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Preferential instability in arrays of coupled lasers

Ruo-ding Li
Université Libre de Bruxelles, Campus de la Plaine, Code Postal 231, 1050 Bruxelles, Belgium

Thomas Erneux
Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, Illinois 60208

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We consider an array of N coupled class-B lasers in a ring geometry. We analyze the stability of the steady-state solutions for small values of the coupling strength and small damping. The problem is motivated by recent studies of laser-diode arrays, but analytical results on the possible instabilities remain limited to the case $N=2$. We consider N arbitrary and use the coupling strength as the bifurcation parameter. As this parameter increases from zero, we show that the first instability leads to a preferential mode of oscillations. For N even, we study this bifurcation to a time-periodic standing-wave solution and determine the direction of bifurcation. We discuss the bifurcation possibilities in terms of the parameter α, known as the linewidth-enhancement factor, in semiconductor lasers. Increasing α destabilizes phase locking between adjacent lasers but leads to a smooth bifurcation to periodic solutions. Inversely, decreasing α stabilizes the laser array, but the first bifurcation leads to a hard transition to time-dependent solutions. The predictions of our analysis are in agreement with the results of a numerical study of the laser equations.

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I. INTRODUCTION

Arrays of semiconductor diode lasers can operate with high output powers and are promising devices for applications that require high optical power from a laser source (high-speed optical recording, high-speed printing, free-space communications, pumping of solid-state lasers) [1]. For all these applications, it is necessary that the array operates in phase so as to produce a single and narrow beam. This has motivated the recent theoretical interest for models predicting the response of coupled diode lasers (called phase-locked arrays). Early studies of phase-locked arrays were able to interpret the experiments [2] but failed to predict the possible oscillating modes of an array of coupled emitters. Later Otsuka [3] and Butler, Ackley, and Botez [4] developed a coupled-mode theory and have shown that an array of N coupled lasers has N normal modes (called array modes). The results of their analysis have been confirmed by many experimental observations [5]. The coupled-mode theory describes the possible modes of a laser array but ignores their stability. Experiments have shown that the population of coupled lasers has a tendency to operate in the 180°-phase shift mode (called out-of-phase mode: the phase difference for the electrical field of two successive coupled lasers is π). Recently, Winful and co-workers [6–8] and Otsuka [9] studied simple models of coupled lasers and have found a rich variety of spatiotemporal responses. Analytical work is limited to the case of two coupled lasers [6–8,10,11] and the case of phase-locked solutions in very large arrays [12].

In this paper, we consider N arbitrary and determine the stability properties of the laser system in the limit of small coupling and small damping. Our analytical results are valid for all class-B lasers which involve semiconductor lasers as well as other lasers of practical interest [13]. Furthermore, we propose a first-bifurcation analysis of the leading instability. This analysis is difficult but leads to a simple expression for the direction of bifurcation. In particular, it reveals a key nonlinear effect of the parameter α defined as the linewidth-enhancement factor in semiconductor lasers. It can range in magnitude from 4 to 6 in semiconductor lasers and couples amplitude and phase across the array. For other laser systems such as detuned gas lasers, α is smaller than 1 or even zero in resonant conditions. We show that increasing α destabilizes the basic steady-state solution but leads to a smooth transition to small-amplitude oscillations. Inversely, decreasing α stabilizes the basic steady-state solution but may lead to a hard transition to large-amplitude oscillations ($\alpha > 1.2$). The first case applies to semiconductor lasers while the second case is appropriate for detuned systems.

Coupled nonlinear oscillators are used as models in different areas of physics [14] and are studied by first deriving a phase equation in the limit of weak coupling. This limit does not apply for class-B lasers because the laser is not a limit-cycle oscillator but rather a weakly damped oscillator. A weak coupling between the lasers is enough to destabilize both the amplitude and the phase of each oscillator. A Hopf bifurcation is the simplest manifestation of this amplitude and phase instability. Recently, this Hopf bifurcation has been analyzed for a population of two lasers [7,8]. In this paper, we consider a population of N lasers and determine steady and time-periodic solutions in terms of the amplitude of the coupling. We find that the first bifurcation of the basic steady state corresponds to a time-periodic and spatially nonuniform solution. The wave number of this solution

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is always equal to $N/2$ (N even) whatever the values of the laser parameters. This is a particular feature of the laser problem which does not appear in other (hydrodynamic, chemical, or laser) stability problems. Thus the first instability corresponds to a preferential mode of instability.

The paper is organized as follows. In Sec. II we formulate the evolution equations for a system of coupled lasers in a ring geometry. In Sec. III we describe the steady-state solutions. Section IV is devoted to their linear stability analysis. In Sec. V we investigate the first Hopf bifurcation in the case N even. Section VI summarizes the results for N odd and discusses the relevance of bifurcation methods for other laser array problems. All mathematical details are described in the Appendix.

II. FORMULATION

We consider a system of N coupled semiconductor lasers described in dimensionless form by the following equations [8]:

$$\frac{dY_j}{d\sigma}=(1-i\alpha)Z_jY_j+i\eta(Y_{j+1}+Y_{j-1}) , \tag{2.1}$$

$$T\frac{dZ_j}{d\sigma}=p-Z_j-(1+2Z_j)|Y_j|^2 . \tag{2.2}$$

In these equations, the variables Y_j and Z_j are defined as the normalized electrical field and normalized excess carrier density in the jth laser, respectively. The basic time σ is defined as $\sigma=t/\tau_p$ where τ_p denotes the photon lifetime. The parameter p is the normalized excess pump current ($p\approx0.05$), η is the coupling strength ($\eta\approx10^{-3}-10^{-4}$), α is the linewidth-enhancement factor ($\alpha\approx5$), and T is equal to the ratio τ_s/τ_p where τ_s is the spontaneous carrier lifetime τ_s ($T=2\times10^3$). Typical values of these parameters are shown in parentheses and were used in numerical studies [8]. As Otsuka [9], we consider a ring geometry (looped coupled waveguide lasers [9]) and introduce periodic boundary conditions given by

$$Y_{N+1}(t)=Y_1(t) \text{ and } Z_{N+1}(t)=Z_1(t) . \tag{2.3}$$

The ring geometry allows us to find all the steady-state solutions of Eqs. (2.1)–(2.3). For the case of a one-dimensional geometry with fixed boundary conditions (open coupled waveguide lasers [9]), the steady-state solutions cannot be obtained for an arbitrary N. They must be determined either numerically or asymptotically for a specific range of values of the parameters (for example, $N=2$ [8] and $N\to\infty$ [12]).

It is convenient to reformulate Eqs. (2.1)–(2.3) in terms of the amplitude (X_j) and phase (ϕ_j) of the electrical field. Inserting

$$Y_j=X_j\exp(i\phi_j) \tag{2.4}$$

into Eqs. (2.1)–(2.3), we find

$$\frac{dX_j}{d\sigma}=Z_jX_j-\eta[X_{j+1}\sin(\phi_{j+1}-\phi_j)+X_{j-1}\sin(\phi_{j-1}-\phi_j)] , \tag{2.5}$$

$$T\frac{dZ_j}{d\sigma}=p-Z_j-(1+2Z_j)X_j^2 , \tag{2.6}$$

$$\frac{d\phi_j}{d\sigma}=-\alpha Z_j+\eta X_j^{-1}[X_{j+1}\cos(\phi_{j+1}-\phi_j)+X_{j-1}\cos(\phi_{j-1}-\phi_j)] , \tag{2.7}$$

$$X_{N+1}(\sigma)=X_1(\sigma),\quad Z_{N+1}(\sigma)=Z_1(\sigma),\quad \phi_{N+1}(\sigma)=\phi_1(\sigma)+m2\pi , \tag{2.8}$$

where $m=0,\pm1,\pm2,\ldots$ is arbitrary.

III. STEADY-STATE SOLUTIONS AND THEIR LINEAR STABILITY

We seek a steady-state solution satisfying the conditions

$$dX_j/d\sigma=dZ_j/d\sigma=d(\Delta_{j+1,j})/d\sigma=0 , \tag{3.1}$$

where $\Delta_{j+1,j}=\phi_{j+1}-\phi_j$ is defined as the phase difference between laser $j+1$ and laser j and which is independent of j. We denote the steady-state solution by $X_j=\bar{X}$, $Z_j=\bar{Z}$, and $\Delta_{j+1,j}=\bar{\Delta}$. From the steady-state equations (3.1) and since $\Delta_{j-1,j}=-\bar{\Delta}$, $\phi_0=\phi_N+m2\pi$, and $\phi_{N+1}=\phi_1+m2\pi$, we obtain the conditions

$$\bar{X}=p^{1/2},\quad \bar{Z}=0 , \tag{3.2}$$

and

$$\sin(\bar{\Delta})=\sin(\phi_1-\phi_N) \text{ and } \cos(\bar{\Delta})=\cos(\phi_1-\phi_N) . \tag{3.3}$$

Since we may write $\phi_N-\phi_1$ as $\phi_N-\phi_1=(N-1)\bar{\Delta}$, Eq. (3.3) implies that

$$\bar{\Delta}=-(N-1)\bar{\Delta}+2\pi m \tag{3.4}$$

or, equivalently,

$$\bar{\Delta}=\frac{1}{N}m2\pi \quad (m=0,1,2,\ldots,N-1) . \tag{3.5}$$

Using (3.5), we then determine an expression for $\phi_j(\sigma)=\bar{\phi}_j(\sigma)$:

$$\bar{\phi}_j=2\eta\cos\left[\frac{1}{N}m2\pi\right]\sigma+\phi_0+j\frac{1}{N}m2\pi . \tag{3.6}$$

In summary, we have found N distinct steady-state solutions characterized by a constant intensity and an identical constant phase difference between each laser [$\bar{\Delta}=\bar{\Delta}(m)$]. We call m the wave number of the steady state. For example, if $m=N/2$ and N is even, we find

$$Y_j(\sigma)=(-1)^jp^{1/2}\exp[i(\phi_0-2\eta\sigma)] ,\quad Z_j=0 \quad (1\le j\le N) . \tag{3.7}$$

We now analyze the linear stability of the steady-state solution $(X_j, Z_j, \phi_j) = (\bar{X}, 0, \bar{\phi}_j)$. To this end, we introduce the deviations u_j, v_j, and w_j defined by

$$X_j = \bar{X} + u_j, \quad Z_j = v_j, \quad \text{and} \quad \phi_j = \bar{\phi}_j + w_j \; . \tag{3.8}$$

Neglecting all quadratic terms, the linearized equations are given by

$$\frac{du_j}{d\sigma} = \bar{X} v_j - \eta \sin\left[\frac{1}{N} 2\pi m\right](u_{j+1} - u_{j-1}) - \eta \bar{X} \cos\left[\frac{1}{N} 2\pi m\right](w_{j+1} + w_{j-1} - 2w_j) \; , \tag{3.9}$$

$$T\frac{dv_j}{d\sigma} = -2\bar{X} u_j - (1 + 2\bar{X}^2) v_j \; , \tag{3.10}$$

$$\frac{dw_j}{d\sigma} = -\alpha v_j - \eta \sin\left[\frac{1}{N} 2\pi m\right](w_{j+1} - w_{j-1}) + \eta \bar{X}^{-1} \cos\left[\frac{1}{N} 2\pi m\right](u_{j+1} + u_{j-1} - 2u_j) \quad (1 \le j \le N) \; . \tag{3.11}$$

The periodic boundary conditions for u_j, v_j, and w_j are formulated as

$$\begin{aligned} &u_0(t) = u_N(t), \quad u_{N+1}(t) = u_1(t) \; , \\ &v_0(t) = v_N(t), \quad v_{N+1}(t) = v_1(t) \; , \\ &w_0(t) = w_N(t), \quad w_{N+1}(t) = w_1(t) \; . \end{aligned} \tag{3.12}$$

The general solution of Eqs. (3.9)–(3.12) is a linear combination of eigenfunctions of the form

$$u_j = p e^{\omega\sigma} \psi_j, \quad v_j = q e^{\omega\sigma} \psi_j, \quad w_j = r e^{\omega\sigma} \psi_j \; , \tag{3.13}$$

where ψ_j satisfies the eigenvalue problem

$$\psi_{j+1} + \psi_{j-1} - 2\psi_j = -k^2 \psi_j, \quad \psi_{N+1} = \psi_1, \quad \psi_0 = \psi_N \; , \tag{3.14}$$

and k^2 is the eigenvalue. Solving (3.14), we find

$$\psi_j = e^{\pm i j \theta}, \quad \theta = \frac{1}{N} 2\pi n \quad (n = 0, \ldots, N-1) \; , \tag{3.15}$$

and

$$k^2 = 2[1 - \cos(\theta)] = 4 \sin^2(\theta/2) \; . \tag{3.16}$$

Note that there are N possible solutions. We call n the wave number of the perturbation. Substituting (3.13) and (3.15) into Eqs. (3.9)–(3.12) leads to a characteristic equation for the growth rate ω. Because T is large and η is small, we introduce the parameters ϵ and b defined by

$$\epsilon = T^{-1/2} \quad \text{and} \quad \eta = \epsilon^2 b \quad [b = O(1)] \; . \tag{3.17}$$

The scalings of η and ω are motivated by the analysis of the case $N = 2$ [8] which indicates that an oscillatory instability appears as $\eta = O(T^{-1})$, with frequency $\omega = O(T^{-1/2})$. In terms of (3.17), the characteristic equation for $\omega = \epsilon \Omega$ is given by

$$\Omega^3 + D_1 \Omega^2 + D_2 \Omega + D_3 = 0 \; , \tag{3.18}$$

where

$$D_1 = \epsilon\left[(1 + 2\bar{X}^2) + 4ib \sin\left[\frac{1}{N} 2\pi m\right] \sin(\theta)\right] , \tag{3.19}$$

$$D_2 = 2\bar{X}^2 + 4i\epsilon^2 b(1 + 2\bar{X}^2) \sin\left[\frac{1}{N} 2\pi m\right] \sin(\theta) + \epsilon^2 b^2 \left[\cos^2\left[\frac{1}{N} 2\pi m\right] k^4 - \sin^2\left[\frac{1}{N} 2\pi m\right] 4 \sin^2(\theta)\right] , \tag{3.20}$$

$$D_3 = \epsilon b\left[-2\alpha \bar{X}^2 \cos\left[\frac{1}{N} 2\pi m\right] k^2 + 4i \sin(\theta) \bar{X}^2 \sin\left[\frac{1}{N} 2\pi m\right]\right] + (1 + 2\bar{X}^2)\epsilon^3 b^2 \left[\cos^2\left[\frac{1}{N} 2\pi m\right] k^4 - 4 \sin^2(\theta) \sin^2\left[\frac{1}{N} 2\pi m\right]\right] . \tag{3.21}$$

Since $\epsilon = O(10^{-2})$ is small, we determine the roots of (3.18) using a perturbation method. Specifically, we seek a solution of the form

$$\Omega(\epsilon) = \Omega_0 + \epsilon \Omega_1 + \epsilon^2 \Omega_2 + \cdots \; . \tag{3.22}$$

Substituting (3.22) into (3.18) and equating to zero the coefficients of each power of ϵ leads to a succession of problems for Ω_0, Ω_1, ... We have found the following three solutions:

(1) One complex root

$$\Omega_1(\epsilon) = -\epsilon (2\bar{X}^2)^{-1} b\left[-2\alpha \bar{X}^2 \cos\left[\frac{1}{N} 2\pi m\right] k^2 + 4i \sin(\theta) \bar{X}^2 \sin\left[\frac{1}{N} 2\pi m\right]\right] + O(\epsilon^2) \tag{3.23}$$

and (2) a pair of complex-conjugate roots

$$\Omega_{2,3}(\epsilon)=\pm i2^{1/2}\bar{X}+\frac{\epsilon}{2}\left[-(1+2\bar{X}^2)-2ib\sin\left[\frac{1}{N}2\pi m\right]\sin(\theta)-b\alpha\cos\left[\frac{1}{N}2\pi m\right]k^2\right]+O(\epsilon^2)\ . \quad (3.24)$$

From (3.23), we find that $\mathrm{Re}(\Omega_1)<0$ $[\mathrm{Re}(\Omega_1)>0]$ if

$$\cos\left[\frac{1}{N}2\pi m\right]<0\ , \quad (3.25)$$

$$\left[\cos\left[\frac{1}{N}2\pi m\right]>0\right]\ . \quad (3.26)$$

Thus we conclude from (3.25) that one condition for stability is the inequalities

$$\frac{N}{4}<m<\frac{3N}{4}\ . \quad (3.27)$$

We now analyze the real part of $\Omega_{2,3}$. From (3.24), we find that $\mathrm{Re}(\Omega_{2,3})<0$ if

$$(1+2\bar{X}^2)+b\alpha\cos\left[\frac{1}{N}2\pi m\right]k^2>0\ . \quad (3.28)$$

and $\mathrm{Re}(\Omega_{2,3})>0$ if

$$(1+2\bar{X}^2)+b\alpha\cos\left[\frac{1}{N}2\pi m\right]k^2<0\ . \quad (3.29)$$

Assuming that the stability condition (3.27) is satisfied, the condition (3.28) implies that the coupling strength b

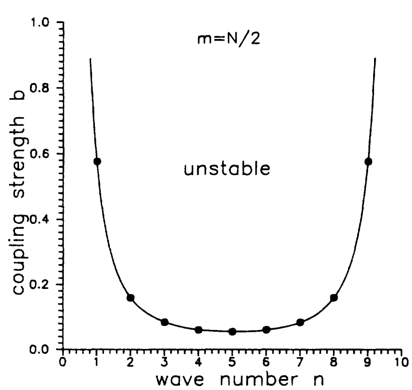

FIG. 1. Neutral stability curve for N even. The full curve represents the Hopf bifurcations b_{mn} for $m=N/2$ and as a function of n taken as a continuous variable. For $m\neq N/2$, the function b_{mn} as a function of n is similar to the curve for $m=N/2$ but the minimum appears at a higher value of b. The dots indicate the permitted wave numbers. The figure illustrates the case $N=10$. Note that the minimum of the curve corresponds exactly to $n=N/2$.

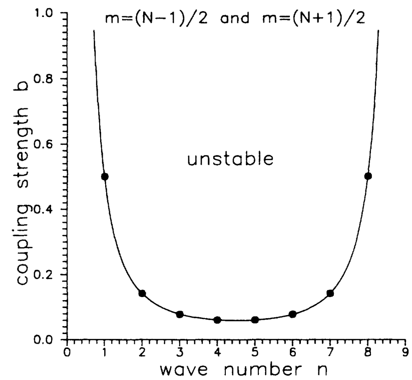

FIG. 2. Neutral stability curve for N odd. The full curve represents the Hopf bifurcations b_{mn} for $m=(N-1)/2$ or $m=(N+1)/2$ and as a function of n taken as a continuous variable. For $m<(N-1)/2$ or $m>(N+1)/2$, the function b_{mn} as a function of n is similar but the minimum appears at a higher value of b. The dots indicate the permitted wave numbers. The figure illustrates the case $N=9$. Note that the first instability corresponds to $n=(N-1)/2$ *and* $(N+1)/2$.

is below a critical value $b=b_{mn}$:

$$b<b_{mn}=-\alpha^{-1}\frac{(1+2\bar{X}^2)}{\cos[(1/N)2\pi m]k^2}\ . \quad (3.30)$$

The critical values $b=b_{mn}$ correspond to Hopf bifurcation points. They are characterized by two wave numbers m and n. Recall that m is the wave number associated with the steady-state solution and n is the wave number of the small-amplitude perturbation. As b progressively increases from zero, the first bifurcation point corresponds to $\min(b_{mn})$. Taking into account the condition (3.27), we find from (3.30) that this minimum appears for the steady state with "$m=N/2$" and with respect to a perturbation of wave number "$n=N/2$" if N is even. If N is odd, m is either equal to $(N-1)/2$ or equal to $(N+1)/2$. Moreover, both the wave numbers $n=(N-1)/2$ and $(N-1)/2$ are possible candidates for an instability because we obtain the same minimal value of b_{mn}. Figures 1 and 2 illustrate the two cases. If N is even, we determine from (3.30) a simple expression for $b_0=\min(b_{mn})$:

$$b=b_0=\alpha^{-1}\tfrac{1}{4}(1+2\bar{X}^2)\ \ (N\ \text{even})\ . \quad (3.31)$$

Using (3.17), we may then obtain an approximation of the first bifurcation point $\eta=\eta_c$. The scaling $\eta_c=O(T^{-1}\alpha^{-1})$ is in agreement with the result of a different analysis given in [12]. Moreover the case $N=2$ can be analyzed in detail and does not require the limit $\epsilon\to 0$ (the laser equations are the same as the equations given in [7] except that η must be replaced by 2η because we consider periodic boundary conditions and $Y_{j+1}=Y_{j-1}$). We have found that the out-of-phase solution becomes unstable if $b>b^*$ where b^* is identical to b_0.

IV. BIFURCATION ANALYSIS FOR $m=N/2$ AND N EVEN

We consider Eqs. (2.5)–(2.8) with $\epsilon^2=T^{-1}$ and $\eta=\epsilon^2 b$, or equivalently,

$$\frac{dX_j}{d\sigma}=Z_jX_j-\epsilon^2 b[X_{j+1}\sin(\phi_{j+1}-\phi_j)+X_{j-1}\sin(\phi_{j-1}-\phi_j)] , \quad (4.1)$$

$$\epsilon^{-2}\frac{dZ_j}{d\sigma}=p-Z_j-(1+2Z_j)X_j^2 , \quad (4.2)$$

$$\frac{d\phi_j}{d\sigma}=-\alpha Z_j+\epsilon^2 bX_j^{-1}[X_{j+1}\cos(\phi_{j+1}-\phi_j)+X_{j-1}\cos(\phi_{j-1}-\phi_j)] . \quad (4.3)$$

These equations can be simplified if we introduce new variables defined as

$$x_j=\left[\frac{2}{p}\right]^{1/2}\epsilon^{-1}Z_j,\quad y_j=(X_j^2-p)p^{-1}, \quad s=(2p)^{1/2}\epsilon\sigma . \quad (4.4)$$

These new variables are motivated by the expressions of the frequency and the eigenvector of the critical mode previously obtained by the linear stability analysis. Similar changes of variables have been proposed for other class-*B* lasers (such as CO_2 lasers). Equations (4.1)–(4.3) then become

$$\frac{dx_j}{ds}=-y_j-\epsilon(2p)^{-1/2}x_j[1+2p(1+y_j)] , \quad (4.5)$$

$$\frac{dy_j}{ds}=(1+y_j)x_j-\epsilon b\left[\frac{2}{p}\right]^{1/2}[(1+y_{j+1})^{1/2}(1+y_j)^{1/2}\sin(\phi_{j+1}-\phi_j)+(1+y_{j-1})^{1/2}(1+y_j)^{1/2}\sin(\phi_{j-1}-\phi_j)] , \quad (4.6)$$

$$\frac{d\phi_j}{ds}=-\tfrac{1}{2}\alpha x_j+\epsilon b(2p)^{-1/2}(1+y_j)^{-1/2}[(1+y_{j+1})^{1/2}\cos(\phi_{j+1}-\phi_j)+(1+y_{j-1})^{1/2}\cos(\phi_{j-1}-\phi_j)] . \quad (4.7)$$

We are interested in determining the periodic solution which is bifurcating from the steady state $m=N/2$. In terms of the new variables, this steady-state solution is given by

$$x_j=y_j=0 \text{ and } \phi_j=-2\epsilon b(2p)^{-1/2}s+\phi_0+j\pi . \quad (4.8)$$

We are now ready for the bifurcation analysis. We first introduce the new variables u_j, v_j, and w_j which are defined as the deviations from the steady state,

$$u_j=x_j,\quad v_j=y_j,\quad w_j=\phi_j-[-2\epsilon b(2p)^{-1/2}s+\phi_0+j\pi] . \quad (4.9)$$

We then seek a small-amplitude solution of the form

$$u_j(S,\nu)=\nu u_{j1}(S)+\nu^2u_{j2}(S)+\cdots , \quad (4.10)$$

$$v_j(S,\nu)=\nu v_{j1}(S)+\nu^2v_{j2}(S)+\cdots , \quad (4.11)$$

$$w_j(S,\nu)=D(\nu)S+\nu w_{j1}(S)+\nu^2w_{j2}(S)+\cdots , \quad (4.12)$$

where the new time S is defined by

$$S=\omega(\nu)s=[\omega_0+\nu^2\omega_2+O(\nu^4)]s . \quad (4.13)$$

All the correction terms $u_{j1},u_{j2},\ldots,v_{j1},v_{j2},\ldots,w_{j1},w_{j2},\ldots$ are 2π-periodic functions of S. Note from (4.9) that w_j corresponds to ϕ_j, the phase of the electrical field. The phase has an expression of the form $\phi_j(S)=R_0+R_1S+R_2(S)$ where R_2 is a 2π-periodic function of S. By introducing the term

$$D(\nu)S=(\nu^2D_2+\nu^3D_3+\cdots)S \quad (4.14)$$

in (4.12), we allow a possible $O(\nu^2)$ correction term for its linear dependence in time. The small parameter ν is proportional to the deviation $b-b_0$ and is defined by

$$b(\nu)-b_0=\nu^2c , \quad (4.15)$$

where $c=1$ if $b-b_0>0$ and $c=-1$ if $b-b_0<0$. We introduce (4.10)–(4.15) into Eqs. (4.5)–(4.7) and equate to zero the coefficients of each power of ν. We then obtain a succession of problems for the unknown coefficients. The three first problems are given in the Appendix. Each problem is then solved by a perturbation analysis valid in the limit $\epsilon\to 0$.

In summary, we have found the following results. From the $O(\nu)$ problem, we obtain that the Hopf bifurcation is located at

$$b_0(\epsilon)=\frac{1}{4\alpha}(1+2p)+O(\epsilon^2) \quad (4.16)$$

and the frequency of the oscillations at the Hopf bifurcation point is

$$\omega_0(\epsilon)=1+0(\epsilon^2) . \quad (4.17)$$

The periodic solution corresponds to a *standing-wave solution* in space and is harmonic in time. It is given by

$$u_j=\nu(\beta e^{i(S+j\pi)}+\text{c.c.})+O(\nu^2) , \quad (4.18)$$

$$v_j=\nu(\beta Be^{i(S+j\pi)}+\text{c.c.})+O(\nu^2) , \quad (4.19)$$

$$w_j=\nu(\beta Ce^{i(S+j\pi)}+\text{c.c.})+O(\nu^2) , \quad (4.20)$$

where B and C are two constant coefficients defined by

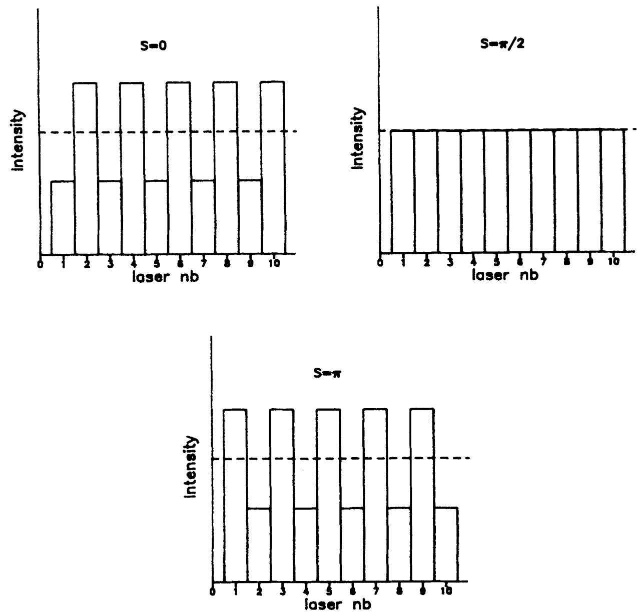

FIG. 3. The standing-wave solution. The intensity of the 2π-time-periodic standing-wave solution is represented for ten coupled lasers and for three different times.

$$B(\epsilon)=-i+O(\epsilon) \quad \text{and} \quad C(\epsilon)=i\frac{\alpha}{2}+O(\epsilon) \ . \tag{4.21}$$

Figure 3 represents the intensity of the standing-wave solution for an array of ten lasers and during half the period (time $S=0$, $\pi/2$, and π). The coefficient $\nu\beta$ is the amplitude of the periodic solution and $\beta=\beta(c)$ is determined from the $O(\nu^3)$ solvability condition. The expression of $|\beta|^2$ is given in (A42) and the fact that it is positive gives the direction of bifurcation. The bifurcation is *supercritical* (i.e., $b>b_0$), if $c=1$ which then requires the condition (A43). On the other hand, the bifurcation is *subcritical* (i.e., $b<b_0$) if $c=-1$ which then requires the condition (A44). A supercritical bifurcation means that the transition to the oscillations is smooth and the amplitude progressively increases from zero. A subcritical bifurcation implies a hard transition near the bifurcation point to large-amplitude solutions which may or may not be periodic and are not approximated by our analysis. From (A42), we find that the supercritical case appears if

$$\alpha>\alpha_c=(\tfrac{13}{9})^{1/2}\approx 1.20 \ . \tag{4.22}$$

If $\alpha<\alpha_c$, the bifurcation is subcritical and leads to unstable periodic solutions.

V. DISCUSSION

We have investigated the first Hopf bifurcation of a system of N coupled class-B lasers. If N is even, we have shown that the bifurcation leads to a specific time-periodic standing-wave solution. It corresponds to a preferential mode of instability because the wave number is equal to $N/2$ for all values of the laser parameters.

Since the wave number is $O(N)$ as $N\rightarrow\infty$, the standing-wave solution varies a lot from one laser to the next in the array. It is not a function of the slow space scale j/N and cannot be determined by using the continuum limit as in [12]. To use the continuum limit as $N\rightarrow\infty$ and determine an approximation of the time-periodic standing-wave solution, we must seek a solution of the discrete laser equations that depends on both the position j and the slow space j/N [15].

The bifurcation results described in this paper will be useful if we investigate the stabilizing effects of additional control mechanisms (injection locking, periodic modulations, parallel coupling). Provided that these additional terms in the laser equations can be treated as weak perturbation terms, we may derive amplitude equations which are slight modifications of the bifurcation equation of Eq. (A40). In particular, the nonlinear term in (A40) will remain unchanged.

Of particular physical interest is the fact that the condition for a smooth transition to small-amplitude time-periodic solutions only depends on α. Increasing α (i.e., $\alpha>\alpha_c\approx 1.2$) has a stabilizing effect even if the value of the bifurcation point $\eta=\eta_c(\alpha)$ is decreased because we may guarantee the existence of stable small-amplitude time-periodic solutions. We have analyzed the predictions for the direction of bifurcation (4.22) by investigating numerically the laser equations. For $T=2000$, $p=0.05$ (the values of the parameters used by Winful and Rahman [8]), and $N=4$, we have determined the branch of periodic solutions which appears at the first Hopf bifurcation point. We have used both a continuation method (AUTO [16]) and a direct integration method. We have found that the bifurcation is supercritical if $\alpha=1.3$ and subcritical if $\alpha=1.1$. This is in agreement with (4.22) since the condition predicts that the bifurcation changes direction at $\alpha_c\approx 1.20$. In addition, we have found numerically that the branch of stable periodic solutions which appears in the supercritical case ($\alpha=1.3$ and 5) admits a *secondary bifurcation* to quasiperiodic solutions at a value of b slightly larger than the Hopf bifurcation point $b=b_0$. This secondary bifurcation reveals the effects of the nearby modes ($n=N/2+1$ and $N/2-1$).

If N is odd, the bifurcation analysis is more complicated because the first instability corresponds to a multiple eigenvalue [17]. Indeed, both modes with wave numbers $n=N/2+1$ and $N/2-1$ become unstable at the same time as the bifurcation parameter is increased. Their interaction leads to multiple branches of solutions (two stable traveling-wave solutions and one unstable standing-wave solution) [15].

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APPENDIX: HOPF BIFURCATION ANALYSIS

In terms of the new variables (4.9), we rewrite Eqs. (4.5)–(4.7) as

$$\frac{du_j}{ds}=-v_j-\epsilon(2p)^{-1/2}[u_j(1+2p)+2pu_jv_j] , \tag{A1}$$

$$\frac{dv_j}{ds}=u_j+u_jv_j+\epsilon b\left[\frac{2}{p}\right]^{1/2}\{[1+\tfrac{1}{2}(v_j+v_{j+1})+\tfrac{1}{4}v_jv_{j+1}-\tfrac{1}{8}(v_j^2+v_{j+1}^2)+\cdots]\sin(w_{j+1}-w_j)$$
$$+[1+\tfrac{1}{2}(v_j+v_{j-1})+\tfrac{1}{4}v_jv_{j-1}-\tfrac{1}{8}(v_j^2+v_{j-1}^2)+\cdots]\sin(w_{j-1}-w_j)\} , \tag{A2}$$

$$\frac{dw_j}{ds}=-\tfrac{1}{2}\alpha u_j+\epsilon b(2p)^{-1/2}\{[1+\tfrac{1}{2}(v_{j+1}-v_j)+\tfrac{3}{8}v_j^2-\tfrac{1}{8}v_{j+1}^2-\tfrac{1}{4}v_{j+1}v_j+\cdots][-1+\tfrac{1}{2}(w_{j+1}-w_j)^2+\cdots]$$
$$+[1+\tfrac{1}{2}(v_{j-1}-v_j)+\tfrac{3}{8}v_j^2-\tfrac{1}{8}v_{j-1}^2-\tfrac{1}{4}v_{j-1}v_j+\cdots][-1+\tfrac{1}{2}(w_{j-1}-w_j)^2+\cdots]\} , \tag{A3}$$

where we have expanded the functions $(1+y_j)^{1/2},\ldots$ and $\cos(\phi_{j+1}-\phi_j),\ldots$ in Taylor series. Introducing (4.10)–(4.15) into Eqs. (A1)–(A3) leads to a succession of problems for the unknown coefficients:

$O(\nu)$,

$$\omega_0u_{j1S}=-\epsilon(2p)^{-1/2}(1+2p)u_{j1}-v_{j1} , \tag{A4}$$

$$\omega_0v_{j1S}=u_{j1}+\epsilon b_0\left[\frac{2}{p}\right]^{1/2}(w_{j+1,1}+w_{j-1,1}-2w_{j1}) , \tag{A5}$$

$$\omega_0w_{j1S}=-\tfrac{1}{2}\alpha u_{j1}-\tfrac{1}{2}\epsilon b_0(2p)^{-1/2}(v_{j+1,1}+v_{j-1,1}-2v_{j1}) ; \tag{A6}$$

$O(\nu^2)$,

$$\omega_0u_{j2S}+\epsilon(2p)^{-1/2}(1+2p)u_{j2}+v_{j2}=-\epsilon(2p)^{1/2}u_{j1}v_{j1} , \tag{A7}$$

$$\omega_0v_{j2S}-u_{j2}-\epsilon b_0\left[\frac{2}{p}\right]^{1/2}(w_{j+1,2}+w_{j-1,2}-2w_{j2})$$
$$=u_{j1}v_{j1}+\epsilon b_0\left[\frac{2}{p}\right]^{1/2}[\tfrac{1}{2}(v_{j1}+v_{j+1,1})(w_{j+1,1}-w_{j1})+\tfrac{1}{2}(v_{j1}+v_{j-1,1})(w_{j-1,1}-w_{j1})] , \tag{A8}$$

$$\omega_0w_{j2S}+\tfrac{1}{2}\alpha u_{j2}+\tfrac{1}{2}\epsilon b_0(2p)^{-1/2}(v_{j+1,2}+v_{j-1,2}-2v_{j2})=\epsilon b_0(2p)^{-1/2}[-\tfrac{3}{8}v_{j1}^2+\tfrac{1}{8}v_{j+1,1}^2+\tfrac{1}{4}v_{j+1,1}v_{j1}+\tfrac{1}{2}(w_{j+1,1}-w_{j1})^2$$
$$-\tfrac{3}{8}v_{j1}^2+\tfrac{1}{8}v_{j-1,1}^2+\tfrac{1}{4}v_{j-1,1}v_{j1}$$
$$+\tfrac{1}{2}(w_{j-1,1}-w_{j1})^2]-\omega_0D_2 ; \tag{A9}$$

$O(\nu^3)$,

$$\omega_0u_{j3S}+\epsilon(2p)^{-1/2}(1+2p)u_{j3}+v_{j3}=\epsilon R_1-\omega_2u_{j1S} , \tag{A10}$$

$$\omega_0v_{j3S}-u_{j3}-\epsilon b_0\left[\frac{2}{p}\right]^{1/2}(w_{j+1,3}+w_{j-1,3}-2w_{j3})$$
$$=(u_{j1}v_{j2}+u_{j2}v_{j1})+\epsilon R_2+\epsilon c\left[\frac{2}{p}\right]^{1/2}(w_{j+1,1}+w_{j-1,1}-2w_{j1})-\omega_2v_{j1S} , \tag{A11}$$

$$\omega_0w_{j3S}+\tfrac{1}{2}\alpha u_{j3}+\tfrac{1}{2}\epsilon b_0(2p)^{-1/2}(v_{j+1,3}+v_{j-1,3}-2v_{j3})=\epsilon R_3-\omega_0D_3-\omega_2u_{j1s} , \tag{A12}$$

where R_1 and R_2 are defined by

$$R_1=-(2p)^{1/2}(u_{j1}v_{j2}+u_{j2}v_{j1}) , \tag{A13}$$

$$R_2=b_0\left[\frac{2}{p}\right]^{1/2}\{\tfrac{1}{2}(v_{j2}+v_{j+1,2})(w_{j+1,1}-w_{j1})+\tfrac{1}{2}(v_{j1}+v_{j+1,1})(w_{j+1,2}-w_{j2})$$
$$+\tfrac{1}{2}(v_{j2}+v_{j-1,2})(w_{j-1,1}-w_{j1})+\tfrac{1}{2}(v_{j1}+v_{j-1,1})(w_{j-1,2}-w_{j2})-\tfrac{1}{6}(w_{j+1,1}-w_{j1})^3$$
$$-\tfrac{1}{6}(w_{j-1,1}-w_{j1})^3+[\tfrac{1}{4}v_{j1}v_{j+1,1}-\tfrac{1}{8}(v_{j1}^2+v_{j+1,1}^2)](w_{j+1,1}-w_{j1})$$
$$+[\tfrac{1}{4}v_{j1}v_{j-1,1}-\tfrac{1}{8}(v_{j1}^2+v_{j-1,1}^2)](w_{j-1,1}-w_{j1})\} . \tag{A14}$$

We do not need to know the expression of R_3.

We solve each problem sequentially. The $O(\nu)$ problem has a 2π-periodic solution if

$$b_0(\epsilon)=b_{00}+O(\epsilon^2)=\frac{1}{4\alpha}(1+2p)+O(\epsilon^2)\ , \quad \text{(A15)}$$

$$\omega_0(\epsilon)=\omega_{00}+O(\epsilon^2)=1+O(\epsilon^2)\ . \quad \text{(A16)}$$

The solution is then given by

$$u_{j1}=\beta e^{i(S+j\pi)}+\text{c.c.}\ , \quad \text{(A17)}$$

$$v_{j1}=\beta B e^{i(S+j\pi)}+\text{c.c.}\ , \quad \text{(A18)}$$

$$w_{j1}=\beta C e^{i(S+j\pi)}+\text{c.c.}\ , \quad \text{(A19)}$$

where β is an unknown amplitude and B and C are two constant coefficients defined by

$$B(\epsilon)=-i-\epsilon(2p)^{-1/2}(1+2p)+O(\epsilon^2)\ , \quad \text{(A20)}$$

$$C(\epsilon)=i\frac{\alpha}{2}-2\epsilon b_{00}(2p)^{-1/2}+O(\epsilon^2)\ . \quad \text{(A21)}$$

We now consider the $O(\nu^2)$ problem. We find that D_2 is a function of $\beta\bar{\beta}$ given by

$$D_2(\epsilon)=\epsilon(2p)^{-1/2}\beta\bar{\beta}[-\alpha(1+2p)+b_{00}2(\alpha^2-1)]+O(\epsilon^2)\ . \quad \text{(A22)}$$

The solution for u_{j2}, v_{j2}, and w_{j2} is then of the form

$$u_{j2}=(\beta_2 e^{i(S+j\pi)}+\text{c.c.})+\beta\bar{\beta}A_0 +(\beta^2 A_2 e^{2i(S+j\pi)}+\text{c.c.})\ , \quad \text{(A23)}$$

$$v_{j2}=(\beta_2 B e^{i(S+j\pi)}+\text{c.c.})+\beta\bar{\beta}B_0 +(\beta^2 B_2 e^{2i(S+j\pi)}+\text{c.c.})\ , \quad \text{(A24)}$$

$$w_{j2}=(\beta_2 C e^{i(S+j\pi)}+\text{c.c.})+\beta\bar{\beta}C_0 +(\beta^2 C_2 e^{2i(S+j\pi)}+\text{c.c.})\ , \quad \text{(A25)}$$

where β_2 is a new undetermined coefficient which multiplies the homogeneous solution and $\bar{\beta}$ is the complex conjugate of β. The coefficients A_0, B_0, C_0, A_2, B_2, C_2 are obtained by substituting (A23)–(A25) into Eqs. (A7)–(A9) and solving the resulting algebraic equations for (A_0,B_0,C_0) and (A_2,B_2,C_2), respectively. As $\epsilon\to 0$, we find that C_0 is arbitrary and

$$A_0(\epsilon)=2\epsilon(2p)^{-1/2}(1+2p)+O(\epsilon^2)\ , \quad \text{(A26)}$$

$$B_0(\epsilon)=-\epsilon^2 p^{-1}(1+2p)+O(\epsilon^3)\ , \quad \text{(A27)}$$

$$A_2(\epsilon)=-\frac{i}{3}+\frac{\epsilon}{9}(2p)^{-1/2}(10p-1)+O(\epsilon^2)\ , \quad \text{(A28)}$$

$$B_2(\epsilon)=-\tfrac{2}{3}+\frac{i}{9}\epsilon(2p)^{-1/2}(4p+5)+O(\epsilon^2)\ , \quad \text{(A29)}$$

$$C_2(\epsilon)=\frac{\alpha}{12}+i\frac{\epsilon}{2\alpha}(2p)^{-1/2}\left[\frac{a^2}{36}(38p+7)-(1+2p)\frac{1}{4}\right] +O(\epsilon^2)\ . \quad \text{(A30)}$$

Note that the solvability condition for the $O(\nu^2)$ is identically satisfied because the right-hand side of Eqs. (A7)–(A9) contains only constant terms or terms multiplying $\exp[\pm 2i(S+j\pi)]$. However, the right side of the $O(\nu^3)$ problem contains terms multiplying $\exp[\pm i(S+j\pi)]$. These terms are of the same form as the solution of the homogeneous problem. Therefore the right-hand side must satisfy a solvability condition. This condition will give equations for β and ω_2. To formulate this condition, we first need to solve the homogeneous adjoint problem. This problem is given by the following equations:

$$-\omega_0 u_{jS}^*+\epsilon(2p)^{-1/2}(1+2p)u_j^*-v_j^*+\frac{\alpha}{2}w_j^*=0\ , \quad \text{(A31)}$$

$$-\omega_0 v_{jS}^*+u_j^*+\tfrac{1}{2}\epsilon b_0(2p)^{-1/2}(w_{j+1}^*+w_{j-1}^*-2w_j^*)=0\ , \quad \text{(A32)}$$

$$-\omega_0 w_{jS}^*-\epsilon b_0\left[\frac{2}{p}\right]^{1/2}(v_{j+1}^*+v_{j-1}^*-2v_j^*)=0\ . \quad \text{(A33)}$$

Equations (A31)–(A33) have two solutions given by

$$u_j^*=e^{\pm i(S+j\pi)},\quad v_j^*=B^* e^{\pm i(S+j\pi)}\ , \quad \text{(A34)}$$

and

$$w_j^*=C^* e^{\pm i(S+j\pi)}\ , \quad \text{(A35)}$$

where the coefficients B^* and C^* are defined by

$$B^*=-i+O(\epsilon^2),\quad C^*=-4\epsilon b_{00}(2/p)^{1/2}+O(\epsilon^2)\ . \quad \text{(36)}$$

Note that the correction term in B^* is $O(\epsilon^2)$ and that $C^*=O(\epsilon)$. These properties will be useful in the formulation of the solvability condition. We now consider the $O(\nu^3)$ problem. The first solvability condition is given by

$$\int_0^{2\pi} e^{-i(S+j\pi)}\left\{(\epsilon R_1-\omega_2 u_{j1S})+\bar{B}^*\left[u_{j1}v_{j2}+u_{j2}v_{j1}+\epsilon R_2+\epsilon c\left[\frac{2}{p}\right]^{1/2}(w_{j+1,1}+w_{j-1,1}-2w_{j1})-\omega_2 v_{j1s}\right] +\bar{C}^*[\epsilon R_3-\omega_0 D_3-\omega_2 u_{j1S}]\right\}ds=0\ , \quad \text{(A37)}$$

where bar means complex conjugate. We do not write the second solvability condition because it is the complex conjugate of (A37). We now solve (A37) by expanding the various functions in power series of ϵ and neglecting all $O(\epsilon^2)$ terms. The algebra is easier if we first note a series of simplifications. First, since C^* is $O(\epsilon)$, we do not have to take into account the contribution of ϵR_3 because $C^*\epsilon R_3$ is $O(\epsilon^2)$. Second, the constant term $\omega_0 D_3$ will not appear after integration. Third, we only need to evaluate the leading terms in ϵR_1 and ϵR_2 because their contribution is $O(\epsilon)$.

Finally, the first correction term in B^* is $O(\epsilon^2)$ and we do not need it. The condition (A37) then takes the form

$$-i\omega_2\beta[2-2i\epsilon(2p)^{-1/2}(1+2p)]+2\alpha\epsilon c\left[\frac{2}{p}\right]^{1/2}\beta +\beta^2\bar{\beta}\left[-\frac{i}{3}+\frac{\epsilon}{9}(2p)^{-1/2}(11+22p)-\epsilon(2p)^{-1/2}\frac{(1+2p)}{6}(5+3\alpha^2)\right]=0 . \quad \text{(A38)}$$

After dividing by β, the imaginary and real parts of (A38) lead to two conditions of the form

$$2\omega_2+\tfrac{1}{3}\beta\bar{\beta}=0 , \quad \text{(A39)}$$

$$-\omega_2 2\epsilon(2p)^{-1/2}(1+2p)+2\alpha\epsilon c\left[\frac{2}{p}\right]^{1/2}+\epsilon(2p)^{-1/2}\beta\bar{\beta}\left[\frac{1}{9}(11+22p)-\frac{(1+2p)}{6}(5+3\alpha^2)\right]=0 . \quad \text{(A40)}$$

Solving first (A39) for ω_2 and then (A40) for $\beta\bar{\beta}$, we find

$$\omega_2=-\tfrac{1}{6}\beta\bar{\beta} , \quad \text{(A41)}$$

$$\beta\bar{\beta}=\frac{72\alpha c}{(1+2p)}\frac{1}{(9\alpha^2-13)} . \quad \text{(A42)}$$

The expression (A41) is the correction of the frequency. The expression (A42) gives the amplitude $|\beta|$. From (A42), we note that $\beta\bar{\beta}>0$ requires that

$$\alpha>\alpha_c=(\tfrac{13}{9})^{1/2}\approx 1.20 \quad \text{if } c=1 \quad \text{(A43)}$$

and

$$\alpha<\alpha_c \quad \text{if } c=-1 . \quad \text{(A44)}$$

Following the Hopf bifurcation theorem, we conclude that the bifurcation is *supercritical* and leads to stable periodic solution if $c=1$ (i.e., $b>b_0$) which requires the condition (A43). On the other hand, the bifurcation is *subcritical* and leads to unstable periodic solutions if $c=-1$ (i.e., $b<b_0$) which requires the condition (A44).

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