ \sqrt{\gamma}c dB^\dagger(t) - \sqrt{\gamma}c^\dagger dB(t) - \frac{\gamma}{2}cc^\dagger\, dB^\dagger(t)dB(t) - \frac{\gamma}{2}c^\dagger c\, dB(t)dB^\dagger(t) + \frac{\gamma}{2}c^2 dB^{\dagger 2} + \frac{\gamma}{2}c^{\dagger 2} dB^2 \Big\}|\varphi,t\rangle. \tag{46}$$

The question now arises—what do we do with the terms that are quadratic in the increments? To understand how to deal with these we must look more carefully at the space in which $dB(t)$, $dB^\dagger(t)$ operate.

D. Description of initial states

Let us now assume that the initial state is not the vacuum, but rather, that we have a nonpure initial state, but one which corresponds to a white-noise situation. Previous authors have written quantum stochastic differential equations (QSDE's) in the Schrödinger picture as equations for the evolution operator $U(t,t0)$, whereas we wish to write down *wave-function* QSDE's. Since the wave function refers to both the system and the bath, we cannot formulate these equations quite as simply as those for $U(t,t0)$—instead of describing the bath by a separate bath density operator, we must consider an *ensemble* of initial states, in which the individual members of the ensemble differ only in the bath space. Mathematically, the description in terms of evolution operator and our direct wave-function description are equivalent, but the wave-function description is more adapted to our needs. Remember now that the operators $b(t)$, $b^\dagger(t)$ are Fourier transforms of the *initial* bath operators, so that $dB(t)$, $dB^\dagger(t)$ form a set of *initial* operators labeled by the parameter t—the time at which this operator will interact with the system. Since these operators commute at different t, the states of different t may be specified independently. Thus we can specify an initial state as a factorized form (for a discrete partition $\tau_0, \tau_1, \tau_2, \ldots$ in which $d\tau_i = \tau_{i+1} - \tau_i$)

$$|B\rangle = |a_0\rangle_{\tau_0} \otimes |a_1\rangle_{\tau_1} \otimes |a_2\rangle_{\tau_2} \otimes |a_3\rangle_{\tau_3} \otimes \cdots, \tag{47}$$

where $|a_i\rangle_{\tau_i}$ is a state in the space in which the operators $dB(\tau_i)$, $dB^\dagger(\tau_i)$ act. This can be viewed as an alternative to the usual description in terms of $b(\omega)$, $b^\dagger(\omega)$, in which there is a basis set that can be factorized in frequency space. Because of the direct connection between $b(t)$, $b^\dagger(t)$ and the incoming part of the initial field at the point $x = c(t - t_0)$, these states labeled by the time τ_i can be viewed as states that describe the incoming field at points $x_i = c(\tau_i - t_0)$. In this respect, this description of the state of the initial field is perhaps more natural than the usual description in terms of frequencies.

The kind of initial state that we want to consider is one in which there is an ensemble of states $|a_i\rangle_{\tau_i}$ at each τ_i occurring with probabilities $p_i(a_i)$, giving a density operator at each τ_i

$$\rho(\tau_i) = \sum_i p_i(a_i)|a_i\rangle_{\tau_i}\langle a_i|_{\tau_i}\,. \tag{48}$$

We will specify thermal statistics at each time by the conditions

$$\mathrm{Tr}_B\{\rho(\tau_i)dB(\tau_i)\} = 0, \tag{49}$$

$$\mathrm{Tr}_B\{\rho(\tau_i)dB^\dagger(\tau_i)\} = 0, \tag{50}$$

$$\mathrm{Tr}_B\{\rho(\tau_i)dB(\tau_i)^2\} = M\, d\tau_i, \tag{51}$$

$$\mathrm{Tr}_B\{\rho(\tau_i)dB^\dagger(\tau_i)^2\} = M^*\, d\tau_i, \tag{52}$$

$$\mathrm{Tr}_B\{\rho(\tau_i)dB^\dagger(\tau_i)dB(\tau_i)\} = N\, d\tau_i, \tag{53}$$

$$\mathrm{Tr}_B\{\rho(\tau_i)dB(\tau_i)dB^\dagger(\tau_i)\} = (N+1)\, d\tau_i, \tag{54}$$

and by the condition that the distribution is Gaussian. [By a Gaussian state we mean one in which the relationships between normally ordered moments are the same as those between the corresponding moments in a classical Gaussian description. In [20] it is shown that this implies that the same relationship is true for antinormally ordered and symmetrically ordered moments, and that the density operator is the exponential of a quadratic in $dB(\tau_i)$ and $dB^\dagger(\tau_i)$.]

Positive definiteness of the density operator means that all allowable M, N satisfy

$$|M|^2 \le N(N+1), \tag{55}$$

with the equality holding only for a pure state [10]. The initial state of the system will also be assumed to factorize into a bath part and a system part—thus there will be an ensemble of initial states, in which the system part is, however, always the definite state $|\varphi, t_0, \mathrm{sys}\rangle$. A typical member of the initial ensemble can be written

$$|\varphi,t_0\rangle = |\varphi,t_0,\mathrm{sys}\rangle \otimes |a_0\rangle_{\tau_0} \otimes |a_1\rangle_{\tau_1} \otimes |a_2\rangle_{\tau_2} \otimes |a_3\rangle_{\tau_3} \otimes \cdots. \tag{56}$$

The assumption that the initial ensemble is described by independent probabilities $p_i(a_i)$ at each time τ_i is the white-noise assumption, equivalent to a Markov assumption. The physical validity of this assumption depends on whether or not we are considering a vacuum input.

For a vacuum input, the only limits on the validity are those used in Sec. II B, in which the δ-function commutator was derived. Since the vacuum state is unique, the ensemble is given by the vacuum for each τ_i, and this is trivially factorizable.

In a situation in which there is a physical noise input, the limitations are given by the shortness of the correlation time of the apparatus which produces the input light, but whatever this is, it will never be as short as the transit time for light to pass from one side of an atom to the other, and in practice may approach the typical time scales of the system under investigation. In such cases the white-noise approximation for the input noise can lose its validity. Non-white-noise inputs have been investigated by the authors elsewhere [21, 22], but will not be treated further in this paper.

The solution for $|\varphi,t\rangle$, (43) involves $B(t')$ and $B^\dagger(t')$ only for $t_0 < t' \le t$ and can be written

$$|\varphi,t\rangle = U(t,t_0)|\varphi,t_0\rangle, \tag{57}$$

where $U(t,t_0)$ does not contain any dependence on the

Ito increments $dB(s), dB^\dagger(s)$ for $s \geq t$. Hence

$$[dB(t), U(t,t_0)] = [dB^\dagger(t), U(t,t_0)] = 0, \tag{58}$$

and the density operator for $dB(t')$, $dB^\dagger(t')$ will be unchanged. Therefore the averages of functions of $dB(t)$, $dB^\dagger(t)$ over the ensemble $|\varphi,t\rangle$ are the same as those over the initial ensemble, i.e., they are given by the forms (49)–(54).

E. Multiplication rules for stochastic increments

Quantum stochastic integration is defined as in (27). We can show that in stochastic integrals we can make the substitutions

$$dB^\dagger(t)dB(t) = N\,dt, \tag{59}$$
$$dB(t)dB^\dagger = (N+1)\,dt, \tag{60}$$
$$dB(t)^2 = M\,dt, \tag{61}$$
$$dB^\dagger(t)^2 = M^*\,dt, \tag{62}$$
$$dtdB(t) = dtdB^\dagger(t) = dt^2 = 0 \tag{63}$$

by the same method as is used in classical stochastic differential equations. For example, we consider

$$|\psi,\mathbf{a}\rangle \equiv \int [dB^\dagger(t)dB(t) - N\,dt]|\varphi,t,\mathbf{a}\rangle. \tag{64}$$

Here $\mathbf{a} = \{a_0, a_1, a_2, \ldots\}$ is a label that defines the particular member of the ensemble as in (47) and (48), and has a probability distribution

$$\mathbf{p}(\mathbf{a}) \equiv \prod_i^\infty p_i(a_i). \tag{65}$$

We discretize this integral in time, and consider that $|\psi\rangle \to 0$ in the limit of infinitely fine discretization whenever

$$\sum_{\mathbf{a}} \mathbf{p}(\mathbf{a})\langle\psi,\mathbf{a}|\psi,\mathbf{a}\rangle \to 0 \tag{66}$$

in this limit. This defines what could be called a mean-square-norm topology, and when this limit is zero, we say that in the mean-square-norm topology

$$\int dB^\dagger(t)dB(t)|\varphi,t\rangle \to \int N|\varphi,t\rangle\,dt. \tag{67}$$

The proof in the classical case (see [18] Sec. 4.2.5) depends on the independence of the noise and the integrand, as it does here, and the assumption of the Gaussian nature of the increment $dW(t)$, which means that $\langle dW(t)^2\rangle \approx dt^2$. Similarly here, all fourth-order moments of $dB(t)$, $dB^\dagger(t)$ must be of order dt^2, which is a property of quantum Gaussian (and in particular thermal) states.

We now use the rules (59)–(63) in Eq. (46), and finally obtain

$$\begin{aligned}\{d|\varphi,t\rangle\}_{\mathcal{I}} = &-\frac{\gamma}{2}\{cc^\dagger N + c^\dagger c(N+1) \\ &-c^2M^* - {c^\dagger}^2 M\}|\varphi,t\rangle dt \\ &+\sqrt{\gamma}\{cdB^\dagger(t) - c^\dagger dB(t)\}|\varphi,t\rangle.\end{aligned} \tag{68}$$

This equation is the most general kind of Ito equation involving only one driving field.

F. Conversion between Ito and Stratonovich integrals

It is only possible to give a rule for conversion between the two different kinds of integrals when it is known what differential equation is satisfied by the integrand. This must be so because the Stratonovich integral involves the integrand at a future time, and this can only be predicted using an equation of motion.

Let us suppose that an Ito equation can be written

$$d|\varphi,t\rangle = \{\alpha dt + \beta dB^\dagger(t) - \beta^\dagger dB(t)\}|\varphi,t\rangle\,. \tag{69}$$

Then we show in the Appendix A that the connection between the Ito and Stratonovich integrals is given by

$$\begin{aligned}\left\{\int dB(t)|\varphi(t)\rangle\right\}_{\mathcal{S}} &= \left\{\int dB(t)|\varphi(t)\rangle\right\}_{\mathcal{I}} - \frac{1}{2}(N+1)\int \beta|\varphi(t)\rangle dt \\ &\quad +\frac{1}{2}M\int \beta^\dagger|\varphi(t)\rangle dt, \\ \left\{\int dB^\dagger(t)|\varphi(t)\rangle\right\}_{\mathcal{S}} &= \left\{\int dB^\dagger(t)|\varphi(t)\rangle\right\}_{\mathcal{I}} \\ &\quad -\frac{1}{2}N\int \beta^\dagger|\varphi(t)\rangle dt \\ &\quad +\frac{1}{2}M^*\int \beta|\varphi(t)\rangle dt, \\ \left\{\int \langle\varphi(t)|dB(t)\right\}_{\mathcal{S}} &= \left\{\int \langle\varphi(t)|dB(t)\right\}_{\mathcal{I}} \\ &\quad -\frac{1}{2}N\int \langle\varphi(t)|\beta dt \\ &\quad +\frac{1}{2}M\int \langle\varphi(t)|\beta^\dagger dt, \\ \left\{\int \langle\varphi(t)|dB^\dagger(t)\right\}_{\mathcal{S}} &= \left\{\int \langle\varphi(t)|dB^\dagger(t)\right\}_{\mathcal{I}} - \frac{1}{2}(N+1)\int \langle\varphi(t)|\beta^\dagger dt \\ &\quad +\frac{1}{2}M^*\int \langle\varphi(t)|\beta dt\,.\end{aligned} \tag{70}$$

Using these rules, it is possible to make an alternative rule for the conversion from the Stratonovich form to the Ito form of the stochastic differential equation, and get the result (68).

G. Ito equations when the input is the vacuum

Suppose that $|\varphi,t_0\rangle$ represents the vacuum of the electromagnetic field. Then, for all ω

$$b(\omega)|\varphi,t_0\rangle = 0 \tag{71}$$

and thus

$$dB(t)|\varphi,t_0\rangle = 0 \qquad \text{for all } t\,. \tag{72}$$

Because $|\varphi,t\rangle = U(t,t_0)|\varphi,t_0\rangle$, and $U(t,t_0)$ commutes with $dB(t)$, we can say that

$$dB(t)|\varphi,t\rangle = 0. \tag{73}$$

This means that as far as $dB(t)$ is concerned, $|\varphi,t\rangle$ is a *vacuum* state and therefore the Ito equation (68) can be simplified to

$$\{d|\varphi,t\rangle\}_{\mathcal{I}} = \left\{-\frac{1}{2}\gamma c^\dagger c\,dt + \sqrt{\gamma}c dB^\dagger(t)\right\}|\varphi,t\rangle. \quad (74)$$

Equation (74) looks nonunitary. However, the time evolution operator $U(t,t_0)$ is the solution of Eq. (74) with initial condition $U(t_0,t_0)=1$. This solution is, in fact, the explicitly unitary operator

$$U(t,t_0) = T\left\{\exp\left(\sqrt{\gamma}\int_{t_0}^{t} dB^\dagger(t')c - \sqrt{\gamma}\int_{t_0}^{t} dB(t')c^\dagger\right)\right\} \quad (75)$$

It is also possible to use the multiplication rules (59)–(62) with $M=N=0$ to show that the form of $|\varphi,t\rangle$ is explicitly preserved by the solution of (74).

H. Physical interpretation of the Ito form of equation

Equation (74) can be interpreted physically quite logically. The term $dB^\dagger$ involves the incoming field evaluated in the immediate future of t. Thus this field is not affected by the system. However, the system does create a self-field, which causes the process of radiation damping by reacting back on the system. This is the meaning of the term $-\frac{\gamma}{2}c^\dagger c|\varphi,t\rangle$, which is the damping induced by radiation. The Stratonovich equation does not have this term because the evaluation of $dB^\dagger(t)$, half in the future and half in the past, itself generates the radiation reaction.

Mollow [16] derived an equation that, apart from notation, is exactly the same as (74) in his pure state treatment of the interaction of atoms and light. He did not, however, use the Ito equation. We can explain Mollow's argument by starting with the Stratonovich equation,

$$\{d|\varphi,t\rangle\}_{\mathcal{S}} = \sqrt{\gamma}\left\{c dB^\dagger(t) - c^\dagger dB(t)\right\}|\varphi,t\rangle\,, \quad (76)$$

and making the assumption that the input is the vacuum. We rewrite the discretized form of the Stratonovich equation (41) as

$$|\varphi,t+h\rangle - |\varphi,t\rangle = \frac{\sqrt{\gamma}}{2}\Delta\{|\varphi,t\rangle + |\varphi,t+h\rangle\} \quad (77)$$

with

$$\Delta \equiv \Delta_h B^\dagger(t)c - \Delta_h B(t)c^\dagger. \quad (78)$$

Solving (77) for $|\varphi,t+h\rangle$ and expanding to second order in Δ, we find

$$|\varphi,t+h\rangle = (1+\sqrt{\gamma}\Delta + \gamma\Delta^2/2)|\varphi,t\rangle. \quad (79)$$

Now note that because $\Delta_h B(t)$ is in the future of t, and we are dealing with a vacuum input,

$$\Delta_h B(t)|\varphi,t\rangle = 0\,, \quad (80)$$

and we can compute $\Delta_h B(t)|\varphi,t+h\rangle$ from (79), (80), and the commutation relation between $\Delta_h B(t)$ and $\Delta_h B^\dagger(t)$. Doing all this we find

$$\Delta_h B(t)|\varphi,t+h\rangle = h\sqrt{\gamma}c|\varphi,t+h\rangle \quad (81)$$

so that for the Stratonovich increment, provided that the input field is a vacuum,

$$\{dB(t)|\varphi,t\rangle\}_{\mathcal{S}} = -\frac{\sqrt{\gamma}}{2}c|\varphi,t\rangle dt. \quad (82)$$

We can then substitute in (76) to get

$$\{d|\varphi,t\rangle\}_{\mathcal{S}} = \left\{-\frac{\gamma}{2}c^\dagger c\,dt + \sqrt{\gamma}c dB^\dagger(t)\right\}|\varphi,t\rangle. \quad (83)$$

This is exactly the same equation as the Ito form (74). But from the rules (70) we see that when $N=M=0$, the $dB^\dagger(t)$ Ito and Stratonovich integrals are the same.

Mollow's type of derivation is a method of explicitly computing the radiation reaction for this special case. Using the Ito form does the same thing for all (white noise) inputs.

I. Density operator equations

From Eq. (68), we can derive the equation of motion for

$$\hat{\rho} \equiv |\varphi,t\rangle\langle\varphi,t| \quad (84)$$

as

$$\begin{aligned} d\hat{\rho} = &-\frac{1}{2}\gamma dt[(N+1)c^\dagger c + Ncc^\dagger - Mc^\dagger c^\dagger - M^* cc,\hat{\rho}]_+ \\ &+\gamma[dB^\dagger(t)c - dB(t)c^\dagger]\hat{\rho}[dB(t)c^\dagger - dB^\dagger(t)c] \\ &+\sqrt{\gamma}[dB^\dagger(t)c - \sqrt{\gamma}dB(t)c^\dagger,\hat{\rho}]\,. \end{aligned} \quad (85)$$

If we now trace over the bath variables, this will execute an average over the $dB(t), dB^\dagger(t)$ operators. We use the cyclic property of the trace, and derive the usual master equation for $\rho = \mathrm{Tr}_B\{\hat{\rho}\}$,

$$\begin{aligned} \frac{d\rho}{dt} = \frac{\gamma}{2}\Big\{ &(N+1)\{2c\rho c^\dagger - \rho c^\dagger c - c^\dagger c\rho\} \\ &+N(2c^\dagger\rho c - \rho cc^\dagger - cc^\dagger\rho) \\ &-M(2c^\dagger\rho c^\dagger - \rho c^\dagger c^\dagger - c^\dagger c^\dagger\rho) \\ &-M^*(2c\rho c - \rho cc - cc\rho)\Big\}\,. \end{aligned} \quad (86)$$

J. Heisenberg quantum Langevin equations

For a system operator a we define a Heisenberg operator $a(t) = U^\dagger(t,t_0)aU(t,t_0)$ that obeys the Ito quantum Langevin equation

$$\begin{aligned} da(t) &\equiv U^\dagger(t,t+dt)a(t)U(t,t+dt) - a(t) \\ &= -\frac{\mathrm{i}}{\hbar}\left[a, H_{\mathrm{sys}} + \mathrm{i}\hbar\sqrt{\gamma}dBc^\dagger - \mathrm{i}\hbar\sqrt{\gamma}dB^\dagger c\right]dt + \frac{1}{2}\gamma(N+1)(2c^\dagger ac - ac^\dagger c - c^\dagger ca)dt \\ &\quad +\frac{1}{2}\gamma N(2cac^\dagger - acc^\dagger - cc^\dagger a)dt - \frac{1}{2}\gamma M(2c^\dagger ac^\dagger - ac^\dagger c^\dagger - c^\dagger c^\dagger a)dt - \frac{1}{2}\gamma M^*(2cac - acc - cca)dt\,. \end{aligned} \quad (87)$$

In deriving Eq. (87) we have expanded $U(t,t+dt)$ to second order in the increments, and used the Ito rules (59)–(63).

In the work of Collett and Gardiner [9, 10] and Barchielli [8] the equations of motion are expressed as operator quantum Langevin equations, in which the driving term is

$$b_{\rm in}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega b(\omega, t_0) e^{-i\omega(t-t_0)} , \tag{88}$$

where $b(\omega, t_0)$ is the Heisenberg-picture field operator evaluated at time t_0. This is, of course, exactly the same thing as our $b(\omega)$, evaluated in the Schrödinger picture, since these two pictures coincide at $t = t_0$. Thus

$$b_{\rm in}(t) \equiv b(t). \tag{89}$$

Collett and Gardiner also define an "out" operator, defined by

$$b_{\rm out}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega b(\omega, t_1) e^{i\omega(t-t_1)}, \tag{90}$$

where t_1 is a final time, chosen in the remote future. The relation between $b(\omega, t_1)$ and $b(\omega, t_0)$ is given by the time evolution operator $U(t,t')$, so that

$$b_{\rm out}(t) = U^\dagger(t_1, t_0) b(t) U(t_1, t_0). \tag{91}$$

We can write the evolution operator as the product of three factors

$$U(t_1, t_0) = U(t_1, t+dt) U(t+dt, t) U(t, t_0). \tag{92}$$

We note that (1) $U(t_1, t+dt)$ commutes with $b(t)$; thus we can actually write

$$b_{\rm out}(t) = U^\dagger(t, t_0) U^\dagger(t+dt, t) b(t) U(t+dt, t) U(t, t_0) . \tag{93}$$

(2) Using the infinitesimal form of $U(t+dt, t)$, we can compute that

$$U^\dagger(t+dt) b(t) U(t+dt) = b(t) + \sqrt{\gamma} c . \tag{94}$$

(3) $U(t, t_0)$ commutes with $b(t)$, but not with c. The effect is to convert c to a Heisenberg operator; thus

$$b_{\rm out}(t) = b(t) + \sqrt{\gamma} c(t) , \tag{95}$$

where $c(t)$ is the Heisenberg picture operator. This is exactly the relationship between "in" and "out" fields derived in Refs. [9, 10].

By construction (90) and because the transformation (91) is unitary and independent of t, the "out" fields satisfy exactly the same commutation relations as the "in" fields.

IV. SIMULATION METHODS

A. Relation to continuous measurement theory

Srinivas and Davies [12] developed a formulation of continuous-measurement theory for the particular process of counting photons which in fact can be considerably generalized. Barchielli [8] has also developed and applied this method, and more recently, Barchielli and Belavkin [23] have formulated a posterior stochastic calculus—our methods can be viewed as an implementation of this calculus. Here we want to restate these formulations for use in more general situations, in which the system experiences what we shall call "*jumps*." A jump is viewed as a sudden change of the system wave function; a particular example of this is the collapse of the wave function when a photon is counted, but there are other possibilities.

We therefore consider a continuous-measurement process, in which the discrete "jumps" occur at isolated points in the time interval (t_0, t) and introduce the *operation* (i.e., two-sided operator) $N(t, t_0, n)$ which represents this as follows.

(1) After n jumps have occurred in (t_0, t) the density operator is

$$\frac{N(t, t_0, n)\rho(t_0)}{\mathrm{Tr}\,\{N(t, t_0, n)\rho(t_0)\}} . \tag{96}$$

(2) The probability that n jumps occur in (t_0, t) is

$$P(t, t_0, n) = \mathrm{Tr}\,\{N(t, t_0, n)\rho(t_0)\} . \tag{97}$$

(3) The process is Markovian—that is, we may partition the time interval at any arbitrary time s inside (t_0, t), and write

$$N(t, t_0, n) = \sum_{\substack{n_1, n_2 \\ n_1+n_2=n}} N(t, s, n_1) N(s, t_0, n_2). \tag{98}$$

This states that the process by which n jumps occur is given by all combinations of n_1 and n_2, jumps in any two subintervals of which the interval is composed.

(4) The density operator at time t is

$$\rho(t) = \sum_n \frac{N(t, t_0, n)\rho(t_0)}{\mathrm{Tr}\,\{N(t, t_0, n)\rho(t_0)\}} P(t, t_0, n) \tag{99}$$

$$= \sum_n \tilde{\rho}_n(t), \tag{100}$$

where

$$\tilde{\rho}_n(t) \equiv N(t, t_0, n)\rho(t_0) \tag{101}$$

after using (97).

B. Differential form of continuous-measurement theory

We need differential forms of these equations, and to derive these we need to assume, for infinitesimal τ,

$$\begin{aligned} &N(t+\tau, t, 0) \approx 1 + A(t)\tau, \\ &N(t+\tau, t, 1) \approx B(t)\tau, \\ &N(t+\tau, t, n) \approx \tau^n \quad \text{for } n > 1 . \end{aligned} \tag{102}$$

These are smoothness assumptions, which basically all follow from the second assumption that requires the rate occurrence of jumps in a small time interval τ to be proportional to τ.

From these equations and (98), we can derive

$$\frac{dN(t,t_0,n)}{dt}$$
$$= \lim_{\tau\to 0}\frac{1}{\tau}\sum_{\substack{n_1,n_2\\ n_1+n_2=n}}[N(t+\tau,t,n_1)-1]N(t,t_0,n_2)$$
$$= A(t)N(t,t_0,n)+B(t)N(t,t_0,n-1)\,. \tag{103}$$

Thus the equation of motion for $\tilde{\rho}_n(t)$ is

$$\frac{d\tilde{\rho}_n(t)}{dt} = A(t)\tilde{\rho}_n(t)+B(t)\tilde{\rho}_{n-1}(t)\,. \tag{104}$$

There is one further condition. From the definitions (99) and (104) we have

$$\frac{d\rho(t)}{dt} = A(t)\rho(t)+B(t)\rho(t)\,, \tag{105}$$

and because $\mathrm{Tr}\{\rho(t)\}=1$ for all $\rho(t)$, we must have

$$\mathrm{Tr}\{A(t)\rho(t)\} = -\mathrm{Tr}\{B(t)\rho(t)\} \tag{106}$$

for any ρ.

C. Relation to the pure state formalism

We can now relate this to the problem we are considering. We can write the equation of motion (68) in an appropriate form by setting

$$|\varphi,t\rangle = \sum_{n=0}^{\infty}|\varphi,n,t\rangle \tag{107}$$

in which

$$d|\varphi,n,t\rangle = -\frac{\gamma}{2}\Big\{cc^\dagger N + c^\dagger c(N+1) - c^2M^* - {c^\dagger}^2M\Big\}|\varphi,n,t\rangle dt + \sqrt{\gamma}\left\{c\,dB^\dagger(t)-c^\dagger dB(t)\right\}|\varphi,n-1,t\rangle \quad (n\geq 1), \tag{108}$$

$$d|\varphi,0,t\rangle = -\frac{\gamma}{2}\{\,cc^\dagger N + c^\dagger c(N+1) - c^2M^* - {c^\dagger}^2M\}|\varphi,0,t\rangle dt\,. \tag{109}$$

The number n is not necessarily the eigenvalue of any operator, but when N and M are zero, and we use the vacuum form (74) of the equations (in which $|\varphi,t_0\rangle$ is the vacuum), it is clear that $|\varphi,n,t\rangle$ is in fact an n photon state.

It is trivial that $|\varphi,t\rangle$ as defined in (107) does satisfy the equation of motion, and that we may choose the initial conditions for the $|\varphi,n,t\rangle$ to be

$$|\varphi,n,t_0\rangle = \delta_{n,0}|\varphi,t_0\rangle. \tag{110}$$

We now define

$$\rho_n(t) = \mathrm{Tr}_B\{|\varphi,n,t\rangle\langle\varphi,n,t|\} \tag{111}$$

and derive the equations of motion. It will be convenient to diagonalize the noise matrix by setting

$$\gamma\begin{bmatrix}N+1 & -M^*\\ -M & N\end{bmatrix} = V\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}V^\dagger, \tag{112}$$

where V is a unitary matrix. We can then define

$$\begin{bmatrix}dX^\dagger(t)\\ dY^\dagger(t)\end{bmatrix} = V^\dagger\begin{bmatrix}dB^\dagger(t)\\ -dB(t)\end{bmatrix} \tag{113}$$

from which it follows that

$$\gamma\begin{bmatrix}dX(t)\\ dY(t)\end{bmatrix}[dX^\dagger(t)\;dY^\dagger(t)] = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}dt. \tag{114}$$

If we also define

$$\begin{bmatrix}a_1\\ a_2\end{bmatrix} \equiv V^T\begin{bmatrix}c\\ c^\dagger\end{bmatrix}\,, \tag{115}$$

then

$$\sqrt{\gamma}\{cdB^\dagger(t)-c^\dagger dB(t)\} = \sqrt{\gamma}\{a_1dX^\dagger(t)+a_2dY^\dagger(t)\}. \tag{116}$$

Notice that, although the right-hand side of (116) does not appear obviously anti-Hermitian, the definitions (113) and (115) do ensure that it is indeed anti-Hermitian. In this form we can write

$$\frac{d\rho_n}{dt} = -\mathrm{i}H_{\mathrm{eff}}\rho_n(t)+\mathrm{i}\rho_n(t)H_{\mathrm{eff}}^\dagger + \lambda_1a_1\rho_{n-1}(t)a_1^\dagger+\lambda_2a_2\rho_{n-1}(t)a_2^\dagger \tag{117}$$

with the anti-Hermitian effective Hamiltonian defined by

$$H_{\mathrm{eff}} = -\frac{\mathrm{i}\gamma}{2}\left\{(N+1)c^\dagger c+Ncc^\dagger - Mc^{\dagger 2}-M^*c^2\right\} \tag{118}$$
$$= -\frac{\mathrm{i}}{2}\{\lambda_1a_1^\dagger a_1+\lambda_2a_2^\dagger a_2\}. \tag{119}$$

This now corresponds exactly to (104) if we identify

$$A(t)\rho \equiv A\rho = -\mathrm{i}H_{\mathrm{eff}}\rho+\mathrm{i}\rho H_{\mathrm{eff}}^\dagger, \tag{120}$$

$$B(t)\rho \equiv B\rho = \lambda_1a_1\rho a_1^\dagger+\lambda_2a_2\rho a_2^\dagger\,. \tag{121}$$

The conclusion one draws is that the definitions (1)–(4) of the continuous-measurement process generate a set of $\tilde{\rho}_n(t)$ which are, for appropriate A and B, exactly equivalent to the differential equations which are satisfied by the partial density operators $\rho_n(t)$ defined in (111). This means that the continuous-measurement theory is a consequence of the pure state formalism, and can be used in any description that does not require direct information about the field.

D. Simulations—theory

The description based on continuous-measurement theory can be used to generate a simulation method that involves only wave functions. To do this, we start with a pure state density operator, and show that a simulation of the density operator with this initial condition gives a density operator that is always a pure state. Let us carry this out.

(1) Start at time t_0 with a pure state density operator

$$\rho(t_0) = |\varphi, t_0\rangle\langle\varphi, t_0| \ . \tag{122}$$

The probability that no jumps occurred in the interval (t_0, t_1) is

$$P(t_1, t_0, 0) = \mathrm{Tr}\,\{\tilde{\rho}_0(t_1)\}\ , \tag{123}$$

with $\tilde{\rho}_0(t_1)$ given by the solution of (104) with $n = 0$; using an explicit form for $A(t)$ in (120),

$$\tilde{\rho}_0(t_1) = |\varphi, t_1, 0\rangle\langle\varphi, t_1, 0| \tag{124}$$

where

$$|\varphi, t_1, 0\rangle = \exp[-\mathrm{i}H_{\mathrm{eff}}(t_1 - t_0)]|\varphi, t_0, 0\rangle \ . \tag{125}$$

(2) The density operator during (t_0, t_1) given that no jumps occurred is computed from (96) with $n = 0$; it is

$$\rho(t) = |\bar{\varphi}, t, 0\rangle\langle\bar{\varphi}, t, 0|, \tag{126}$$

where

$$|\bar{\varphi}, t\rangle = |\varphi, t, 0\rangle / \parallel |\varphi, t, 0\rangle \parallel \ . \tag{127}$$

(3) We use the notation $\rho^-(t_1) = |\bar{\varphi}, t_1, 0\rangle\langle\bar{\varphi}, t_1, 0|$ to represent the density operator just before the occurrence of the jump at time t_1.

(4) The probability of occurrence of a jump in $(t_1, t_1 + dt_1)$ with an initial density operator at t_1 given by $\rho^-(t_1)$ is, using (97) and the infinitesimal forms (102),

$$\mathrm{Tr}\,\{B(t_1)\rho^-(t_1)\}dt_1 \tag{128}$$

and the density operator after this jump is, from (96),

$$\rho^+(t_1) = \frac{B(t_1)\rho^-(t_1)}{\mathrm{Tr}\,\{B(t_1)\rho^-(t_1)\}}. \tag{129}$$

If we use the explicit form for $B(t)$ (121), we can formulate the following algorithm. The density operator after the occurrence of the jump at time t_1 can be written

$$\rho^+(t_1) = |\varphi, t_1, 1\rangle\langle\varphi, t_1, 1| \tag{130}$$

in which $|\varphi, t_1, 1\rangle$ can take on either of two forms

$$|\varphi, t_1, 1\rangle = \frac{\sqrt{\lambda_i}a_i|\varphi, t_1, 0\rangle}{\parallel \sqrt{\lambda_i}a_i|\varphi, t_1, 0\rangle \parallel} \quad (i = 1, 2) \tag{131}$$

with probabilities

$$p_i = \frac{\mathrm{Tr}\{\lambda_i a_i \rho^-(t_1) a_i^\dagger\}}{\mathrm{Tr}\{B\rho^-(t_1)\}} \ . \tag{132}$$

(5) The joint probability that no jump occurred in $(0, t_1)$, and a jump occurred in $(t_1, t_1 + dt_1)$ is then obtained by multiplying (123) and (128), i.e.,

$$P_\varphi(t_1)dt_1 = \mathrm{Tr}\,\{B(t_1)\tilde{\rho}_0(t_1)\}dt_1. \tag{133}$$

This is the probability density of the decay times t_1. If we define

$$s(t_1) = \langle\varphi, t_1, 0|\varphi, t_1, 0\rangle \equiv \mathrm{Tr}\,\{\tilde{\rho}_0(t)\}\ . \tag{134}$$

then $s(t)$ is a function of t_1, with range $(0, 1)$, and the probability density of $s(t)$ is given by

$$p(s)|ds| = P_\varphi(t_1)dt_1 \ . \tag{135}$$

Using the definition (134) and (104), we get

$$p(s)\left|\mathrm{Tr}\left\{\frac{d\tilde{\rho}_0}{dt_1}\right\}\right| = p(s)|\mathrm{Tr}\,\{A(t_1)\tilde{\rho}_0(t_1)\}|dt_1$$
$$= p(s)\mathrm{Tr}\,\{B(t_1)\tilde{\rho}_0(t_1)\}dt_1\ , \tag{136}$$

and thus, comparing with (133) and (135), we find

$$p(s) = 1, \tag{137}$$

which means that $s(t) = \langle\varphi, t_1, 0|\varphi, t_1, 0\rangle$ is uniformly distributed on the unit interval.

E. Simulations—application

The application to simulations is then straightforward.

(1) Choose a random number S_1 from the uniform distribution on $[0, 1]$.

(2) Choose an initial state $|\phi, t_0\rangle$, and compute

$$|\phi, t, 0\rangle = \exp[-iH_{\mathrm{eff}}(t - t_0)]|\phi, t_0\rangle \ . \tag{138}$$

(3) Determine the time of the jump t_1 by the condition

$$S_1 = \langle\phi, t_1, 0|\phi, t_1, 0\rangle. \tag{139}$$

(4) The wave function during the time $[t_0, t_1]$ is now

$$|\bar{\phi}, t, 0\rangle = \frac{|\phi, t, 0\rangle}{\||\phi, t, 0\rangle\|} \ . \tag{140}$$

(5) The wave function immediately after the jump is one of

$$|\phi, t_1, 1\rangle = \frac{\sqrt{\lambda_i}a_i|\phi, t, 0\rangle}{\|\sqrt{\lambda_i}a_i|\phi, t_1, 0\rangle\|} \ . \tag{141}$$

The particular choice is determined randomly according to the probabilities p_i in (132).

(6) Using the initial time t_1, and the initial state $|\phi, t_1, 1\rangle$, repeat the process with a new random number S_2, to compute t_2.

(7) The result is a sequence of jump times t_1, t_2, t_3, . . ., t_n, . . ., in which, for $t_1 < t < t_{n+1}$

$$|\phi, t, n\rangle = \frac{\exp[-iH_{\mathrm{eff}}(t - t_n)]|\phi, t_n, n\rangle}{\|\exp[-iH_{\mathrm{eff}}(t - t_n)]|\phi, t_n, n\rangle\|}, \tag{142}$$

$$|\phi, t_n, n\rangle = \frac{\sqrt{\lambda_i}a_i|\phi, t_n, n-1\rangle}{\|\sqrt{\lambda_i}a_i|\phi, t_n, n-1\rangle\|} \ . \tag{143}$$

V. SIMULATION OF MEASUREMENTS OF OUTPUTS

In quantum-optical situations we commonly drive the system with an input that is completely different from the output that is to be measured. Thus, for example, an atom may be driven by a laser beam, which appears to the atom as almost a plane wave. The driven atom radiates into all directions, and possibly at different frequencies. Measurements can be carried out on any of these outputs, or even on the output corresponding to the driven output, though this is less common. (The

most significant example is in the production of squeezed light.)

We can model this quite simply by intoducing two independent fields $dB(t)$ and $dF(t)$. The input field corresponding to the measured output $dF(t)$ is assumed to correspond to the vacuum, and in this case we will assume the input driving field $dB(t)$ corresponds to squeezed white noise, with parameters M and N, as in Eqs. (59)–(63). The operators that introduce the coupling between the system and the fields are c_1 for the measured field $dF(t)$ and c_2 for the driving field $dB(t)$. The resulting equation of motion for $|\phi,t\rangle$ is then

$$d|\varphi,t\rangle = -iH_{\rm eff}|\varphi,t\rangle dt + \sqrt{\gamma_1}c_1 dF^\dagger(t)|\varphi,t\rangle + \sqrt{\gamma_2}[c_2 dB^\dagger(t) - c_2^\dagger dB(t)]|\varphi,t\rangle, \tag{144}$$

where in this case we take

$$H_{\rm eff} = -\frac{i\gamma_1}{2}c_1^\dagger c_1 - \frac{i\gamma_2}{2}\left[Nc_2c_2^\dagger + (N+1)c_2^\dagger c_2 - M^* c_2{}^2 - M c_2^{\dagger 2}\right]. \tag{145}$$

It is important to note that both fields actually affect the equation of motion. The field $dF(t)$ is a vacuum, but it affects the equation of motion because the system radiates energy into it. As we see, then, the existence of a measurable output affects the system. If c_1 and c_2 are the same, then as far as the system is concerned, this behaves as if driven by single field $\frac{\sqrt{\gamma_1}dF(t)+\sqrt{\gamma_2}dB(t)}{\sqrt{\gamma_1+\gamma_2}}$, even though the fields $dF(t)$ and $dB(t)$ are quite distinct. For example, in resonance fluorescence of a two-level atom, the only difference between $dF(t)$ and $dB(t)$ is one of direction; the actual coupling operators c_1, c_2 are the same operator c for both.

A. Measurement of output spectrum

We can now carry out a number of possible measurements on the light emitted from the system. The most straightforward is to compute the spectrum of the output field $dF_{\rm out}(t)$. This output field can be related to the input field in a simple manner via the transformation

$$dF_{\rm out}(s) = U^\dagger(t,t_0)\, dF(s) U(t,t_0) \tag{146}$$

provided $t_0 \le s < t$, as shown in Sec. III J. We define the spectrum first in the Heisenberg picture as

$$S(\omega) = \frac{\lim_{t\to\infty}\int_{t_0}^t\int_{t_0}^t e^{-i\omega(s-s')}\langle\varphi|dF_{\rm out}^\dagger(s)dF_{\rm out}(s')|\varphi\rangle}{t-t_0}, \tag{147}$$

and making use of the transformation (146), we can convert this to the Schrödinger picture to yield

$$S(\omega) = \lim_{t\to\infty}\frac{1}{t-t_0}\langle\varphi,t|r^\dagger(\omega,t)r(\omega,t)|\varphi,t\rangle \tag{148}$$

where

$$r(\omega,t) = \int_{t_0}^t e^{-i\omega(t-s)}dF(s). \tag{149}$$

We can see that the evaluation of the spectrum amounts to the evaluation of the norm of the vector $|\beta,t\rangle$, defined by

$$|\beta,t\rangle = r(\omega,t)|\varphi,t\rangle. \tag{150}$$

We now notice that we can write a stochastic equation for $r(\omega,t)$, namely

$$dr(\omega,t) = -i\omega r(\omega,t)dt + dF(t)\ . \tag{151}$$

If we define a vector of states by

$$|\psi,t\rangle = \begin{bmatrix}|\varphi,t\rangle\\ |\beta,t\rangle\end{bmatrix}, \tag{152}$$

we can use the Ito rules to derive an equation of motion,

$$d|\psi,t\rangle = -i\mathcal{H}_{\rm eff}|\psi,t\rangle dt + \sqrt{\gamma_1}c_1 dF^\dagger(t)|\psi,t\rangle + \sqrt{\gamma_2}\left\{c_2 dB^\dagger(t) - c_2^\dagger dB(t)\right\}|\psi,t\rangle\ , \tag{153}$$

in which

$$\mathcal{H}_{\rm eff} = \begin{bmatrix} H_{\rm eff} & 0\\ i\sqrt{\gamma_1}c_1 & H_{\rm eff}+\omega\end{bmatrix}\ . \tag{154}$$

Notice that this equation is equivalent to (144) as far as $|\varphi,t\rangle$ is concerned; as would be expected, $|\varphi,t\rangle$ is not affected by $|\beta,t\rangle$, but of course the equation for $|\beta,t\rangle$ is coupled to that for $|\varphi,t\rangle$ by the off-diagonal element in $H_{\rm eff}$.

The spectrum is now given by

$$S(\omega) = \lim_{t\to\infty}\frac{\langle\beta,t|\beta,t\rangle}{t-t_0}\ . \tag{155}$$

Because the existence of the limit in (155) implies that $\langle\beta,t|\beta,t\rangle$ is asymptotic to a linear function of t, we can also write

$$S(\omega) = \lim_{t\to\infty}\frac{d\langle\beta,t|\beta,t\rangle}{dt} \tag{156}$$

and we can use the stochastic equation (153) for $|\beta,t\rangle$ to deduce

$$S(\omega) = \lim_{t\to\infty}\sqrt{\gamma_1}\left\{\langle\beta,t|c_1|\varphi,t\rangle + \langle\varphi,t|c_1^\dagger|\beta,t\rangle\right\}. \tag{157}$$

For computational purposes this formula has certain advantages.

B. Simulation algorithm for the output spectrum

Equation (153) for $|\psi,t\rangle$ is very similar in form to that for $|\varphi,t\rangle$, and it is possible to modify the simulation algorithm to generate information on $|\beta,t\rangle$ and $|\varphi,t\rangle$ simultaneously. Introducing $dX(t)$ and $dY(t)$ as done previously, we rewrite (153) in a form exactly like that used before. Defining

$$|\psi,t\rangle=\sum_{n=1}^{\infty}|\psi,n,t\rangle\ , \tag{158}$$

we get

$$\begin{aligned}d|\psi,n,t\rangle=&-\mathrm{i}\mathcal{H}_{\mathrm{eff}}|\psi,n,t\rangle dt+\sqrt{\gamma_1}c_1dF^\dagger(t)|\psi,n-1,t\rangle\\&+\Big\{\sqrt{\gamma_2}\mathcal{A}_1dX^\dagger(t)\\&+\sqrt{\gamma_2}\mathcal{A}_2dY^\dagger(t)\Big\}|\psi,n-1,t\rangle\end{aligned} \tag{159}$$

in which

$$\begin{bmatrix}\mathcal{A}_1\\ \mathcal{A}_2\end{bmatrix}=V^T\begin{bmatrix}c_2\\ c_2^\dagger\end{bmatrix}\ . \tag{160}$$

If we now define

$$\mathcal{M}_n(t)=\mathrm{Tr}_B\{|\psi,n,t\rangle\langle\psi,n,t|\}\equiv\begin{bmatrix}\rho_n(t) & \eta_n^\dagger(t)\\ \eta_n(t) & \mu_n(t)\end{bmatrix} \tag{161}$$

and

$$\mathcal{M}(t)=\sum_{n=0}^{\infty}\mathcal{M}_n(t)\equiv\begin{bmatrix}\rho(t) & \eta^\dagger(t)\\ \eta(t) & \mu(t)\end{bmatrix}\ , \tag{162}$$

we get the equation

$$\begin{aligned}\frac{d\mathcal{M}_n}{dt}=&-\mathrm{i}\mathcal{H}_{\mathrm{eff}}\mathcal{M}_n(t)+\mathrm{i}\mathcal{M}_n(t)\mathcal{H}_{\mathrm{eff}}^\dagger+\gamma_1c_1\mathcal{M}_{n-1}c_1^\dagger\\&+\lambda_1\mathcal{A}_1\mathcal{M}_{n-1}(t)\mathcal{A}_1^\dagger+\lambda_2\mathcal{A}_2\mathcal{M}_{n-1}(t)\mathcal{A}_2^\dagger.\end{aligned} \tag{163}$$

Because of the analogous form of these equations to those for $|\varphi,n,t\rangle$ and $\rho_n(t)$, we can quite straightforwardly generalize the continuous measurement theory. There are three differences.

(1) There are now *three* possible jumps, corresponding to emission into $dF(t)$, and absorption from and emission into $dB(t)$.

(2) We need the initial condition for $|\beta,t_0\rangle=0$, which follows from (149).

(3) The equation of motion for $\mathcal{M}=\sum_n\mathcal{M}_n$ does not preserve $\mathrm{Tr}\{\mathcal{M}\}$, but of course does still preserve $\mathrm{Tr}\{\rho\}$. This means that we should determine all probabilities and normalizations in terms of $\mathrm{Tr}\{\rho\}$, *not* in terms of $\mathrm{Tr}\{\mathcal{M}\}$. Thus, in carrying out a simulation, we still determine the time of the first jump by (139), using $|\varphi,t_1,0\rangle$, *not* $|\psi,t_1,0\rangle$, and in making the choice of which particular jump takes place, the relative probabilities are proportional to $\mathrm{Tr}\{\gamma_1c_1\rho c_1^\dagger\}$, $\mathrm{Tr}\{\lambda_1\mathcal{A}_1\rho\mathcal{A}_1^\dagger\}$, and $\mathrm{Tr}\{\lambda_2\mathcal{A}_2\rho\mathcal{A}_2^\dagger\}$.

Thus the algorithm follows from that in Sec. IV E with the substitutions

$$|\varphi,n,t\rangle\to|\psi,n,t\rangle, \tag{164}$$

$$\|\,|\varphi,n,t\rangle\|\to\|\,|\varphi,n,t\rangle\|, \tag{165}$$

$$a_i\to(\mathcal{A}_i,c_1)\ . \tag{166}$$

From this we can deduce simulated values for $\mathcal{M}(t)$, leading to values for $\rho(t)$ and $\mu(t)$.

By averaging over simulations, we obtain the values of $\rho(t),\eta(t)$, and $\mu(t)$, from which the spectrum can be calculated through

$$S(\omega)=\lim_{t\to\infty}\frac{\mathrm{Tr}\,\{\mu(t)\}}{t-t_0}. \tag{167}$$

However, much more information is available than only the spectrum. Since $\mu(t)$ is an operator in the system space, we can also compute objects like $\mathrm{Tr}\,\{f\mu(t)\},\mathrm{Tr}\,\{f\rho(t)\},\mathrm{Tr}\,\{f\eta(t)\}$, where f is a system operator. Furthermore, we can use formula (157) to write the alternative equation for the spectrum

$$S(\omega)=\lim_{t\to\infty}\sqrt{\gamma_1}\mathrm{Tr}\{c_1\eta(t)\}+\sqrt{\gamma_1}\mathrm{Tr}\{c_1^\dagger\eta^\dagger(t)\} \tag{168}$$

$$=2\sqrt{\gamma_1}\mathrm{Re}[\mathrm{Tr}\{c_1\eta(t)\}]\ . \tag{169}$$

C. Higher-order spectra

We can write

$$\begin{aligned}&|\beta_0,t\rangle=|\varphi,t\rangle,\\&|\beta_1,t\rangle=r(\omega_1,t)|\varphi,t\rangle\equiv|\beta,t\rangle,\\&|\beta_n,t\rangle=r(\omega_n,t)|\beta_{n-1},t\rangle,\end{aligned} \tag{170}$$

and take in this case

$$|\varphi_s,t\rangle=\begin{bmatrix}|\beta_0,t\rangle\\ |\beta_1,t\rangle\\ \vdots\\ |\beta_J,t\rangle\end{bmatrix}, \tag{171}$$

where J is some positive integer. From these we can derive a straightforward generalization of the simulation algorithm. The equation of motion for $|\varphi_s,t\rangle$ takes on the same form as (153), with the substitution

$$\mathcal{H}_{\mathrm{eff}}\to\begin{bmatrix}H_{\mathrm{eff}} & 0 & 0 & 0 & \cdots\\ \mathrm{i}\sqrt{\gamma_1}c_1 & H_{\mathrm{eff}}+\omega_1 & 0 & 0 & \cdots\\ 0 & 2\mathrm{i}\sqrt{\gamma_1}c_1 & H_{\mathrm{eff}}+\omega_1+\omega_2 & 0 & \cdots\\ 0 & 0 & 3\mathrm{i}\sqrt{\gamma_1}c_1 & H_{\mathrm{eff}}+\omega_1+\omega_2+\omega_3 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}\ . \tag{172}$$

The simulation algorithm looks exactly the same as in Sec. V B, and we can now compute higher-order spectra such as

$$S(\omega',\omega)=\lim_{t\to\infty}\frac{\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t}e^{-i\omega'(s_1-s_2)-i\omega(s_3-s_4)}\langle\varphi|dF_{\rm out}^{\dagger}(s_1)dF_{\rm out}^{\dagger}(s_3)dF_{\rm out}(s_4)dF_{\rm out}(s_2)|\varphi\rangle}{(t-t_0)^2}$$
$$=\lim_{t\to\infty}\frac{\langle\beta_2,t|\beta_2,t\rangle}{(t-t_0)^2}. \tag{173}$$

Notice that in this simulation, there are exactly the same jumps as occur in the computation of the single frequency spectrum, or indeed of simply the density operator. The operators $\mathcal{A}_1,\mathcal{A}_2,c_1$ are exactly the same. This is so because these are determined by the physical setup, which is described by the equation of motion (144). This includes, of course, the driving field $dB(t)$ and the measured output field $dF(t)$. The latter, it must be emphasized, is no fictitious field introduced for the purposes of computation, but represents an actual output channel which we may choose to measure.

As can be seen from the above, measurements of arbitrary complexity can be introduced, at the cost of more algebra at each simulation step, since the dimensionality of the problem is J times the dimensionality of $|\varphi,t\rangle$. It is clear also that off-diagonal spectra can be obtained from the off-diagonal matrix elements like $\langle\beta_m|\beta_n,t\rangle$. In essence then, all statistical properties up to order $2J$ of any output field are determined by this procedure, and J can be made as large as necessary—the only limitation is computer time and memory.

D. Squeezing spectrum

In systems that produce squeezing, it is of interest to compute the squeezing spectrum, which depends on the operators

$$X(\omega)=\frac{v}{2}r(\omega)+\frac{v^*}{2}[r(\omega)]^{\dagger}, \tag{174}$$
$$Y(\omega)=\frac{v}{2i}r(\omega)-\frac{v^*}{2i}[r(\omega)]^{\dagger}, \tag{175}$$

where v is a phase that can be chosen arbitrarily. The average values $\langle X(\omega)^2\rangle$, $\langle Y(\omega)^2\rangle$, and $\langle X(\omega)Y(\omega)\rangle$ are of interest. It is clear that these can be computed by setting

$$|\varphi,t\rangle=\begin{bmatrix}|\varphi,t\rangle\\|\beta,t\rangle\\|\beta^{\dagger},t\rangle\end{bmatrix} \tag{176}$$

with $|\beta,t\rangle$ defined as in (150), and

$$|\beta^{\dagger},t\rangle=[r(\omega,t)]^{\dagger}|\varphi,t\rangle. \tag{177}$$

The equation of motion takes a slightly different form,

$$d|\varphi,t\rangle=-i\mathcal{H}_{\rm eff}|\varphi,t\rangle+\mathcal{C}_1dF^{\dagger}(t)|\varphi,t\rangle$$
$$+\sqrt{\gamma_2}\left\{c_2dB^{\dagger}(t)-c_2^{\dagger}dB(t)\right\}|\varphi,t\rangle \tag{178}$$

with

$$\mathcal{H}_{\rm eff}=\begin{bmatrix}H_{\rm eff}&0&0\\\sqrt{\gamma}_1c_1&H_{\rm eff}+\omega&0\\0&0&H_{\rm eff}-\omega\end{bmatrix} \tag{179}$$

and

$$\mathcal{C}_1=\begin{bmatrix}\sqrt{\gamma}_1c_1&0&0\\0&\sqrt{\gamma}_1c_1&0\\1&0&\sqrt{\gamma}_1c_1\end{bmatrix}. \tag{180}$$

The procedure for simulation is now the same as in Sec. V A, apart from the fact that we now have $\sqrt{\gamma_1}c_1\to\mathcal{C}_1$, and there are three subspaces. The relevant spectra are of the form

$$\lim_{t\to\infty}\frac{\langle r^{\dagger}(\omega,t)r(\omega,t)\rangle}{t-t_0}=\lim_{t\to\infty}\frac{\langle\beta,t|\beta,t\rangle}{t-t_0},$$
$$\lim_{t\to\infty}\frac{\langle r(\omega,t)r(\omega,t)\rangle}{t-t_0}=\lim_{t\to\infty}\frac{\langle\beta^{\dagger},t|\beta,t\rangle}{t-t_0}, \tag{181}$$

and the averages of the Hermitian conjugates, from which squeezing spectra may be obtained by appropriate linear combinations.

E. Photon-counting spectra

To compute photon-counting spectra, that is, the Fourier transforms of correlation functions like $\langle f^{\dagger}(t)f^{\dagger}(t')f(t')f(t)\rangle$, requires an extension of quantum stochastic calculus to bring in the so-called *gauge process* $d\Lambda(t)$, which is directly related to photon counting [7,8]. The gauge process arises by noting that the total number of photons counted between t_1 and t_2 (assuming a perfect detector) in the field $f(t)$ is

$$\mathcal{N}(t_1,t_2)=\int_{t_1}^{t_2}f^{\dagger}(t)f(t)dt. \tag{182}$$

We normally make the substitutions

$$f(t)dt\to dF(t), \tag{183}$$
$$f^{\dagger}(t)dt\to dF^{\dagger}(t), \tag{184}$$

which in this case leads to a rather curious definition

$$\mathcal{N}(t_1,t_2)=\int_{t_1}^{t_2}\frac{dF^{\dagger}(t)dF(t)}{dt} \tag{185}$$
$$\equiv\int_{t_1}^{t_2}d\Lambda(t). \tag{186}$$

Thus $d\Lambda(t)$ is the operator whose eigenvalues are the number of photons in the interval dt. It is only possible to make sense of $d\Lambda(t)$ in situations in which M and N are zero. The reason for this is, physically, quite understandable: the existence of nonzero M or N would mean that strictly the field $f(t)$ had infinite bandwidth, hence any

photon count would yield an infinite number of counts in any finite time interval. Substituting $dF^\dagger(t)dF(t) = N\,dt$ into (184) shows this, albeit nonrigorously. This does not mean that $d\Lambda(t)$ is a useless concept, since most output fields are not white-noise outputs, apart from vacuum noise, which does not lead to any counts.

We will also need the Ito rules for $d\Lambda(t)$. We can deduce these by noting that for any optical field that has no white-noise component, the probability of finding more than one photon in the time interval dt will go to zero at least as fast as dt^2. We can use this fact to show that if we discretize the integral

$$|A\rangle = \int [d\Lambda(t)^2 - d\Lambda(t)]|\varphi,t\rangle, \tag{187}$$

then in the limit of infinitely fine discretization the mean of the norm of $|A\rangle$ goes to zero. This means that in the mean-norm topology, introduced in Sec. III E, $|A\rangle \to 0$, and thus we can formally write in integrals

$$d\Lambda(t)^2 = d\Lambda(t). \tag{188}$$

This equation ensures that the only eigenvalues of $d\Lambda(t)$ are 0 and 1, and that we can only count either one or no photons in a time interval dt.

Similarly

$$d\Lambda(t)dF(t) = 0, \tag{189}$$

$$d\Lambda(t)dF^\dagger(t) = dF^\dagger(t), \tag{190}$$

and taking the Hermitian conjugate

$$dF^\dagger(t)d\Lambda(t) = 0, \tag{191}$$

$$dF(t)d\Lambda(t) = dF(t). \tag{192}$$

These rules are to be taken together with the Ito rules for $dF(t)$:

$$dF(t)dF^\dagger(t) = dt, \tag{193}$$

$$dF^\dagger(t)dF(t) = dF(t)dt = dF^\dagger(t)dt = 0\,. \tag{194}$$

Remember, too, that $d\Lambda(t)$ is an Ito increment, and thus commutes with the evolution operator $U(t_0,t)$. An output is typically a field with a finite correlation time, for which Barchielli [8] has shown that a perfectly straightforward application of the gauge process $d\Lambda(t)$ can be made.

We can compute the photon-counting spectrum by considering

$$R(\omega,t) = \int_{t_0}^{t} e^{-i\omega(t-s)}d\Lambda(s) \tag{195}$$

$$= \int_{t_0}^{t} e^{-i\omega(t-s)}f^\dagger(s)f(s)ds. \tag{196}$$

If we define

$$|\beta_p,t\rangle = R(\omega,t)|\varphi,t\rangle \tag{197}$$

we can compute an equation of motion

$$d|\beta_p,t\rangle = dR(\omega,t)|\varphi,t\rangle + R(\omega,t)d|\varphi,t\rangle + dR(\omega,t)d|\varphi,t\rangle. \tag{198}$$

Using the Ito rules, this becomes

$$\begin{aligned} d|\beta_p,t\rangle = {} & -i[H_{\text{eff}}+\omega]|\beta_p,t\rangle dt \\ & +\sqrt{\gamma_2}[c_2 dB^\dagger - c_2^\dagger dB]|\beta_p,t\rangle \\ & +\sqrt{\gamma_1}c_1 dF^\dagger|\beta_p,t\rangle + \sqrt{\gamma}c_1 dF^\dagger(t)|\varphi,t\rangle\,, \end{aligned} \tag{199}$$

where a term $d\Lambda(t)|\varphi,t\rangle$ has been dropped, since $|\varphi,t\rangle$ is a vacuum for increments $dF(t), dF^\dagger(t), d\Lambda(t)$. We thus find an equation of motion for

$$|\psi_p,t\rangle = \begin{bmatrix} |\varphi,t\rangle \\ |\beta_p,t\rangle \end{bmatrix} \tag{200}$$

in the usual form

$$\begin{aligned} d|\psi_p,t\rangle = {} & -i\mathcal{H}_{\text{eff}}|\psi_p,t\rangle dt \\ & +\sqrt{\gamma_2}\left[c_2 dB^\dagger(t) - c_2^\dagger dB(t)\right]|\psi_p,t\rangle \\ & +\sqrt{\gamma}\mathcal{C}_p dF^\dagger(t)|\psi_p,t\rangle \end{aligned} \tag{201}$$

with

$$\mathcal{H}_{\text{eff}} = \begin{bmatrix} H_{\text{eff}} & 0 \\ 0 & H_{\text{eff}}+\omega \end{bmatrix} \tag{202}$$

and

$$\mathcal{C}_p = \begin{pmatrix} c_1 & 0 \\ c_1 & c_1 \end{pmatrix}. \tag{203}$$

The usual normally ordered photon-counting spectrum is given by

$$\lim_{t\to\infty} \frac{\int_{t_0}^{t} ds \int_{t_0}^{t} ds' e^{i\omega(t-s)}e^{-i\omega(t-s')}\langle\varphi,t|f^\dagger(s)f^\dagger(s')f(s')f(s)|\varphi,t\rangle}{t-t_0} \tag{204}$$

$$= \lim_{t\to\infty} \frac{\int_{t_0}^{t}\int_{t_0}^{t} \langle\varphi,t|d\Lambda(s)d\Lambda(s')|\varphi,t\rangle\, e^{i\omega(t-s)}e^{-i\omega(t-s')}}{t-t_0} - \lim_{t\to\infty} \frac{\int_{t_0}^{t} ds\langle\varphi,t|d\Lambda(s)|\varphi,t\rangle}{t-t_0} \tag{205}$$

$$= \lim_{t\to\infty} \frac{\langle\beta_p,t|\beta_p,t\rangle}{t-t_0} - \lim_{t\to\infty} \frac{\langle\varphi,t|\beta_p,t\rangle|_{\omega=0}}{t-t_0}\,. \tag{206}$$

F. Correlation functions

Instead of studying spectra, we can study correlation functions directly. We define an operator $r([\epsilon],t)$, where $\epsilon(t)$ is an arbitrary function of t, by

$$r([\epsilon],t)=\int_{t_0}^{t}\epsilon(s)dF(s) \tag{207}$$

and correspondingly define

$$|\beta,[\epsilon],t\rangle = r([\epsilon],t)|\varphi,t\rangle \tag{208}$$

from which an equation of motion for $|\beta,[\epsilon],t\rangle$ is

$$\begin{aligned} d|\beta,[\epsilon],t\rangle = &-\mathrm{i}H_{\mathrm{eff}}|\beta,[\epsilon],t\rangle+\sqrt{\gamma_1}c_1dF^{\dagger}(t)|\beta,[\epsilon],t\rangle \\ &+\sqrt{\gamma_2}[c_2dB^{\dagger}(t)-c_2^{\dagger}dB(t)]|\beta,[\epsilon],t\rangle \\ &+\epsilon(t)\sqrt{\gamma_1}c_1|\varphi,t\rangle. \end{aligned} \tag{209}$$

A relation corresponding to (157) is that

$$\sqrt{\gamma_1}\langle\varphi,t|c_1^{\dagger}|\beta,[\epsilon],t\rangle=\int_{t_0}^{t}\epsilon(s)ds\langle\varphi,t|f^{\dagger}(t)f(s)|\varphi,t\rangle. \tag{210}$$

This equation is proved by using the definition of $|\beta,[\epsilon],t\rangle$ by means of (207), (208), and going to the Heisenberg picture, in which we can use the relationship

$$f_{\mathrm{out}}(t)=f_{\mathrm{in}}(t)+\sqrt{\gamma_1}c_1(t)\ . \tag{211}$$

This formulation is quite general, but a most useful special case comes when we set

$$\epsilon(s)=\delta_{\tau_0}(s)\equiv\delta(s-\tau_0) \tag{212}$$

so that

$$\sqrt{\gamma_1}\langle\varphi,t|c_1^{\dagger}|\beta,[\delta_{\tau_0}],t\rangle=\langle\varphi,t|f^{\dagger}(t)f(\tau_0)|\varphi,t\rangle\quad(t\geq\tau_0). \tag{213}$$

The simulation of this is straightforward. We pick a random time τ_0 while integrating the simulation of $|\varphi,t\rangle$. Equation (209) is exactly the same as that for $|\varphi,t\rangle$, apart from the term involving $\epsilon(t)$, which is in this case a δ function. Since the initial condition is $|\beta,[\delta_{\tau_0}],t_0\rangle=0$, this simply imposes a later initial condition

$$|\beta,[\delta_{\tau_0}],\tau_0\rangle=\sqrt{\gamma_1}c_1|\varphi,\tau_0\rangle, \tag{214}$$

which then gives a contribution to the correlation function for various $t>t_0$. We repeat the procedure sufficiently frequently to build up the average correlation function.

VI. CONCLUSIONS

The aim of this paper has been to develop the wave-function formulation of quantum stochastic differential equations from a physical basis, and demonstrate the strong relationship between this formulation and the methods of quantum-jump simulations. We have formulated everything from the starting point of an appropriately simplified description of a localized system interacting with the electromagnetic field, and shown how the input field commutators can be taken in a δ function form. From there, we formulate the concept of quantum stochastic integration in the two forms: Ito and Stratonovich.

In formulating the concept of white noise in quantum stochastic differential equations, it is important to distinguish between the vacuum noise and the noise arising from an actual physical input. As shown in Sec. II, the vacuum noise can in almost all realistic situations be approximated by white noise. Only in the study of very fast pulses would this cease to be valid. However, this is not the case for other inputs, whose time scales are often significantly slower. Nevertheless, we have considered white-noise inputs other than the vacuum, because (1) the results are interesting, and straightforwardly derived; (2) all possible quantum white-noise inputs can be characterized by the parameters N and M, as in (59)–(63); (3) there are situations where the white-noise approximation to physical inputs is appropriate, such as in the study of inhibition of atomic phase decays by squeezed light; and (4) coherent components to an input can be easily added by making a canonical transformation so that they appear as classical driving terms in H_{sys}. These coherent inputs can be time dependent and even random.

The emphasis on the Ito formulation of the quantum stochastic differential equations, with its rather curious calculus, is deliberate. The fact that the Ito formulation includes radiation damping explicitly, and uses noise terms $dB(t)$ and $dB^{\dagger}(t)$ whose state is independent of that of the system under consideration, is an enormous simplification, and it is well worth the small amount of inconvenience required to get used to the fact that in expanding infinitesimals we must keep track of quadratic terms in $dB(t)$ and $dB^{\dagger}(t)$. Furthermore, these terms are physically very important—they give rise to the terms in the master equation which generate quantum jumps.

This is the other advantage of the Ito formulation—the simple derivation of the appropriate master equation for the system, and the very direct correlation between the quantum-jump simulation methods and the quantum stochastic differential equation. The only drawback at the moment is the lack of any good way to simulate the quantum stochastic differential equation directly, but this is compensated for by the ease with which correlation functions and spectra for output fields may be generated. We have shown in Sec. V that all output spectra may be simulated directly at the same time as the density operator by the method of quantum jumps, so that in this sense the quantum jump simulation can be considered to be a complete description of the systems under study.

It is good to indicate what cannot yet be done using the quantum-jump methods. Inputs that are not white noise or random superpositions of coherent states are still inaccessible. There is as yet no good numerical way of simulating, for example, an antibunched input field.

However, the quantum-jump simulations are a very powerful technique for a wide range of problems, as is shown in [13], where laser cooling was studied, and in the second paper of this pair [17], where resonance fluorescence, an atom driven by squeezed light, and a strongly

coupled atom-cavity system driven by thermal light are all studied numerically.

There is a final problematic point—are these quantum jumps real? By this we mean to ask the question: does the technique of quantum-jump simulation give a picture of what would actually happen if certain measurements were carried out? Can the system really be regarded as evolving smoothly for most of the time, with occasional jumps to different quantum states as the result of a measurement process that is represented by the simulation? For a vacuum field the index n can be identified with the photon number, allowing an interpretation in terms of photon-counting measurements. However, in the general case there is in fact some arbitrariness in the jumps. For example, we can make a unitary transformation on the operators a_1, a_2 (115) in the form

$$\sqrt{\lambda_i}a_i \to \sqrt{\lambda_i}\bar{a}_i = \sum_j R_{ij}\sqrt{\lambda_j}a_j, \tag{215}$$

where R is any unitary matrix, and the master equation remains the same. All correlation functions and spectra will be unchanged, but the jumps that occur will be different, and thus all the sequences of wave functions will be different. At the same time, from the point of view of the stochastic differential equations, the system is being driven by two different fields

$$\frac{d\bar{X}_i(t)}{\sqrt{\lambda_i}} = \sum_j R_{ji}\frac{dX_j(t)}{\sqrt{\lambda_j}}. \tag{216}$$

For example, the system may be driven by thermal light, so that we could regard the basic fields as being $dB(t), -dB^\dagger(t)$. Carrying out these two transformations, we obtain a picture in which the system is driven by two *squeezed* fields, which when mixed produce a thermal field. This is reasonable—a thermal field *can* be regarded as a mixture of two squeezed fields, and in many different ways, in much the same way as polarized light can be regarded as being a mixture of two kinds of circularly polarized light, or as a mixture of left and right circularly polarized light. There is no way of telling the difference—there is no difference—between the two points of view. In this sense the jumps between the states are not necessarily a realization of any actual measurement, but can be regarded simply as a convenient method of simulating the equations of motion. Carmichael [15], however, in his discussion of quantum simulations has emphasized a point of view and formulation in which the jumps are expressed completely in terms of actual (photodetector) measurements.

Note added. We have recently received a copy of unpublished work from K. Mølmer, Y. Castin, and J. Dalibard which also describes applications of the Monte Carlo wave-function approach to problems in quantum optics, but with emphasis on mechanical light effects.

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APPENDIX A: CONVERSION TO ITO EQUATIONS

1. Vacuum input

We can also express the equation of motion in an Ito form, that is, as an integral equation defined in terms of Ito integrals. The procedure is analogous to that employed in the conversion between classical Ito and Stratonovich integrals [18]. The only modifications occur in the use of noncommuting $\Delta B, \Delta B^\dagger$ operators.

Let us suppose that the Ito equation corresponding to

$$\{d|\varphi,t\rangle\}_{\mathcal{S}} = \sqrt{\gamma}\{dB^\dagger(t)c - dB(t)c^\dagger\}|\varphi,t\rangle \tag{A1}$$

can be written

$$d|\varphi,t\rangle = \{\alpha dt + \beta dB^\dagger(t) - \beta^\dagger dB(t)\}|\varphi,t\rangle\ . \tag{A2}$$

We consider an arbitrary Stratonovich integral of a function $|\varphi(t)\rangle$ which obeys Eq. (A2):

$$\left\{\int dB(t)|\varphi(t)\rangle\right\}_{\mathcal{S}} = \lim\left\{\sum \Delta B_i \frac{|\varphi,t_i\rangle + |\varphi,t_{i+1}\rangle}{2}\right\}$$
$$= \lim\left\{\sum \Delta B_i\left[|\varphi,t_i\rangle + \frac{1}{2}(\alpha\Delta t_i + \beta\Delta B_i^\dagger - \beta^\dagger\Delta B_i)|\varphi,t_i\rangle\right]\right\} \tag{A3}$$

and using the rules (36)

$$\left\{\int dB(t)|\varphi(t)\rangle\right\}_{\mathcal{S}} = \lim\sum\left\{\Delta B_i|\varphi,t_i\rangle - \frac{1}{2}\beta\Delta t_i|\varphi,t_i\rangle\right\}$$
$$= \left\{\int dB(t)|\varphi(t)\rangle\right\}_{\mathcal{I}} - \frac{1}{2}\int\beta|\varphi,t\rangle dt\ . \tag{A4}$$

In this case (the vacuum state), we also find

$$\left\{\int dB^\dagger(t)|\varphi(t)\rangle\right\}_{\mathcal{S}} = \left\{\int dB^\dagger(t)|\varphi(t)\rangle\right\}_{\mathcal{I}},$$
$$\left\{\int\langle\varphi(t)|dB(t)\right\}_{\mathcal{S}} = \left\{\int\langle\varphi(t)|dB(t)\right\}_{\mathcal{I}}, \tag{A5}$$
$$\left\{\int\langle\varphi(t)|dB^\dagger(t)\right\}_{\mathcal{S}} = \left\{\int\langle\varphi(t)|dB^\dagger(t)\right\}_{\mathcal{I}} - \frac{1}{2}\int\langle\varphi(t)|\beta^\dagger dt\ .$$

Remembering now that (A1) is shorthand for the form (39), an integral equation, we see that (A2) is equivalent to (A1) if

$$\beta = \sqrt{\gamma}c, \qquad \beta^\dagger = \sqrt{\gamma}c^\dagger, \qquad \alpha = -\frac{\gamma}{2}c^\dagger c, \tag{A6}$$

so that the Ito equation equivalent to (A1) is

$$\{d|\varphi,t\rangle\}_{\mathcal{I}} = \left\{ -\frac{1}{2}\gamma c^\dagger c dt + \sqrt{\gamma}dB^\dagger(t)c - \sqrt{\gamma}dB(t)c^\dagger \right\}|\varphi,t\rangle \, . \tag{A7}$$

2. Nonvacuum inputs

In making these transformations we have assumed that the quantum state of $dB(t), dB^\dagger(t)$ is the vacuum. The usual input-output description allows a more general set of relations than (36), namely (59)–(63). The same methods as used for the vacuum input yield in this case:

$$\begin{aligned}
\left\{\int dB(t)|\varphi(t)\rangle\right\}_{\mathcal{S}} &= \left\{\int dB(t)|\varphi(t)\rangle\right\}_{\mathcal{I}} - \frac{1}{2}(N+1)\int \beta|\varphi(t)\rangle dt + \frac{1}{2}M\int \beta^\dagger|\varphi(t)\rangle dt, \\
\left\{\int dB^\dagger(t)|\varphi(t)\rangle\right\}_{\mathcal{S}} &= \left\{\int dB^\dagger(t)|\varphi(t)\rangle\right\}_{\mathcal{I}} - \frac{1}{2}N\int \beta^\dagger|\varphi(t)\rangle dt + \frac{1}{2}M^*\int \beta|\varphi(t)\rangle dt, \\
\left\{\int \langle\varphi(t)|dB(t)\right\}_{\mathcal{S}} &= \left\{\int \langle\varphi(t)|dB(t)\right\}_{\mathcal{I}} - \frac{1}{2}N\int \langle\varphi(t)|\beta dt + \frac{1}{2}M\int \langle\varphi(t)|\beta^\dagger dt, \\
\left\{\int \langle\varphi(t)|dB^\dagger(t)\right\}_{\mathcal{S}} &= \left\{\int \langle\varphi(t)|dB^\dagger(t)\right\}_{\mathcal{I}} - \frac{1}{2}(N+1)\int \langle\varphi(t)|\beta^\dagger dt + \frac{1}{2}M^*\int \langle\varphi(t)|\beta dt \, .
\end{aligned} \tag{A8}$$

Under these conditions, the Stratonovich equation (A2) becomes

$$\{d|\varphi,t\rangle\}_{\mathcal{I}} = \left\{ -\frac{1}{2}\gamma[(N+1)c^\dagger c + Ncc^\dagger - Mc^\dagger c^\dagger - M^* cc]dt + \sqrt{\gamma}dB^\dagger(t)c - \sqrt{\gamma}dB(t)c^\dagger \right\}|\varphi,t\rangle. \tag{A9}$$

APPENDIX B: EXAMPLE—RESONANCE FLUORESCENCE OF A TWO-LEVEL ATOM

We want to consider a two-level atom driven by a coherent driving field, and compare this briefly with Mollow's [16] pure state treatment of this problem.

We do not consider separate driving and observation fields in this case, since, after making a unitary transformation, the effective input field is a vacuum. We make the identification

$$c = \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad c^\dagger = \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{B1}$$

and we choose

$$H_{\rm sys} = -\frac{1}{2}\hbar\Delta\sigma_z = -\frac{1}{2}\hbar\Delta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{B2}$$

The input is assumed to be a coherent field of amplitude $\mathcal{E}e^{-i\Omega t}$, which is realized by making a unitary transformation, so that (in the interaction picture)

$$H_{\rm sys} \to +i\Omega\left(\sigma^+ - \sigma^-\right) , \tag{B3}$$

and the Ito equation for the wave function becomes (on the assumption of a vacuum input) from (74),

$$\begin{aligned}
d|\varphi,t\rangle &= \left\{ -\frac{i}{\hbar}H_{\rm sys} - \frac{1}{2}\gamma\sigma^+\sigma^- \right\}|\varphi,t\rangle dt \\
&\quad + \sqrt{\gamma}dB^\dagger(t)\sigma^-|\varphi,t\rangle \, . \\
&\equiv -iH_{\rm eff}|\varphi,t\rangle dt + \sqrt{\gamma}dB^\dagger(t)\sigma^-|\varphi,t\rangle \, .
\end{aligned} \tag{B4}$$

If we consider first the division into n-jump states, then (117) for this particular case is

$$\frac{d\rho_n(t)}{dt} = -iH_{\rm eff}\rho_n(t) + i\rho_n(t)H_{\rm eff}^\dagger - \gamma\sigma^-\rho_{n-1}(t)\sigma^+, \tag{B5}$$

which yields Eqs. (7.4) of Mollow [16] describing the evolution of the reduced atomic density operator in the different n subspaces.

Similarly, the rate of emission of photons at frequency ω,

$$\frac{d}{dt}{\rm Tr}\{\beta(t)\} = \sqrt{\gamma}{\rm Tr}\{\sigma^-\eta^\dagger(t)\} + {\rm c.c.} \, , \tag{B6}$$

together with the equations for $d\eta(t)/dt$ and $d\rho(t)/dt$ are equivalent to Mollow's Eqs. (5.7)–(5.10) with the explicit identification

$$|\varphi,t\rangle \equiv \begin{pmatrix} |A(t)\rangle_1 \\ |A(t)\rangle_0 \end{pmatrix}, \quad |\beta,t\rangle \equiv \begin{pmatrix} |B_k(t)\rangle_1 \\ |B_k(t)\rangle_0 \end{pmatrix} \tag{B7}$$

and with $k = \omega/c$.

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